

Chapter 6. Diving

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- 13 • *Am I comfortable as I fall toward a black hole?*
- 14 • *How fast am I going when I reach the event horizon? Who measures my*
15 *speed?*
- 16 • *How long do I live, measured on my wristwatch, as I fall into a black*
17 *hole?*
- 18 • *How much does the mass of a black hole increase when a stone falls into*
19 *it? when I fall into it?*
- 20 • *How close to a black hole can I stand on a spherical shell and still*
21 *tolerate the “acceleration of gravity”?*

CHAPTER

6

Diving

Edmund Bertschinger & Edwin F. Taylor *

23 *Many historians of science believe that special relativity could have*
24 *been developed without Einstein; similar ideas were in the air at the*
25 *time. In contrast, it's difficult to see how general relativity could*
26 *have been created without Einstein – certainly not at that time, and*
27 *maybe never.*

28 —David Kaiser

6.1 ■ GO STRAIGHT: THE PRINCIPLE OF MAXIMAL AGING IN GLOBAL COORDINATES

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31 *“Go straight!” spacetime shouts at the stone.*
32 *The stone’s wristwatch verifies that its path is straight.*

33 Section 5.7 described how an observer passes through a sequence of local
34 inertial frames, making each measurement in only one of these local frames.
35 Special relativity describes motion in each such local inertial frame. The
36 observer is just a stone that acts with purpose. Now we ask how a
37 (purposeless!) free stone moves in global coordinates.

38 Section 1.6 introduced the Principle of Maximal Aging that describes
39 motion in a single inertial frame. To describe global motion, we need to extend
40 this principle to a *sequence* of adjacent local inertial frames. Here, without
41 proof, is the simplest possible extension, to a *single adjacent pair* of local
42 inertial frames.

DEFINITION 1. Principle of Maximal Aging (curved spacetime)

44 The *Principle of Maximal Aging* states that a free stone follows a
45 worldline through spacetime such that its wristwatch time (aging) is a
46 maximum when summed across every adjoining pair of local inertial
47 frames along its worldline.

Definition: **Principle of Maximal Aging**
in curved spacetime

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Box 1. What Then Is Time?

What then is time? If no one asks me, I know what it is. If I wish to explain it to him who asks me, I do not know.

The world was made, not in time, but simultaneously with time. There was no time before the world.

—St. Augustine (354–430 C.E.)

Time takes all and gives all.

—Giordano Bruno (1548–1600 C.E.)

Everything fears Time, but Time fears the Pyramids.

—Anonymous

Philosophy is perfectly right in saying that life must be understood backward. But then one forgets the other clause—that it must be lived forward.

—Søren Kierkegaard

As if you could kill time without injuring eternity.

Time is but the stream I go a-fishing in.

—Henry David Thoreau

Although time, space, place, and motion are very familiar to everyone, . . . it is useful to distinguish these quantities into absolute and relative, true and apparent, mathematical and common.

—Isaac Newton

Time is defined so that motion looks simple.

—Misner, Thorne, and Wheeler

Nothing puzzles me more than time and space; and yet nothing troubles me less, as I never think about them.

—Charles Lamb

Either this man is dead or my watch has stopped.

—Groucho Marx

“What time is it, Casey?”

“You mean right now?”

—Casey Stengel

It's good to reach 100, because very few people die after 100.

—George Burns

The past is not dead. In fact, it's not even past.

—William Faulkner

Time is Nature's way to keep everything from happening all at once.

—Graffito, men's room, Pecan St. Cafe, Austin, Texas

What time does this place get to New York?

—Barbara Stanwyck, during trans-Atlantic crossing on the steamship *Queen Mary*



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Objection 1. *Now you have gone off the deep end! In Chapter 1, Speeding, you convinced me that the Principle of Maximal Aging was nothing more than a restatement of Newton's First Law of Motion, the observation that in flat spacetime the free stone moves at constant speed along a straight line in space. But in curved spacetime the stone's path will obviously be curved. You have violated your own Principle.*



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On the contrary, we have changed the Principle of Maximal Aging as little as possible in order to apply it to curved spacetime. We require the free stone to move along a straight worldline across *each one* of the pair of adjoining local inertial frames, as demanded by the special relativity Principle of Maximal Aging in each frame. We allow the stone only the choice of one map coordinate of the event, at the boundary between these

Section 6.2 Map Energy from the Principle of Maximal Aging 6-3

60 two frames. That single generalization extends the Principle of Maximal
 61 Aging from flat to curved spacetime. And the result is a single kink in the
 62 worldline. When we shrink all adjoining inertial frames along the worldline
 63 to the calculus limit, then the result is what you predict: a curved worldline
 64 in global coordinates.

65 Now we can use the more general Principle of Maximal Aging to discover
 66 a constant of motion for a free stone, what we call its *map energy*.

6.2.2 MAP ENERGY FROM THE PRINCIPLE OF MAXIMAL AGING

68 *The global metric plus the Principle of Maximal Aging leads to map energy as*
 69 *a constant of motion.*

Map energy: a
constant of motion

70 This section uses the Principle of Maximal Aging from Section 6.1, plus the
 71 Schwarzschild global metric to derive the expression for map energy of a free
 72 stone near a nonspinning black hole. For a free stone, map energy is a constant
 73 of motion; its value remains the same as the stone moves. Our derivation uses
 74 a stone that falls inward along the r -direction, but at the end we show that
 75 the resulting expression for map energy also applies to a stone moving in any
 76 direction; energy is a *scalar*, which has no direction.

?

77 **Objection 2.** *Here is a fundamental objection to the Principle of Maximal*
 78 *Aging: You nowhere derive it, yet you expect us readers to accept this*
 79 *arbitrary Principle. Why should we believe you?*

!

80 Guilty as charged! Our major tool in this book is the metric, which—along
 81 with the topology of a spacetime region—tells us everything we can know
 82 about the shape of spacetime in that region. But the shape of spacetime
 83 revealed by the metric tells us nothing whatsoever about how a free stone
 84 moves in this spacetime. For that we need a second tool, the Principle of
 85 Maximal Aging which, like the metric, derives from Einstein's field
 86 equations. In this book the metric plus the Principle of Maximal
 87 Aging—both down one step from the field equations—are justified by their
 88 immense predictive power. Until we derive the metric in Chapter 22, we
 89 apply the slogan, "Handsome is as handsome does!"

Find maximal aging:
find natural motion.

90 The Principle of Maximal Aging maximizes the stone's total wristwatch
 91 time across *two adjoining* local inertial frames. Figure 1 shows the Above
 92 Frame A (of average map coordinate \bar{r}_A) and adjoining Below Frame B (of
 93 average map coordinate \bar{r}_B). The stone emits initial flash 1 as it enters the top
 94 of Frame A, emits middle flash 2 as it transits from Above Frame A to Below
 95 Frame B, and emits final flash 3 as it exits the bottom of Below Frame B. We
 96 use the three *flash emission events* to find maximal aging.

97 *Outline of the method:* Fix the r - and ϕ -coordinates of all three flash
 98 emissions and fix the t -coordinates of upper and lower events 1 and 3. Next
 99 vary the t -coordinate of the middle flash emission 2 to maximize the total
 100 *wristwatch time* (aging) of the stone across both frames.

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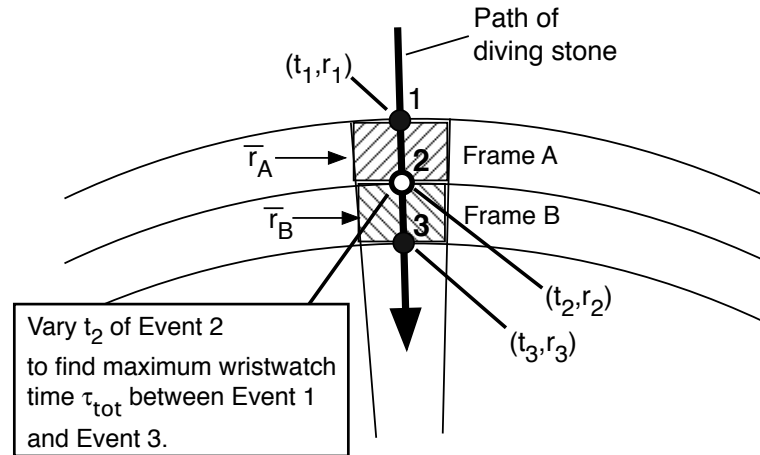


FIGURE 1 Use the Principle of Maximal Aging to derive the expression for Schwarzschild map energy. The diving stone first crosses the Above Frame A, then crosses the Below Frame B, emitting flashes at events 1, 2, and 3. Fix all three coordinates of events 1 and 3; but fix only the r - and ϕ -coordinates of intermediate event 2. Then vary the t -coordinate of event 2 to maximize the *total wristwatch time* (aging) across both frames between fixed end-events 1 and 3. This leads to expression (8) for the stone’s map energy, a constant of motion.

