

# Standing waves in stretched, elastic string

## I. THEORETICAL BACKGROUND

In the measurement we will investigate the propagation of waves in a string with both end fixed. We assume that the propagation is one dimensional (along the string), the wave is transverse (the displacement vectors of the points of the string is perpendicular to the string), in-plane polarized (the displacement vectors lie in the same plane). It follows that displacement of a point from its equilibrium can be described with only one scalar value. Assuming the string is parallel to the  $x$  axis, the wave can be described with

$$c^2 \frac{\partial^2 \Psi(x, t)}{\partial x^2} = \frac{\partial^2 \Psi(x, t)}{\partial t^2}, \quad (1)$$

the one-dimensional wave-equation, where  $x$  is the coordinate,  $t$  is the time,  $\Psi(x, t)$  is the wave-function, which describes the displacement as a function of place and time, i.e. the propagation of the wave, and  $c$  is the speed of the propagation. Deriving the wave-equation for the case of a simple, one-dimensional string, the speed of the propagation,  $c$  only depends only on the stretching force (tension),  $T$  and the linear mass-density of the string,  $\mu$ :

$$c = \sqrt{\frac{T}{\mu}}. \quad (2)$$

The waves formed by an external excitation in the string are generally very complicated. Although it is known that exciting at given frequencies stationary, so called *standing waves* are formed due to the reflection from the ends of the string. A property of the standing waves is that the points of the string are moving in-phase, and amplitude of the oscillation is only the function of the place. Mathematically it means that eq. (1) has a solution, which is the product of a time-dependent and a place-dependent functions (the eq. (1) is separable partial differential equation). If the excitation is harmonic, the solution can be written as

$$\Psi(x, t) = \varphi(x) \sin(\omega t + \alpha), \quad (3)$$

where  $\omega = 2\pi\nu$  is the angular frequency of the oscillation ( $\nu$  is the frequency, Hz), and  $\alpha$  is the initial phase.

Substituting eq. (3) into eq. (1) the time-dependent part cancels, and place-dependent part, which describes the amplitude along the string, gives the following ordinary differential equation:

$$\frac{d^2 \varphi(x)}{dx^2} + k^2 \varphi(x) = 0, \quad (4)$$

where  $k = \omega/c$ , the wavenumber is introduced.

The general solution of eq. (4) is

$$\varphi(x) = A \sin(kx) + B \cos(kx), \quad (5)$$

where  $A$  and  $B$  are arbitrary constants, determined by the boundary conditions. In this case both end of the string is fixed, so the displacement at these points is always zero, i.e.

$$\begin{aligned} \varphi(0) &= 0 \\ \varphi(L) &= 0, \end{aligned} \quad (6)$$

where one end of the string lies in origin of the coordinate system, and length of the string is  $L$ .

The first condition of eq. (6) implies that  $B = 0$ , the solution can have form of

$$\varphi(x) = A \sin(kx). \quad (7)$$

The second condition if eq. (6) restricts the allowed values of the wavenumber to

$$k_n = n \frac{\pi}{L}, \quad (n = 1, 2, 3, \dots) \quad (8)$$

Since the wavenumber,  $k$  is in direct connection with the  $\lambda$  wavelength ( $k = 2\pi/\lambda$ ), eq. (8) implies that standing waves are only formed in case of given discrete value of wavelengths:

$$\lambda_n = \frac{2L}{n}, \quad (n = 1, 2, 3, \dots) \quad (9)$$

Combining this result with the  $\nu = c/\lambda$  formula, in case of fixed speed of propagation (it means fixed tension and linear density according to eq. (2)) the frequency values cannot be arbitrary, either, only given discrete frequency values of

$$\nu_n = n \frac{c}{2L} \quad (n = 1, 2, 3, \dots) \quad (10)$$

are allowed. These frequencies are the resonance frequencies of the string.

As it follows from the discussion above, the solutions, which suit the boundary conditions can be written as

$$\varphi_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right) \quad (n = 1, 2, 3, \dots) \quad (11)$$