101 So much for t -coordinates. How do we find *wristwatch times* across the two
 102 frames? The Schwarzschild metric ties the increment of wristwatch time to
 103 changes in r - and t -coordinates for a stone that falls inward along the
 104 r -coordinate. Write down the approximate form of the global metric twice,
 105 first for Above frame A (at average \bar{r}_A) and second for the Below frame B (at
 106 average \bar{r}_B). Take the square root of both sides:

Approximate the Schwarzschild metric for each frame.

$$\tau_A \approx \left[\left(1 - \frac{2M}{\bar{r}_A} \right) (t_2 - t_1)^2 + (\text{terms without } t\text{-coordinate}) \right]^{1/2} \quad (1)$$

$$\tau_B \approx \left[\left(1 - \frac{2M}{\bar{r}_B} \right) (t_3 - t_2)^2 + (\text{terms without } t\text{-coordinate}) \right]^{1/2} \quad (2)$$

107 We are interested only in those parts of the metric that contain the map
 108 t -coordinate, because we take derivatives with respect to that t -coordinate. To
 109 prepare for the derivative that leads to maximal aging, take the derivative of
 110 τ_A with respect to t_2 of the intermediate event 2. The denominator in the
 111 resulting derivative is just τ_A :

$$\frac{d\tau_A}{dt_2} \approx \left(1 - \frac{2M}{\bar{r}_A} \right) \frac{(t_2 - t_1)}{\tau_A} \quad (3)$$

112 The corresponding expression for $d\tau_B/dt_2$ is:

Section 6.2 Map Energy from the Principle of Maximal Aging **6-5**

$$\frac{d\tau_B}{dt_2} \approx - \left(1 - \frac{2M}{\bar{r}_B}\right) \frac{(t_3 - t_2)}{\tau_B} \quad (4)$$

113 Add the two wristwatch times to obtain the summed wristwatch time τ_{tot}
 114 between first and last events 1 and 3:

$$\tau_{\text{tot}} = \tau_A + \tau_B \quad (5)$$

Maximize aging summed across both frames.

115 Recall that we keep constant the total t -coordinate separation across both
 116 frames. To find the maximum total wristwatch time, take the derivative of
 117 both sides of (5) with respect to t_2 , substitute from (3) and (4), and set the
 118 result equal to zero in order to find the maximum:

$$\frac{d\tau_{\text{tot}}}{dt_2} = \frac{d\tau_A}{dt_2} + \frac{d\tau_B}{dt_2} \approx \left(1 - \frac{2M}{\bar{r}_A}\right) \frac{(t_2 - t_1)}{\tau_A} - \left(1 - \frac{2M}{\bar{r}_B}\right) \frac{(t_3 - t_2)}{\tau_B} \approx 0 \quad (6)$$

119 From the last approximate equality in (6),

$$\left(1 - \frac{2M}{\bar{r}_A}\right) \frac{(t_2 - t_1)}{\tau_A} \approx \left(1 - \frac{2M}{\bar{r}_B}\right) \frac{(t_3 - t_2)}{\tau_B} \quad (7)$$

120 The expression on the left side of (7) depends only on parameters of the
 121 stone's motion across the Above Frame A; the expression on the right side
 122 depends only on parameters of the stone's motion across the Below Frame B.
 123 Hence the value of either side of this equation must be independent of *which*
 124 adjoining pair of frames we choose to look at: this pair can be *anywhere* along
 125 the worldline of a stone. Equation (7) displays a quantity that has the same
 126 value on *every* local inertial frame along the worldline. We have found the
 127 expression for a quantity that is a constant of motion.

Map energy of a stone in Schwarzschild coordinates

128 Now shrink differences $(t_2 - t_1)$ and $(t_3 - t_2)$ in (7) to their differential
 129 limits. In this process the average r -coordinate becomes exact, so $\bar{r} \rightarrow r$. Next
 130 use the result to *define* the stone's **map energy per unit mass**:

$$\frac{E}{m} \equiv \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \quad (\text{map energy of a stone per unit mass}) \quad (8)$$

Far from the black hole, map energy takes special relativity form.

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 133 Why do we call the expression on the right side of (8) *energy* (per unit mass)?
 134 Because when the mass M of the center of attraction becomes very small—or
 135 when the stone is very far from the center of attraction—the limit $2M/r \rightarrow 0$
 136 describes a stone in flat spacetime. That condition reduces (8) to
 137 $E/m = dt/d\tau$, which we recognize as equation (23) in Section 1.7 for E/m in
 138 flat spacetime. Hence we take the right side of (8) to be the general-relativistic
 139 generalization, near a nonspinning black hole, of the special relativity
 140 expression for E/m .

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Map energy E
same unit as m

141 Note that the right side of (8) has no units; therefore both E and m on
142 the left side must be expressed in the *same* unit, a unit that we may choose for
143 our convenience. *Both* numerator and denominator in E/m may be expressed
144 in kilograms or joules or electron-volts or the mass of the proton, or any other
145 common unit.

Map energy
expression valid
for *any* motion
of the stone.

146 Our derivation of map energy employs only the t -coordinate in the metric.
147 It makes no difference in the outcome for map energy—expression
148 (8)—whether dr or $d\phi$ is zero or not. This has an immediate consequence: The
149 expression for map energy in Schwarzschild global coordinates is valid for a
150 free stone moving on *any* orbit around a spherically symmetric center of
151 attraction, not just along the inward r -direction. We will use this generality of
152 (8) to predict the general motion of a stone in later chapters.

6.3. ■ UNICORN MAP ENERGY VS. MEASURED SHELL ENERGY

154 *Map energy is like a unicorn: a mythical beast*

Map energy E/m
is a unicorn:
a mythical beast.

155 The expression on the right side of equation (8) is like a unicorn: a mythical
156 beast. Nobody measures directly the r - or t -coordinates in this expression,
157 which are Schwarzschild global map coordinates: entries in the mapmaker’s
158 spreadsheet or accounting form. Nobody measures E/m on the left side of (8)
159 either; the map energy is also a unicorn. If this is so, why do we bother to
160 derive expression (8) in the first place? Because E/m has an important virtue:
161 It is a constant of motion of a free stone in Schwarzschild global coordinates; it
162 has the same value at every event along the global worldline of a stone. The
163 value of E/m helps us to predict its global motion (Chapters 8 and 9). But it
164 does not tell us the value of the energy measured by an observer in a local
165 inertial frame.

166 What is the stone’s energy measured by the shell observer? The special
167 relativity energy expression is valid for the shell observer. Equation (9) in
168 Section 5.7 gives us:

$$\Delta t_{\text{shell}} = \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \tag{9}$$

169 Then:

$$\frac{E_{\text{shell}}}{m} = \lim_{\Delta\tau \rightarrow 0} \frac{\Delta t_{\text{shell}}}{\Delta\tau} = \lim_{\Delta\tau \rightarrow 0} \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \frac{\Delta t}{\Delta\tau} \tag{10}$$

170 As we shrink increments to the differential calculus limit, the average
171 r -coordinate becomes exact: $\bar{r} \rightarrow r$. The result is:

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{dt}{d\tau} \quad (\text{shell energy of a stone per unit mass}) \tag{11}$$

172 Into this equation substitute expression (8) for the stone’s map energy to
173 obtain:

Section 6.4 Raindrop Crosses the Event Horizon 6-7

$$\frac{E_{\text{shell}}}{m} = \frac{1}{(1 - v_{\text{shell}}^2)^{1/2}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{E}{m} \quad (12)$$

Shell energy

Different shell observers compute same map energy.

How to get inside the event horizon?

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where we have added the special relativity expression (23) in Section 1.7. Equation (12) tells us how to use the map energy—a unicorn—to predict the frame energy directly measured by the shell observer as the stone streaks past. Expression (12) for shell energy E_{shell} applies to a stone moving in any direction, not just along the r -coordinate. Why? Energy—including map energy E —is a *scalar*, a property of the stone independent of its direction of motion. The shell observer knows only his local shell frame coordinates, which are restricted in order to yield a local inertial frame. He observes a stone zip through his local frame and disappear from that frame; he has no global view of the stone’s path. However, equation (12) is valid for a stone in *every* local shell frame and for *every* direction of motion of the stone in that frame. The shell observer uses this equation and his local r —stamped on every shell—to compute the map energy E/m , then radios his result to every one of his fellow shell observers, for example, “The green-colored free stone has map energy $E/m = 3.7$.” A different shell observer, at different map r , measures a different value of shell energy E_{shell}/m of the green stone as it streaks through his own local frame, typically in a different direction. However, armed with (12), every shell observer verifies the constant value of map energy of the green stone, for example $E/m = 3.7$. In brief, each local shell observer carries out a real measurement of shell energy; from this result plus his knowledge of his r -coordinate he derives the value of the map energy E/m , then uses this map energy—a constant of motion—to predict results of shell energy measurements made by shell observers distant from him. The result is a multi-shell account of the entire orbit of the stone. The entire scheme of shell observers depends on the existence of local shell frames, which cannot be built inside the event horizon. Now we turn to the experience of the diver who passes inward across the event horizon.

6.4 ■ RAINDROP CROSSES THE EVENT HORIZON

Convert t -coordinate to raindrop wristwatch time.

The Schwarzschild metric satisfies Einstein’s field equations everywhere in the vicinity of a nonrotating black hole (except on its singularity at $r = 0$). Map coordinates alone may satisfy Schwarzschild and Einstein, but they do not satisfy us. We want to make every measurement in a local inertial frame. Shell frames serve this purpose nicely outside the event horizon, but we cannot construct stationary shells inside the event horizon. Moreover, the expression $(1 - 2M/r)^{-1/2}$ in energy equation (12) becomes imaginary inside the event

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214 horizon, which provides one more indication that shell energy does not apply
215 there.

216 Yet everyone tells us that an unfortunate astronaut who crosses inward
217 through the event horizon at $r = 2M$ inevitably arrives at the lethal central
218 singularity $r = 0$. In the following chapter we build a local frame around a
219 falling astronaut. To prepare for such a local diving frame, we start here as
220 simply as possible: We ask the stone wearing a wristwatch that began our
221 study of relativity (Section 1.1) to take a daring dive, to drop from initial rest
222 far from the black hole and plunge inward to $r = 0$. We call this diving,
223 wristwatch-wearing stone a **raindrop**, because on Earth raindrops fall from
224 rest at a great height. By definition, the raindrop has no significant spatial
225 extent; it has no frame, it is just a stone wearing a wristwatch.

Raindrop defined:
stone dropped
from rest far away

226 **DEFINITION 2. Raindrop**

227 **A raindrop is a stone wearing a wristwatch, that freely falls inward**
228 **starting from initial rest far from the center of attraction.**

Map energy of
a raindrop

229 Examine the map energy (8) of a raindrop. Far from the black hole
230 $r \gg 2M$ so that $(1 - 2M/r) \rightarrow 1$. For a stone at rest there, $dr = d\phi = 0$ and
231 the Schwarzschild metric tells us that $d\tau \rightarrow dt$. As a result, (8) becomes:

$$\frac{E}{m} \equiv \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \rightarrow 1 \quad (\text{raindrop: released from rest at } r \gg 2M) \quad (13)$$

232 The raindrop, released from rest far from the black hole, must fall inward
233 along a radial line. In other words, $d\phi = 0$ along the raindrop worldline.
234 Formally we write:

$$\frac{d\phi}{d\tau} = 0 \quad (\text{raindrop}) \quad (14)$$

235 The raindrop-stone, released from rest at a large r map coordinate, begins
236 to move inward, gradually picks up speed, finally plunges toward the center.
237 As the raindrop hurtles inward, the value of $E/m (= 1)$ remains constant.
238 Equation (12) then tells us that as r decreases, $2M/r$ increases, and so E_{shell}
239 must also increase, implying an increase in v_{shell} . The local shell observer
240 measures this increased speed directly. Equation (12) with $E/m = 1$ for the
241 raindrop yields:

Shell energy of
the raindrop

$$\frac{E_{\text{shell}}}{m} = (1 - v_{\text{shell}}^2)^{-1/2} = \left(1 - \frac{2M}{r}\right)^{-1/2} \quad (\text{raindrop}) \quad (18)$$

242 It follows immediately that:

$$v_{\text{shell}} = - \left(\frac{2M}{r}\right)^{1/2} \quad (\text{raindrop shell velocity}) \quad (19)$$

243 where the negative value of the square root describes the stone's inward
244 motion. Equation (19) shows that the shell-measured speed of the

Box 2. Slow speed + weak field \implies Mass + Newtonian KE and PE

"If you fall, I'll be there."—Floor

The map energy E/m may be a unicorn in general relativity, but it is a genuine race horse in Newtonian mechanics. We show here that the map energy E/m of a stone moving at non-relativistic speed in a weak gravitational field reduces to the mass of the stone plus the familiar Newtonian energy (kinetic + potential). Rearrange (12) to read:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)^{1/2} (1 - v_{\text{shell}}^2)^{-1/2} \quad (15)$$

For $r \gg 2M$ (weak gravitational field) and $v_{\text{shell}}^2 \ll 1$ (non relativistic stone speed) use the approximation inside the front cover twice:

$$\left(1 - \frac{2M}{r}\right)^{1/2} \approx 1 - \frac{M}{r} \quad (r \gg 2M) \quad (16)$$

$$(1 - v_{\text{shell}}^2)^{-1/2} \approx 1 + \frac{1}{2}v_{\text{shell}}^2 \quad (v_{\text{shell}}^2 \ll 1)$$

Substitute these into (15) and drop the much smaller product $(M/2r)v_{\text{shell}}^2$. The result is an approximation:

$$E \approx m + \frac{1}{2}mv_{\text{shell}}^2 - \frac{Mm}{r} \quad (17)$$

$(r \gg 2M, v_{\text{shell}}^2 \ll 1)$

In this equation $-Mm/r$ is the gravitational potential energy of the stone. (In conventional mks units it would read $-GM_{\text{kg}}m_{\text{kg}}/r$.) We recognize in (17) Newtonian's kinetic energy (KE) plus his potential energy (PE) of a stone, with the added stone's mass m .

As a jockey in curved spacetime, you must beware of riding the unicorn map energy E/m ; gravitational potential energy is a fuzzy concept in general relativity. Dividing energy into separate kinetic and potential forms works only under special conditions, such as those given in equation (16).

Except for these special conditions, we expect the map constant of motion E to differ from E_{shell} : The local shell frame is inertial and excludes effects of curved spacetime. In contrast, map energy E —necessarily expressed in map coordinates—includes curvature effects, which Newton attributes to a "force of gravity."

The approximation in (17) is quite profound. It reproduces a central result of Newtonian mechanics without using the concept of force. In general relativity, we can always eliminate gravitational force (see inside the back cover).

245 raindrop—the magnitude of its velocity—increases to the speed of light at the
 246 event horizon. This is a limiting case, because we cannot construct a
 247 shell—even in principle—at the exact location of the event horizon.



248 **Objection 3.** *I am really bothered by the idea of a material particle such as*
 249 *a stone traveling across the event horizon as a particle. The shell observer*
 250 *sees it moving at the speed of light, but it takes light to travel at light speed.*
 251 *Does the stone—the raindrop—become a flash of light at the event*
 252 *horizon?*



253 No. Be careful about limiting cases. No shell can be built at the event
 254 horizon, because the initial gravitational acceleration increases without
 255 limit there (Section 6.7). An observer on a shell just outside the event
 256 horizon clocks the diving stone to move with a speed *slightly less* than the
 257 speed of light. Any directly-measured stone speed less than the speed of
 258 light is perfectly legal in relativity. So there is no contradiction.

Raindrop dr/dt 259 Compare the shell velocity (19) of the raindrop with the value of dr/dt at
 260 a given r -coordinate. To derive dr/dt , solve the right-hand equation in (13) for
 261 $d\tau$ and substitute the result into the Schwarzschild metric with $d\phi = 0$. Result
 262 for the raindrop:

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Sample Problems 1. The Neutron Star Takes an Aspirin

Neutron Star Gamma has a total mass 1.4 times that of our Sun and a map $r_0 = 10$ kilometers. An aspirin tablet of mass one-half gram falls from rest at a large r coordinate onto the surface of the neutron star. An advanced civilization converts into useful energy the entire kinetic energy of the aspirin tablet, measured in the local surface rest frame. Estimate how long this energy will power a 100-watt bulb. Repeat the analysis to find the useful energy for the case of an aspirin tablet falling from a large r coordinate onto the surface of Earth.

SOLUTION

From the value of the mass of our Sun (inside the front cover), the mass of the neutron star is $M \approx 2 \times 10^3$ meters. Hence $2M/r_0 \approx 2/5$. Far from the neutron star the total map energy of the aspirin tablet equals its rest energy, namely its mass, hence $E/m = 1$. From (18), the shell energy of the aspirin tablet just before it hits the surface of the neutron star rises to the value

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r_0}\right)^{-1/2} \approx 1.3 \quad (\text{Neutron Star}) \tag{20}$$

The shell *kinetic energy* of the half-gram aspirin tablet is 0.3 of its rest energy. The rest energy is $m = 0.5$ gram = 5×10^{-4} kilogram or $mc^2 = 4.5 \times 10^{13}$ joules. The fraction 0.3 of this is 1.35×10^{13} joules. One watt is one joule/second; a 100-watt bulb consumes 100 joules per second. At that rate, the bulb can burn for 1.35×10^{11} seconds on the kinetic energy of the aspirin tablet. One year is about 3×10^7 seconds. Result: The kinetic energy of the half-gram aspirin tablet falling to the surface of Neutron Star Gamma from a large r coordinate provides energy sufficient to light a 100-watt bulb for approximately 4500 years!

What happens when the aspirin tablet falls from a large r coordinate onto Earth's surface? Set the values of M and r_0 to those for Earth (inside front cover). In this case $2M \ll r_E$, so equation (20) becomes, to a very good approximation:

$$\frac{E_{\text{shell}}}{m} \approx \left(1 + \frac{M}{r_0}\right) \approx 1 + 6.97 \times 10^{-10} \quad (\text{Earth}) \tag{21}$$

Use the same aspirin tablet rest energy as before. The lower fraction of kinetic energy yields 3.14×10^4 joules. At 100 joules per second the kinetic energy of the aspirin tablet will light the 100-watt bulb for 314 seconds, or a little more than 5 minutes.

$$\frac{dr}{dt} = - \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r}\right)^{1/2} \quad (\text{raindrop}) \tag{22}$$

Raindrop dr/dt :
a unicorn!

263 Equation (22) shows an apparently outrageous result: as the raindrop
264 reaches the event horizon at $r = 2M$, its Schwarzschild dr/dt drops to zero.
265 (This result explains the strange spacing of event-dots along the orbit
266 approaching the event horizon in Figure 3.6.) Does any local observer witness
267 the stone coasting to rest? No! Repeated use of the word “map” reminds us
268 that map velocities are simply spreadsheet entries for the Schwarzschild
269 mapmaker and need not correspond to direct measurements by any local
270 observer. Figure 2 shows plots of both shell speeds and map $dr/d\tau$ of the
271 descending raindrop. Nothing demonstrates more clearly than the diverging
272 lines in Figure 2 the radical difference between (unicorn) map entries and the
273 results of direct measurement.