The  $A_n$  constants, the maximal values of the amplitudes are determined by the details of the excitation (the initial conditions), but for the given measurement it is not relevant. Assuming that actual standing wave only contains one given  $n$  component, the solution of eq. (1) can be written as

$$\Psi_n(x, t) = A_n \sin\left(\frac{2\pi}{\lambda_n}x\right) \sin(\omega_n t + \alpha_n), \quad (12)$$

where  $n$  is an arbitrary positive integer. The amplitude-distribution (place-dependence) of the standing wave is given by eq. (11), some examples are shown in fig. 1, for  $n = 1, 2$  and  $3$ . As it can be seen from fig. 1 and eq. (11),

the solution for a given  $n$  contains  $n$  antinodes and  $n + 1$  nodes (taking into account the fixed ends of the string). From eq. (11) the distance of two consecutive nodes,

$$d_n = \frac{L}{n} = \frac{\lambda_n}{2}. \quad (13)$$

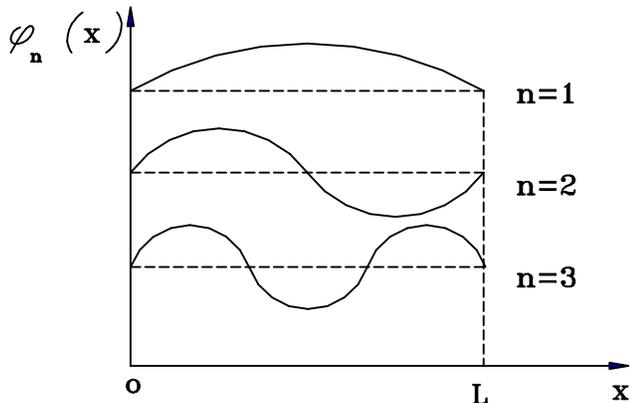


FIG. 1.

According to eq. (10), different values of  $n$  result in different frequencies (pitches). The  $n = 1$  is called fundamental frequency, and higher  $n$  values are called higher harmonics (overtones).

It is useful to note that exciting the string with an ordinary method (e.g. plucking or with a bow) generally more harmonics appear at the same time. (Mathematically it means that the full solution of eq. (1) is the sum of eq. (12)-like solutions, with different values of  $n$ ). But a string has a well defined pitch, since usually the amplitude of the fundamental frequency is much larger than the amplitude of the higher harmonics. The higher harmonics are always present, and the ratio for the different higher harmonics gives the tone of the string. (I.e. the difference of the sound of different musical instruments.)

In the measurement sinusoidal excitation will be used, which enables us to form standing waves with different  $n$  by setting the proper frequencies. Since the excitation itself is a complex process, the frequency of the standing wave will not be the same as the excitation frequency, furthermore higher harmonics will appear, so the resonance frequency has to be determined independently by measuring it.

## II. MEASUREMENT SETUP

The measurement setup (see fig. 2) is a stretched steel string (2) fixed on a board (1). The ends of the string is fixed to an aluminum block (4), which is movable with a screw (3), and to a two-armed lever (5). The length of the oscillating section of the string can be set with two stands (6). The oscillation is induced by an exciter

(driver) coil (8), which is driven by a function generator, via magnetic coupling. The excited wave is transverse, almost in-plane polarized. The oscillation is detected by a detector coil, whose signal is visualized on an oscilloscope along with the excitation signal. The stretching force can be changed by putting weight (11) on the longer (horizontal) arm of the lever (5).

Fixing the string: The copper-pieced end of the string has to be hooked to the shorter (vertical) arm of the lever. The loop on the other end has to be hooked to the screw on the aluminum block. After fixing the string, it has to be tightened by moving the screw (3), so the longer arm of the lever is horizontal.

Relevant tunable parameters of the setup

- The length of the string can be changed by moving the stands (6).
- The stretching force can be changed by changing the position of the weight on the longer arm of the lever. This arm is designed so putting the weight in the first slot, the stretching force is equal to the weight of the weight, putting in the second slot it is twice as large, and so on. Each time changing the stretching force, the longer arm should be fine tuned with the screw (3) to be horizontal.
- The linear mass-density of the string can be changed by using another string with different diameter. The linear mass-density has to be calculated from the diameter, which has to be measured by a screw-micrometer (the density of steel is  $7800 \text{ kg/m}^3$ ).
- The shape of the standing wave can be changed by changing the excitation frequency on the function generator.

The function generator has to be set to sine-wave output and the exciting coil has to be positioned close to one of the stands (about 5 cm distance), since it is most efficient close to a node. Initially the detector coil goes to the center of the the investigated section of the string, then its position has to be changed according to the tasks (the highest output signal is produced close to an antinode).

To find the different standing waves, the excitation frequency should be slowly increased from about 50 Hz, while constantly checking the detector signal on the oscilloscope. Reaching the resonance the detector signal and the sound of the string has a maximum. If the signal is too small, try to change the position of the detector coil, or increase the amplitude of the excitation. After finding the resonance the shape of the standing wave (and so  $n$ ) can be examined by moving the detector coil along the string.

The frequency of the standing wave has to be determined from the detector signal, either directly measuring the period on the oscilloscope, or comparing it to the excitation signal.