274 Does the raindrop cross the event horizon or not? To answer that question
275 we need to track the descent with its directly-measured wristwatch time, not
276 the global t -coordinate. Use equation (13) to convert global coordinate
277 differential dt to wristwatch differential $d\tau$. With this substitution, (22)
278 becomes:

Section 6.4 Raindrop Crosses the Event Horizon 6-11

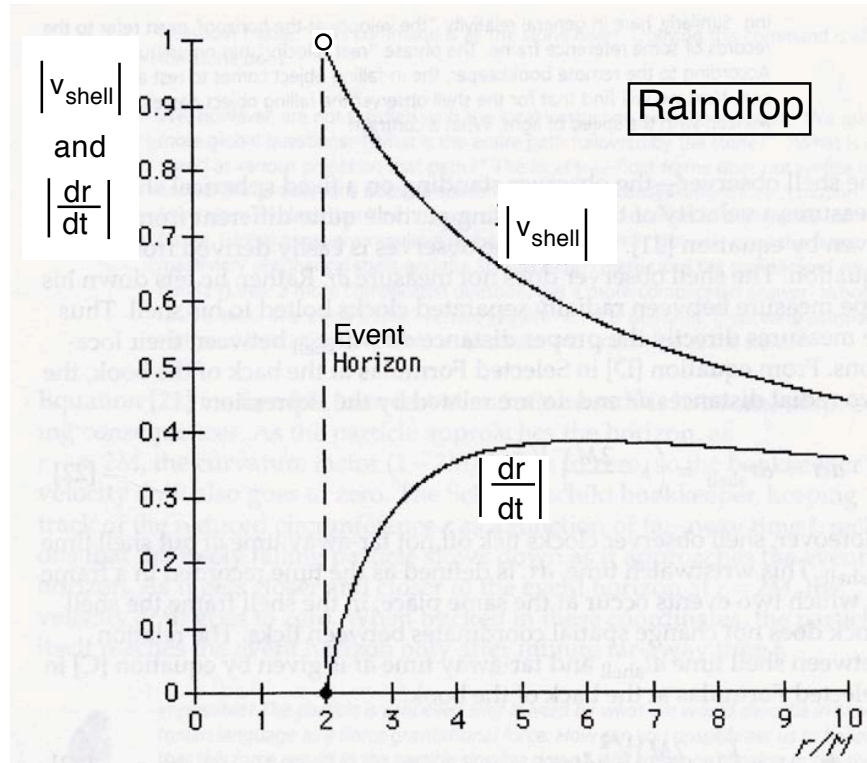


FIGURE 2 Computer plot of the speed $|v_{\text{shell}}|$ of a raindrop directly measured by shell observers at different r -values, from (19), and its Schwarzschild map speed $|dr/dt|$ from (22). Far from the black hole the raindrop is at rest, so both speeds are zero, but both speeds increase as the raindrop descends. Map speed $|dr/dt|$ is not measured but computed from spreadsheet records of the Schwarzschild mapmaker. At the event horizon, the measured shell speed rises to the speed of light, while the computed map speed drops to zero. The upper open circle at $r = 2M$ reminds us that this is a limiting case, since no shell can be constructed at the event horizon. (Why not? See the Appendix, Section 6.7.)

$$\frac{dr}{d\tau_{\text{raindrop}}} = - \left(\frac{2M}{r} \right)^{1/2} \tag{23}$$

Raindrop crosses the event horizon.

279 Expression (23) combines a map quantity dr with the differential advance of
 280 the wristwatch $d\tau_{\text{raindrop}}$. It shows that the raindrop's r -coordinate decreases
 281 as its wristwatch time advances, so the raindrop passes inward through the
 282 event horizon. Indeed, inside the event horizon the magnitude of $dr/d\tau_{\text{raindrop}}$
 283 becomes greater than one, and increases without limit as $r \rightarrow 0$. But this need
 284 not worry us: Both r and dr are map quantities, so $dr/d\tau$ is just an entry on
 285 the mapmaker's spreadsheet, not a quantity measured by anyone.

Comment 1. How do we find the value of dr inside the event horizon?

The numerator dr on the left side of (23) has a clear meaning only *outside* the

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Box 3. Newton Predicts the Black Hole?

It's remarkable how well much of Newton's mechanics works—sort of—on the stage of general relativity. One example is that Newton appears to predict the r -coordinate of the event horizon $r = 2M$. Yet the meaning of that barrier is strikingly different in the two pictures of gravity, as the following analysis shows.

A stone initially at rest far from a center of attraction drops inward. Or a stone on the surface of Earth or of a neutron star is fired outward along r , coming to rest at a large r coordinate. In either case, Newtonian mechanics assigns the same total energy (kinetic plus potential) to the stone. We choose the gravitational potential energy to be zero at the large r coordinate, and the stone out there does not move. From (17), we then obtain

$$\frac{E}{m} - 1 = \frac{v^2}{2} - \frac{M}{r} = 0 \quad (\text{Newton}) \quad (24)$$

From (24) we derive the diving (or rising) speed at any r -coordinate:

$$|v| = \left(\frac{2M}{r}\right)^{1/2} \quad (\text{Newton}) \quad (25)$$

which is the same as equation (19) for the shell speed of the raindrop. One can predict from (25) the r -value at which the speed reaches one, the speed of light, which yields $r = 2M$, the black hole event horizon. For Newton the speed of light is the **escape velocity** from the event horizon.

Newton assumes a single universal inertial reference frame and universal time, whereas (19) applies only to shell separation divided by shell time. A quite different expression (22) describes dr/dt —map differential dr divided by map differential dt —for raindrops.

Does Newton correctly describe black holes? No. Newton predicts that a stone launched radially outward from the event horizon with a speed less than that of light will rise to higher r , slow, stop without escaping, then fall back. In striking contrast, Einstein predicts that nothing, not even light, can be successfully launched outward from inside the event horizon, and that light launched outward *exactly* at the event horizon hovers there, balanced as on a knife-edge (Box 4).

288 event horizon, where every shell displays the stamped value of r . Box 7 in
 289 Section 7.8 describes one practical method by which a descending rain observer
 290 can measure map r , both outside and inside the event horizon.

6.5 ■ GRAVITATIONAL MASS

292 *A new way to measure total energy*

Mass m of the stone

293 This book uses the word *mass* in two different ways. Symbol m in equations
 294 (8) and (11) represents the inertial mass of a test particle, which we call a
 295 *stone*. By definition, the mass of a stone is too small to curve spacetime by a
 296 detectable amount. Expression (8) measures the stone's map energy E and
 297 mass m in the same units.

Mass M of the center of attraction

298 The mass M of the center of attraction is quite different: It is the
 299 gravitational mass that curves spacetime, as reflected in the global metric
 300 expression $(1 - 2M/r)$.

Drop a stone of mass m into a star of mass M .

301 What happens when a stone of mass m falls into a black hole of mass M ?
 302 Does the swallowed mass m increase the black hole's mass? Our new
 303 understanding of energy helps us to calculate how much the mass of a black
 304 hole grows when it swallows matter—and yields a surprising result. To begin,
 305 start with a satellite orbiting close to a star. How can we measure the total
 306 gravitational mass of the star-plus-satellite system? We make this
 307 measurement using the initial acceleration of a distant test particle so remote

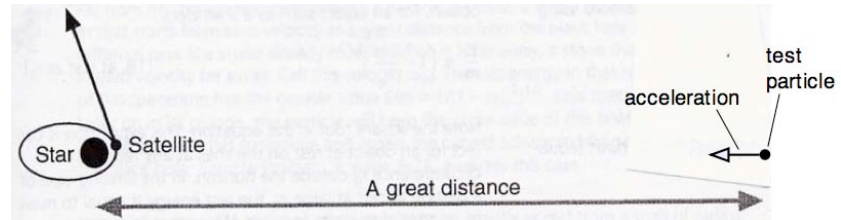


FIGURE 3 Measure the total mass-energy M_{total} of a central star-satellite system using the acceleration of a test particle at a large r coordinate, analyzed using Newtonian mechanics.

308 that Newtonian mechanics gives a correct result (Figure 3). In units of inverse
 309 meters, Newton’s expression for this acceleration is:

$$a = -\frac{M_{\text{total}}}{r^2} \quad (\text{Newton}) \quad (26)$$

Newton says,
 “Add m to M_{star} .”

310 What is M_{total} ? In Newtonian mechanics total mass equals the mass M_{star} of
 311 the original star plus the mass m of the satellite orbiting close to it:

$$M_{\text{total}} = M_{\text{star}} + m \quad (\text{Newton}) \quad (27)$$

Birkhoff’s theorem

312 Could this also be true in general relativity? The answer is no, but proof
 313 requires a sophisticated analysis of Einstein’s equations.

314 A mathematical theorem of general relativity due to G. D. Birkhoff in
 315 1923 states that the spacetime outside any spherically symmetric distribution
 316 of matter and energy is completely described by the Schwarzschild metric with
 317 a *constant* gravitational mass M_{total} , no matter whether that spherically
 318 symmetric source is at rest or, for example, moving inward or outward along
 319 the r -coordinate.

M_{total} includes
 contracting bubble
 of dust.

320 In order to apply Birkhoff’s theorem, we approximate the moving satellite
 321 of Figure 3 by the inward-falling uniform spherical bubble of Figure 4, a
 322 bubble composed of unconnected particles—dust—whose total mass m is the
 323 same as that of the satellite in Figure 3. (We use the label “bubble” instead of
 324 “shell” to avoid confusion with the stationary concentric shells we construct
 325 around a black hole on which we make measurements and observations.) This
 326 falling uniform dust bubble satisfies the condition of Birkhoff’s theorem, so the
 327 Schwarzschild metric applies outside this inward-falling bubble.

How does dust
 bubble increase
 M_{total} ?

328 Unfortunately, Birkhoff’s theorem does not tell us how to calculate the
 329 value of M_{total} , only that it is a constant for any spherically symmetric
 330 configuration of mass/energy. What property of the dust bubble remains
 331 constant as it falls inward? Its inertial mass m ? Not according to special
 332 relativity! Inertial mass is *not* conserved; it can be converted into energy. We
 333 had better look for a conserved energy for our infalling dust bubble. Equation
 334 (12) is our guide: At a given r -coordinate every particle of dust in the
 335 collapsing bubble falls inward at the same rate, so the measure of the total

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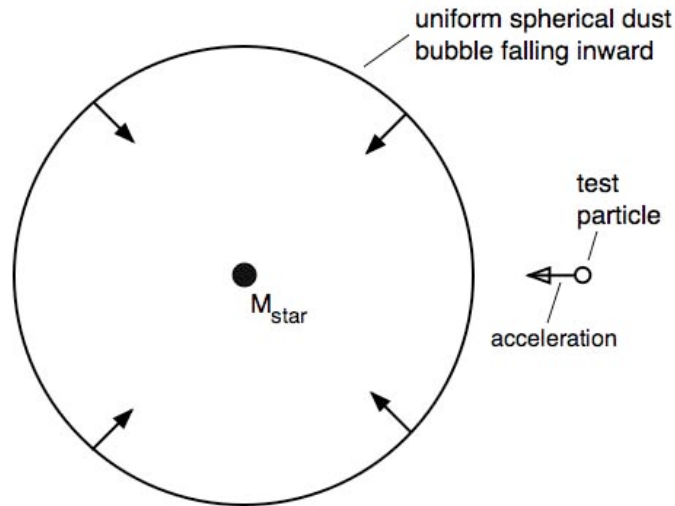


FIGURE 4 Replace the moving satellite of Figure 3 with an inward-falling uniform spherical bubble of dust that satisfies the condition of Birkhoff’s theorem, so the Schwarzschild metric applies outside the contracting dust bubble.

336 shell energy E_{shell} of the bubble at a given r -coordinate is the sum over the
 337 individual particles of the dust bubble. Clearly from (12), successive shell
 338 observers at successively smaller r -coordinates measure successively higher
 339 values of E_{shell} as the collapsing dust bubble falls past them, so we cannot use
 340 shell energy in the Birkhoff analysis.

341 However, the Schwarzschild map energy E *does* remain constant during
 342 this collapse. So instead of the Newtonian expression (27) we have the trial
 343 general relativity replacement:

Einstein: “Add dust
 bubble E to M_{star}
 to find M_{total} .”

$$M_{\text{total}} = M_{\text{star}} + E \quad (\text{Einstein}) \quad (28)$$

344

345 How do we know whether or not the total map energy E of the dust
 346 bubble is the correct constant to add to M_{star} in order to yield the total mass
 347 M_{total} of the system? One check is that when the satellite/dust bubble is far
 348 from the star ($r \gg 2M_{\text{total}}$) but the remote test particle is still exterior to the
 349 dust bubble, then $E \rightarrow E_{\text{shell}}$ from (12). In addition, for a slow-moving
 350 satellite/dust bubble, $E \rightarrow E_{\text{shell}} \rightarrow m$, and we recover Newton’s formula (27),
 351 as we should in the limits $r \gg 2M$ and $v_{\text{shell}}^2 \ll 1$. And when the satellite/dust
 352 bubble falls inward so that our stationary shell observer measures $E_{\text{shell}} > m$,
 353 then equation (28) remains valid, because $E(\approx m)$ does not change. Note that
 354 Birkhoff’s Theorem is satisfied in this approximation.

Check validity
 of (28).

Result: Convert
stone map E
into gravitational
mass.

355 If (28) is correct, then general relativity merely replaces Newton's m in
356 (27) with total map energy E , a constant of motion for the satellite/bubble.
357 Thus the mass of a star or black hole grows by the value of the map energy E
358 of a stone or collapsing bubble that falls into it. *The map energy of the stone*
359 *is converted into gravitational mass.* Earlier we called map energy E "a
360 unicorn, a mythical beast." Now we must admit that this unicorn can add its
361 mass-equivalence to the mass of a star into which it falls.

?

362 **Objection 4.** *You checked equation (28) only in the Newtonian limit, where*
363 *the remote dust bubble is at rest or falls inward with small kinetic energy. Is*
364 *(28) valid for all values of E ? Suppose that the dust bubble in Figure 4 is*
365 *launched inward (or outward) at relativistic speed. In this case does total E*
366 *still simply add to M_{star} to give total mass M_{total} for the still more distant*
367 *observer?*

!

368 Yes it does, but we have not displayed the proof, which requires solution of
369 Einstein's equations. Let a massive star collapse, then explode into a
370 supernova. If this process is spherically symmetric, then a distant observer
371 will detect no change in gravitational attraction in spite of the radical
372 conversions among different forms of energy in the explosion. Indeed, the
373 distant observer has no way to know about these transformations before
374 the outward-blasting bubble of radiation and neutrinos passes her. As they
375 pass, she detects a decline in the gravitational acceleration of the local test
376 particle, because some of the original energy of the central attractor is
377 carried to an r -value greater than hers.

Gravity waves
carry off energy.

378 Is the Birkhoff restriction to spherical symmetry important? It can be: A
379 satellite orbiting around or falling into a star or black hole will emit
380 gravitational waves that carry away some energy, decreasing M_{total} . Chapter
381 16 notes that a spherically symmetric distribution cannot emit gravitational
382 waves, no matter how that spherical distribution pulses in or out. As a result,
383 equation (28) is okay to use only when the emitted gravitational wave energy
384 is very much less than M_{total} . When that condition is met, the cases shown in
385 Figures 3 and 4 are observationally indistinguishable.

Measuring E
from far away.

386 As long as gravitational wave emission is negligible and we are sufficiently
387 far away, we can, in principle, use (28) to measure the map energy E of
388 *anything* circulating about, diving into, launching itself away from, or
389 otherwise interacting with a center of attraction. Simply use Newtonian
390 mechanics to carry out the measurement depicted in Figure 3, first with the
391 satellite absent, second with the satellite in orbit near the star. Subtract the
392 second value from the first for the acceleration (26) and use (28) to determine
393 the value of $E = M_{\text{total}} - M_{\text{star}}$. As in Box 2, this example shows that E (and
394 not E_{shell}) includes effects of curved spacetime.

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Box 4. Event Horizon vs. Particle Horizon

The *event horizon* around any black hole separates events that can affect the future of observers outside the event horizon from events that cannot do so. Barring quantum mechanics, the event horizon never reveals what is hidden behind it. (For a possible exception, see Box 5 on Hawking radiation.)

We can now define a black hole more carefully: *A black hole is a singularity cloaked by an event horizon.*

In Chapter 14 we learn about another kind of horizon, called a **particle horizon**. Some astronomical objects are so far from

us that the light they have emitted since they were formed has not yet reached us. In principle more and more such objects swim into our distant field of view every day, as our cosmic particle horizon sweeps past them. In contrast to the event horizon, the particle horizon yields up its hidden information to us—gradually!

In order to avoid confusion among these different kinds of horizons, we try to be consistent in using the full name of the *event horizon* that cloaks a black hole.

6.6 ■ OVER THE EDGE: ENTERING THE BLACK HOLE

396 *No jerk. No jolt. A hidden doom.*

397 Except for the singularity at $r = 0$, no feature of the black hole excites more
 398 curiosity than the event horizon at $r = 2M$. It is the point of no return beyond
 399 which no traveler can find the way back—or even send a signal—to the outside
 400 world. What is it like to fall into a black hole? No one from Earth has yet
 401 experienced it. Moreover, we predict that future explorers who do so will not
 402 be able to return to report their experiences or to transmit messages about
 403 their experience to us outsiders—so we believe! In spite of the impossibility of
 404 receiving a final report, there exists a well-developed and increasingly
 405 well-verified body of theory that makes clear predictions about our experience
 406 as we approach and cross the event horizon of a black hole. Here are some of
 407 those predictions.

Predict what
no one can verify.

We are not sucked
into a black hole.

408 **We are not “sucked into” a black hole.** Unless we get close to its
 409 event horizon, a black hole will no more grab us than our Sun grabs Earth. If
 410 our Sun should suddenly collapse into a black hole without expelling any mass,
 411 Earth and the other planets would continue on their courses undisturbed (even
 412 though, after eight minutes, continuous night would prevail for us on Earth!).
 413 The Schwarzschild solution (plus the Principle of Maximal Aging) would still
 414 continue to describe Earth’s worldline around our Sun, just as it does now. In
 415 Section 6.7 you show that for an orbit at r -coordinate greater than about
 416 $300M$, Newtonian mechanics predicts gravitational acceleration with an
 417 accuracy of about 0.3 percent. We also find (Section 8.5) that no stable
 418 circular orbit is possible at r less than $6M$. Even at an r -value between $6M$
 419 and the event horizon at $2M$, we can always escape the grip of the black hole,
 420 given sufficient rocket power. Only when we reach or cross the event horizon
 421 are we irrevocably swallowed, our fate sealed.

No jolt as we
cross the
event horizon.

422 **We detect no special event as we fall inward through the event**
 423 **horizon.** Even when we drop across the event horizon at $r = 2M$, we
 424 experience no shudder, jolt, or jar. True, tidal forces are ever-increasing as we
 425 fall inward, and this increase continues smoothly as we cross the event horizon.