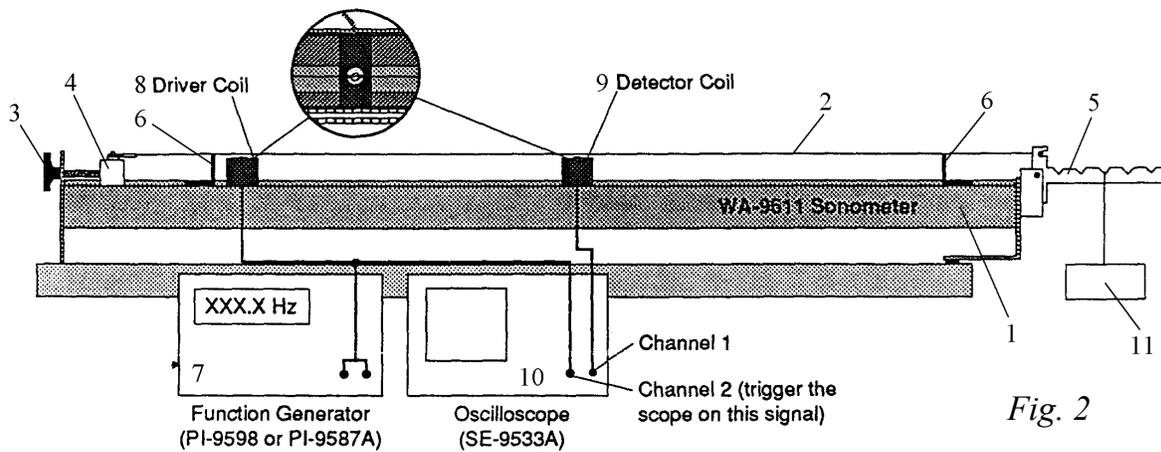


Fig. 2

### III. MEASUREMENTS

1. Measure the diameter of the given string, and calculate the frequency and the speed of the propagating wave if the length of the string is 60 cm and the tension in the string is 60 N. The density of steel is  $7800 \text{ kg/m}^3$ .

Set the length of the string to 60 cm, and stretch it with 60 N (e.g. by putting 2 kg to the third slot of the lever)! Changing the excitation frequency, find the first three ( $n = 1, 2, 3$ ) standing waves. For each of them read the frequency on the screen of the function generator, and measure the coordinates (from one of the stands) of the nodes and antinodes (by moving the detector coil), and determine the wavelength of the standing wave from the measured coordinates. Make a table and check eq. (9). The oscilloscope reads twice as much frequency for the excited signal (the string) than for the driven signal (the function generator). Since reading the screen of the function generator is more accurate than reading the oscilloscope, the frequency of the string is twice that of the function generator.

2. Plot the frequency of the standing waves ( $\nu_n$ ) as a function of  $n$ , fit a linear function to the data. From the slope, determine the speed of sound,  $c$ , and compare it with value calculated from eq. (2).
3. Find the  $n = 1$  standing wave for 40, 50, 60, 70 and 80 cm length using tension of 60 N, and read the resonance frequency in each cases. For finding the frequencies, use that we already know the frequency for the string of length 60 cm. Plot the frequency

as a function of the reciprocal of the length, fit a linear function, and determine the speed of sound from the slope. Compare this value with previous ones.

4. Set the length back to 60 cm, and put the 2 kg weight into the first slot of the lever, and set the first standing wave corresponding to  $n = 1$ . Again, use that we know the frequency for the 60 N tension. Measure the resonance frequency and wavelength and calculate the speed of sound in the string by using the  $c = \lambda\nu$  formula. Make a table and write the values of the tension  $T$ , the linear mass-density  $\mu$ , the fundamental frequency  $\nu$ , and the speed  $c$ . Repeat the measurement using four different tensions by putting the weight farther. Write the measured values into the table.
5. Set the standing wave corresponding to  $n = 1$  with 60 cm length and 60 N tension for other strings given by the leader. Do not forget to measure the diameter of each wire. Give the wavelength and the frequency of the standing wave, and calculate the speed of sound in the strings by using the  $c = \lambda\nu$  formula. Make a table and write the values of the tension  $T$ , the linear mass-density  $\mu$ , the fundamental frequency  $\nu$ , and the speed  $c$ .
6. Using the data collected in tasks 4 and 5 verify eq. (2). (According to the equation, if  $\mu$  is constant,  $c - T^{1/2}$  function is linear, and if  $T$  is constant,  $c - \mu^{-1/2}$  function is linear.) The corresponding data fall along a straight line, the slope of the line is  $(1/\mu)^{1/2}$  in the first case, and  $T^{1/2}$  in the second case.