Box 5. Escape from the Black Hole? Hawking Radiation

Einstein's field equations predict that nothing, not even a light signal, escapes from inside the event horizon of a black hole. In 1973, Stephen Hawking demonstrated an exception to this conclusion using quantum mechanics. For years quantum mechanics had been known to predict that particle-antiparticle pairs—such as an electron and a positron—are continually being created and recombined in “empty” space, despite the frugidity of the vacuum. These processes have indirect, but significant and well-tested, observational consequences. Never in cold flat spacetime, however, do such events present themselves to direct observation. For this reason the pairs receive the name “virtual particles.” When such a particle-antiparticle pair is produced near, but outside, the event horizon of a black hole, Hawking showed, one member of the pair will occasionally be swallowed by the black hole, while the other one escapes to a large r coordinate—

now a *real* particle. Escaped particles form what is called **Hawking radiation**. Before particle emission, we had just the black hole; after particle emission, we have the black hole plus the distant real particle outside the event horizon. Where did the energy of this distant particle come from? In order to conserve mass/energy, the mass of the black hole must decrease in this process. This loss of mass causes the black hole to “evaporate.” As the mass of the black hole decreases, the loss rate grows until eventually it becomes explosive, destroying the black hole. For a black hole of several solar masses, however, Hawking's theory predicts that the Earth-time required to achieve this explosive state exceeds the age of the Universe by a fantastic number of powers of ten. For this reason, we ignore Hawking radiation in our description of black holes.

No shell frames inside the event horizon.

Packages can move inward, not outward.

426 We are not suddenly squashed or torn apart at $r = 2M$, because the event
 427 horizon is not a *physical* singularity, as explained in Box 3, Section 3.1. There
 428 is no sudden discontinuity in our experience as we pass through the event
 429 horizon.

430 **Inside the event horizon no shell frames are possible.** Outside the
 431 event horizon we have erected, in imagination, a set of nested spherical shells
 432 concentric to the black hole. We say “in imagination” because no known
 433 material is strong enough to withstand the “pull of gravity,” which increases
 434 without limit as we approach the event horizon from outside (Section 6.7).
 435 Locally such a stationary shell can be replaced by a spaceship with rockets
 436 blasting in the inward direction to keep it at the same r and ϕ coordinates.
 437 Inside the event horizon, however, nothing remains at rest. No shell, no rocket
 438 ship can remain at constant r -coordinate there, however ferocious the blast of
 439 its engines. The material composing the original star, no matter how strong,
 440 was itself unable to resist the collapse that formed the black hole. The same
 441 irresistible collapse forbids any stationary structure or object inside the event
 442 horizon.

443 **“Outsiders” can send packages to “insiders.”** Inside the event
 444 horizon, different local frames can still move past one another with measurable
 445 relative speeds. Here are some examples. *First local frame:* One traveler may
 446 drop from rest just outside the event horizon. *Second local frame:* An
 447 unpowered spaceship may fall in from far away. *Third local frame:* Another
 448 unpowered spaceship may be hurled inward from outside the event horizon.
 449 Light and radio waves can carry messages inward to us. We who have fallen
 450 inside the event horizon can still see the stars, though with directions, colors,
 451 and intensities that change as we fall (Chapters 11 through 13). Packages and
 452 communications sent inward across the event horizon? Yes. How about moving

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Box 6. Baked on the Shell?

As you stand on a spherical shell close to the event horizon of a black hole, you are crushed by an unsupportable local gravitational acceleration directed downward toward the center (Section 6.7). If that is not enough, you are also enveloped by an electromagnetic radiation field. William G. Unruh used quantum field theory to show that the temperature T of this radiation field (in degrees Kelvin) experienced on the shell is given by the equation

$$T = \frac{hg_{\text{conv}}}{4\pi^2 k_B c} \quad (29)$$

Here g_{conv} is the local acceleration of gravity expressed in conventional units, meters/second²; h is Planck's constant; c is the speed of light; and k_B is **Boltzmann's constant**, which has the value 1.381×10^{-23} kilogram-meters²/(second²degree Kelvin). The quantity $k_B T$ has the unit joules and gives the average ambient thermal energy of this radiation field. (The same radiation field surrounds you when you accelerate at the rate g_{conv} in flat spacetime.)

Section 6.7 derives an expression for the local gravitational acceleration on a shell at r . Equation (46) gives the magnitude of this acceleration, expressed in the unit meter⁻¹:

$$g_{\text{shell}} = \frac{g_{\text{conv}}}{c^2} = \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1/2} \quad (30)$$

Substitute g_{conv} from (30) into (29) to obtain

$$T = \frac{hc}{4\pi^2 k_B} \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1/2} \quad (31)$$

with M in meters. This temperature increases without limit as you approach the event horizon. Therefore one would expect the radiation field near the event horizon to shine brighter than any star when viewed by a distant observer. Why doesn't this happen? In a muted way it does happen.

Remember that radiation is gravitationally red-shifted as it moves away from any center of attraction. Every frequency is red-shifted by the factor $(1 - 2M/r)^{1/2}$, which cancels the corresponding factor in (31). For radiation coming from near the event horizon, let $r \rightarrow 2M$ in the resulting equation. The distant viewer sees the radiation temperature

$$T_H = \frac{hc}{16\pi^2 k_B M} \quad (\text{distant view of event horizon}) \quad (32)$$

with M in meters. The temperature T_H is called the **Hawking temperature** and characterizes the Hawking radiation from a black hole (Box 5). Notice that this temperature *increases* as the mass M of the black hole *decreases*. Even for a black hole whose mass is only a few times that of our Sun, this temperature is extremely low, so from far away such a black hole really looks *almost* black.

The radiation field described by equations (29) through (32), although perfectly normal, leads to strange conclusions. Perhaps the strangest is that this radiation goes entirely undetected by a free-fall observer. The free-fall diving traveler observes no such radiation field, while for the shell observer the radiation is a surrounding presence. This paradox cannot be resolved using the classical general relativity theory presented in this book; see Kip Thorne's *Black Holes and Time Warps: Einstein's Outrageous Legacy*, page 444.

How realistic is the danger of being baked on a shell near the event horizon of a black hole? In answer, compute the local acceleration of gravity for a shell where the radiation field reaches a temperature equal to the freezing point of water, 273 degrees Kelvin. From (29) you can show that $g_{\text{conv}} = 6.7 \times 10^{22}$ meters/second², or almost 10^{22} times the acceleration of gravity on Earth's surface. Evidently we will be crushed by gravity long before we are baked by radiation!

453 outward through the event horizon? No. Box 4 tells us—and Section 7.7
 454 demonstrates—that when a diver fires a light flash radially outward at the
 455 instant she passes inward through the event horizon, that light flash hovers at
 456 the same r -coordinate at the event horizon. Nothing moves faster than light,
 457 so if light cannot move outward through the event horizon, then packages and
 458 stones definitely cannot move outward there either.

459 **Inside the event horizon life goes on—for a while.** Make a daring
 460 dive into an already mature black hole? No. We and our exploration team
 461 want to be still more daring, to follow a black hole as it forms. We go to a
 462 multiple-galaxy system so crowded that it teeters on the verge of gravitational
 463 collapse. Soon after our arrival at the outskirts, it starts the collapse, at first
 464 slowly, then more and more rapidly. Soon a mighty avalanche thunders

Surf a collapsing galaxy group.

Box 7. General relativity is a classical (non-quantum) theory.

Newton's laws describe the motion of a stone in flat spacetime at speeds very much less than the speed of light. For higher speeds we need relativity. Newton's laws correctly describe slow-speed motion of a "stone" more massive than, say, a proton. To describe behavior of smaller particles we need quantum physics.

Does this mean that we have no further use for Newton's laws of motion? Not at all! Newton's laws are *classical*, that is non-quantum. In this book we repeatedly use Newton's mechanics as a simple, intuitive first cut at prediction and observation. And with it we check every prediction of relativity in the limit of slow speed and vanishing spacetime curvature. We expect

that Newton's laws of motion will be scientifically useful as long as humanity survives.

General relativity is also a *classical*—non-quantum—theory. General relativity does not predict Hawking radiation (Box 5) or the Hawking temperature (Box 6). These are predictions of quantum field theory, predictions that we mention as important asides to our classical analysis.

General relativity does not correctly represent every property of the black hole, any more than Newton's mechanics correctly predicts the motion of fast-moving particles. We still expect that general relativity—along with Newton's mechanics—will be scientifically useful during the long future of humanity.

465 (silently!) toward the center from all directions, an avalanche of objects and
 466 radiation, a cataract of momentum-energy-pressure. The matter of the
 467 galaxies and with it our group of enterprising explorers pass smoothly across
 468 the event horizon at Schwarzschild $r = 2M$.

"Publish *and* perish."

469 From that moment onward we lose all possibility of signaling to the outer
 470 world. However, radio messages from that outside world, light from familiar
 471 stars, and packages fired after us at sufficiently high shell speed continue to
 472 reach us. Moreover, communications among us explorers take place now as
 473 they did before we crossed the event horizon. We use the familiar categories of
 474 space and time to share our findings. With our laptop computers we turn out
 475 an exciting journal of observations, measurements, and conclusions. (Our
 476 motto: "Publish *and* perish.")

Killer tides.

477 **Tides become lethal.** Nothing rivets our attention more than the tidal
 478 forces that pull heads up and feet down with ever-increasing tension (Sections
 479 1.11 and 7.9). Before much time has passed on our wristwatch, we can predict,
 480 this differential pull will reach the point where we can no longer survive.
 481 Moreover, we can foretell still further ahead and with certainty the instant of
 482 total crunch. That crunch swallows up not only the stars beneath us, not only
 483 we explorers, but time itself. All worldlines inside the event horizon terminate
 484 on the singularity. For us an instant comes after which there is no "after."
 485 Chapters 7 and 21 give more details of life inside the event horizon.

After crunch there is no "after."

6.7 ■ APPENDIX: INITIAL SHELL GRAVITATIONAL ACCELERATION FROM REST

487 *Unlimited gravitational acceleration on a shell near the event horizon.*

Is gravity real or fictitious?

488 When you stand on a shell near a black hole, you experience gravity—a pull
 489 downward—just as you do on Earth. On the shell this gravity can be great:
 490 near the event horizon it increases without limit, as we shall see. On the other
 491 hand, "In general relativity . . . gravity is *always* a fictitious force which we

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492 can eliminate by changing to a local frame that is in free fall . . .” (inside the
 493 back cover). So is this “gravity” real? Falls kill and injure many people every
 494 year. Anything that can kill you is definitely real, not fictitious! Here we avoid
 495 philosophical issues by asking a practical question: “When the shell observer
 496 drops a stone from rest, what *initial* acceleration does he measure?”

Practical experiment
 to define gravity

497 To begin, we behave like an engineer: Use a thought experiment to define
 498 what we mean by the initial gravitational acceleration of a stone dropped from
 499 rest on a shell at r_0 . Use the heavy machinery of general relativity to find the
 500 magnitude of the newly-defined acceleration experienced by a shell observer.

501 Figure 5 presents our method to measure quantities used to define initial
 502 gravitational acceleration on a shell. The shell is at map r_0 . At a shell distance
 503 $|\Delta y_{\text{shell}}|$ below the shell lies a stationary platform onto which the shell observer
 504 drops a stone. The time lapse Δt_{shell} for the drop is measured as follows:

Specific instructions
 for experiment
 to define gravity

- 505 1. The shell observer starts his clock at the instant he drops the stone.
- 506 2. When the stone strikes the platform, it fires a laser flash upward to the
 507 shell clock.
- 508 3. The shell observer determines the shell time lapse between drop and
 509 impact, Δt_{shell} , by deducting flash transit shell time from the time
 510 elapsed on his clock when he receives the laser flash.

511 To calculate the “flash transit shell time” in Step 3, the shell observer divides
 512 the shell distance $|\Delta y_{\text{shell}}|$ by the shell speed of light. (In an exercise of
 513 Chapter 3, you verified that the shell observer measures light to move at its
 514 conventional speed—value one—in an inertial frame.)

Define g_{shell}

515 The shell observer substitutes Δy_{shell} and Δt_{shell} into the expression that
 516 defines uniform acceleration g_{shell} :

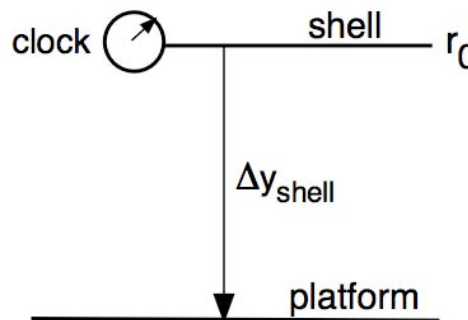


FIGURE 5 Notation for thought experiment to define initial gravitational acceleration from rest in a shell frame. The shell observer at r_0 releases a stone from rest and measures its shell time of fall Δt_{shell} onto a lower stationary platform that he measures to be a distance $|\Delta y_{\text{shell}}|$ below the shell. From these observations he defines and calculates the value of the stone’s initial acceleration g_{shell} , equation (33).

Section 6.7 Appendix: Initial Shell Gravitational Acceleration from Rest **6-21**

$$\Delta y_{\text{shell}} = -\frac{1}{2}g_{\text{shell}}\Delta t_{\text{shell}}^2 \quad (\text{uniform } g_{\text{shell}}) \quad (33)$$

517 Thus far our engineering definition of g_{shell} has little to do with general
518 relativity. The fussy procedure of this thought experiment reflects the care
519 required when general relativity is added to the analysis, which we do now.

Mapmaker demands
constant map energy
for falling stone.

520 What does the Schwarzschild mapmaker say about the acceleration of a
521 dropped stone? She insists that, whatever motion the free stone executes, its
522 map energy E/m must remain a constant of motion. So start with the map
523 energy of a stone bolted to the shell at r_0 . From map energy equation (15)
524 with $v_{\text{shell}} = 0$ and $r = r_0$, we have:

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} \quad (\text{stone released from rest at } r_0) \quad (34)$$

525 Now release the stone from rest. The mapmaker insists that as the stone
526 falls its map energy remains constant, so equate the right sides of (34) and (8),
527 square the result, and solve for $d\tau^2$:

$$d\tau^2 = \left(1 - \frac{2M}{r_0}\right)^{-1} \left(1 - \frac{2M}{r}\right)^2 dt^2 \quad (35)$$

528 Substitute this expression for $d\tau^2$ into the Schwarzschild metric for radial
529 motion ($d\phi = 0$), namely

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \quad (36)$$

530 Divide corresponding sides of equations (36) and (35), then solve the resulting
531 equation for $(dr/dt)^2$:

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r_0}\right)^{-1} \left(1 - \frac{2M}{r}\right)^2 \left(\frac{2M}{r} - \frac{2M}{r_0}\right) \quad (\text{from rest at } r_0) \quad (37)$$

532 We want the acceleration of the stone in Schwarzschild map coordinates.
533 Take the derivative of both sides with respect to the t -coordinate and cancel
534 the common factor $2(dr/dt)$ from both sides of the result to obtain:

$$\frac{d^2r}{dt^2} = -\left(\frac{M}{r^2}\right) \left(1 - \frac{2M}{r}\right) \left(1 - \frac{2M}{r_0}\right)^{-1} \left(\frac{4M}{r_0} + 1 - \frac{6M}{r}\right) \quad (38)$$

535 This equation gives the map acceleration at r of a stone released from rest at
536 r_0 . This acceleration depends on r , so is clearly *not* uniform as the stone falls,
537 but *decreases* as r gets smaller, going to zero as r reaches the event horizon.
538 We know that map acceleration is a unicorn, a result of Schwarzschild map
539 coordinates, not measured by any inertial observer. We are interested in the

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540 *initial* acceleration at the instant of release from rest. Set $r = r_0$ in equation
 541 (38), which then reduces to the relatively simple form:

$$\left(\frac{d^2r}{dt^2}\right)_{r_0} = -\frac{M}{r_0^2} \left(1 - \frac{2M}{r_0}\right) \quad (\text{initial, from rest at } r_0) \quad (39)$$

Acceleration
 in map
 coordinates

542 What is the meaning of this acceleration in Schwarzschild map
 543 coordinates? It is only a spreadsheet entry, an accounting analysis by the
 544 mapmaker, not the result of a direct observation by anyone. Observation
 545 requires an experiment on the shell, which we have already designed, leading
 546 to the expression (33). What is the relation between our engineering definition
 547 of acceleration and acceleration (39) in Schwarzschild coordinates? To compare
 548 the two expressions, expand the Schwarzschild r -coordinate of the dropped
 549 stone close to the radial position r_0 using a Taylor series for a short lapse Δt :

$$r = r_0 + \left(\frac{dr}{dt}\right)_{r_0} \Delta t + \frac{1}{2} \left(\frac{d^2r}{dt^2}\right)_{r_0} (\Delta t)^2 + \frac{1}{6} \left(\frac{d^3r}{dt^3}\right)_{r_0} (\Delta t)^3 + \dots \quad (40)$$

550 Because Δt is small, we disregard terms higher than quadratic in Δt . This
 551 allows us to approximate uniform gravity (constant acceleration) and to
 552 compare mapmaker accounting entries with observed shell acceleration. Since
 553 the stone drops from rest at r_0 , the initial map speed is zero: $(dr/dt)_{r_0} = 0$.
 554 With these considerations, insert (39) into (40) and obtain:

$$r - r_0 = \Delta r \approx -\frac{1}{2} \left[\left(1 - \frac{2M}{r_0}\right) \frac{M}{r_0^2} \right] (\Delta t)^2 \quad (41)$$

555 This equation has a form similar to that of our experimental definition
 556 (33) of shell gravitational acceleration, except the earlier equation employs
 557 vertical shell separation Δy_{shell} and shell time lapse Δt_{shell} . Convert these to
 558 Schwarzschild quantities using standard transformations—equations (5.8) and
 559 (5.9):

$$\Delta y_{\text{shell}} = \left(1 - \frac{2M}{r_0}\right)^{-1/2} \Delta r \quad \text{and} \quad \Delta t_{\text{shell}}^2 = \left(1 - \frac{2M}{r_0}\right) (\Delta t)^2 \quad (44)$$

560 With these substitutions, and after rearranging terms, equation (33) becomes:

$$\Delta r = -\frac{1}{2} \left[\left(1 - \frac{2M}{r_0}\right)^{3/2} g_{\text{shell}} \right] (\Delta t)^2 \quad (45)$$

Initial shell
 acceleration

561 As we go to the limit $\Delta t \rightarrow 0$, the extra terms in (40) become increasingly
 562 negligible, so (41) approaches an equality and we can equate square-bracket
 563 expressions in (41) and (45). Replacing the notation r_0 with r yields the
 564 magnitude of the initial acceleration of a stone dropped from rest on a shell at
 565 any r -coordinate:

Sample Problems 2. Initial Gravitational Acceleration on a Shell

1. On a shell at $r/M = 4$ near a black hole, the initial gravitational acceleration from rest is how many times that predicted by Newton?
2. On a shell at $r/M = 2.1$ near a black hole, the initial gravitational acceleration is how many times that predicted by Newton?
3. What is the minimum value of r/M so that, at or outside of that r -coordinate, Newton's formula for gravitational acceleration yields values that differ from Einstein's by less than ten percent? by less than one percent?
4. Compute the weight in pounds of a 100-kilogram astronaut on the surface of a neutron star with mass equal to $1.4M_{\text{Sun}}$ and $M/r_0 = 2/5$.

in error (it will be too low) by less than ten percent. At or outside $r/M = 100$ Newton's prediction will be too low by less than one percent.

4. The Newtonian acceleration in conventional units is:

$$g_{\text{Newton conv}} = \left(\frac{GM_{\text{kg}}}{c^2 r_0^2} \right) c^2 = \left(\frac{M}{r_0^2} \right) c^2 \quad (42)$$

$$= \left(\frac{M}{r_0} \right)^2 \frac{c^2}{M} = \left(\frac{2}{5} \right)^2 \frac{c^2}{1.4 \times M_{\text{Sun}}}$$

Insert values of c^2 and M_{Sun} (in meters) to yield $g_{\text{Newton conv}} \approx 7.0 \times 10^{12}$ meters/second². From (46),

$$\text{weight} = mg_{\text{shell}} = \left(1 - \frac{4}{5} \right)^{-1/2} mg_{\text{Newton}} \quad (43)$$

$$\approx 16 \times 10^{14} \text{ Newtons}$$

One Newton = 0.225 pounds, so our astronaut weighs approximately 3.5×10^{14} pounds, or 350 trillion pounds (USA measure of weight). It is surprising that, even at the surface of this neutron star, the general relativity result in (43) is greater than Newton's by the rather small factor $5^{1/2} = 2.24$.

SOLUTIONS

1. At $r/M = 4$ the factor $(1 - 2M/r)^{-1/2}$ in (46) predicts a gravitational acceleration $2^{1/2} = 1.41$ times that predicted by Newton.
2. Even at $r/M = 2.1$ the gravitational acceleration is still the relatively mild multiple of 4.6 times the Newtonian prediction.
3. Setting $(1 - 2M/r)^{-1/2} = 1.1$ yields $r/M = 11.5$. At or outside this r -coordinate, Newton's prediction will be

$$g_{\text{shell}} = \left(1 - \frac{2M}{r} \right)^{-1/2} \frac{M}{r^2} \quad (\text{initial, drop from rest}) \quad (46)$$

566

567 Sample Problems 2 explore initial shell accelerations under different
 568 conditions. It is surprising how accurate Newton's expression $g_{\text{Newton}} = M/r^2$
 569 is even quite close to the event horizon of a black hole—an intellectual victory
 570 for Newton that we could hardly have anticipated.

571

QUERY 1. Gravitational acceleration on Earth's surface

Use values for the constants M_E and r_E for the Earth listed inside the front cover to show that equation (46) correctly predicts the value of the gravitational acceleration g_E at Earth's surface. Check your calculated values against those also listed inside the front cover.

- A. Show that in units of length this acceleration has the value $g_E = 1.09 \times 10^{-16}$ meter⁻¹.
- B. Show that in conventional units this acceleration has the value $g_{E,\text{conv}} = 9.81$ meters/second².

578

6-24 Chapter 6 Diving579 **A GRAVITYLESS DAY**

580 *I am sitting here 93 million miles from the sun on a rounded rock which*
 581 *is spinning at the rate of 1,000 miles an hour, and roaring through space*
 582 *to nobody-knows-where, to keep a rendezvous with nobody-knows-what . .*
 583 *. and my head pointing down into space with nothing between me and*
 584 *infinity but something called gravity which I can't even understand, and*
 585 *which you can't even buy anyplace so as to have some stored away for a*
 586 *gravityless day . . .*

587

—Russell Baker

6.8 ■ EXERCISES589 **1. Diving from Rest Far Away**

590 Black Hole Alpha has a mass $M = 10$ kilometers. A stone starting from rest
 591 far away falls radially into this black hole. In the following, express all speeds
 592 as a decimal fraction of the speed of light.

- 593 A. What is the speed of the stone measured by the shell observer at
 594 $r = 50$ kilometers?
- 595 B. Write down an expression for $|dr/dt|$ of the stone as it passes $r = 50$
 596 kilometers?
- 597 C. What is the speed of the stone measured by the shell observer at $r = 25$
 598 kilometers?
- 599 D. Write down an expression for $|dr/dt|$ of the stone as it passes $r = 25$
 600 kilometers?
- 601 E. In two or three sentences, explain why the change in the speed between
 602 Parts A and C is qualitatively different from the change in $|dr/dt|$
 603 between Parts B and D.

604 **2. Maximum Raindrop $|dr/dt|$**

605 A stone is released from rest far from a black hole of mass M . The stone drops
 606 radially inward. Mapmaker records show that the the value of $|dr/dt|$ of the
 607 stone initially increases but declines toward zero as the stone approaches the
 608 event horizon. The value of $|dr/dt|$ must therefore reach a maximum at some
 609 intermediate r . Find this r -value for this maximum. Find the numerical value
 610 of $|dr/dt|$ at that r -value. Who measures this value?

611 **3. Hitting a Neutron Star**

612 A particular nonrotating neutron star has a mass $M = 1.4$ times the mass of
 613 our Sun and $r = 10$ kilometers. A stone starting from rest far away falls onto
 614 the surface of this neutron star.

Section 6.8 Exercises **6-25**

- 615 A. If this neutron star were a black hole, what would be the map r -value
616 of its event horizon? What fraction is this of the r -value of the neutron
617 star?
- 618 B. With what speed does the stone hit the surface of the neutron star as
619 measured by someone standing (!) on the surface?
- 620 C. With what value of $|dr/dt|$ does the stone hit the surface?
- 621 D. With what kinetic energy per unit mass does the stone hit the surface
622 according to the surface observer?

623 Earlier it was thought that astronomical gamma-ray bursts might be caused by
624 stones (asteroids) impacting neutron stars. Carry out a preliminary analysis of
625 this hypothesis by assuming that the stone is made of iron. The impact kinetic
626 energy is very much greater than the binding energy of iron atoms in the
627 stone, greater than the energy needed to completely remove all 26 electrons
628 from each iron atom, and greater even than the energy needed to shatter the
629 iron nucleus into its component 26 protons and 30 neutrons. So we neglect all
630 these binding energies in our estimate. The result is a vaporized gas of 26
631 electrons and 56 nucleons (protons and neutrons) per incident iron atom. We
632 want to find the average energy of photons (gamma rays) emitted by this gas.

- 633 E. Explain briefly why, just after impact, the electrons have very much
634 less kinetic energy than the nucleons. So in what follows we neglect the
635 initial kinetic energy of the electron gas just after impact.
- 636 F. The hot gas emits thermal radiation with characteristic photon energy
637 approximately equal to the temperature. What is the characteristic
638 energy of photons reaching a distant observer, in MeV?

639 NOTE: It is now understood that astronomical gamma-ray bursts release much
640 more energy than an asteroid falling onto a neutron star. Gamma ray bursts
641 are now thought to arise from the birth of new black holes in distant galaxies.

4. A Stone Glued to the Shell Breaks Loose

642 A stone of mass m glued to a shell at r_0 has map energy given by equation
643 (34). Later the glue fails so that the stone works loose and drops to the center
644 of the black hole of mass M .
645

- 646 A. By what amount ΔM does the mass of the black hole increase?
- 647 B. A distant observer measures the mass of black hole plus stone at rest at
648 r_0 using the method of Figure 3. How will the value of this total mass
649 change after the stone has fallen into the black hole?
- 650 C. Apply your result of Part A to find the numerical value of the constant
651 K in the equation $\Delta M = Km$ for the three cases: (a) $r_0 \gg 2M$, (b)
652 $r_0 = 8M$ and (c) r_0 is just outside the event horizon. In all cases the
653 observer in Figure 3 is much farther away than r_0 .

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654 **5. Wristwatch Time to the Center**

655 An astronaut drops from rest off a shell at r_0 . How long a time elapses, as
 656 measured on her wristwatch, between letting go and arriving at the center of
 657 the black hole? If she drops off the shell just outside the event horizon, what is
 658 her event-horizon-to-crunch wristwatch time?

659 *Several hints:* The first goal is to find $dr/d\tau$, the rate of change of r -coordinate
 660 with wristwatch time τ , in terms of r and r_0 . Then form an integral whose
 661 variable of integration is r/r_0 . The limits of integration are from $r/r_0 = 1$ (the
 662 release point) to $r/r_0 = 0$ (the center of the black hole). The integral is

$$\frac{\tau}{M} = -\frac{1}{2^{1/2}} \left(\frac{r_0}{M}\right)^{3/2} \int_1^0 \frac{(r/r_0)^{1/2} d(r/r_0)}{(1 - r/r_0)^{1/2}} \quad (47)$$

663 Solve this integral using tricks, nothing but tricks: Simplify by making the
 664 substitution $r/r_0 = \cos^2 \psi$ (The “angle” ψ is not measured anywhere; it is
 665 simply a variable of integration.) Then $(1 - r/r_0)^{1/2} = \sin \psi$ and
 666 $d(r/r_0) = -2 \cos \psi \sin \psi d\psi$. The limits of integration are from $\psi = 0$ to
 667 $\psi = \pi/2$. With these substitutions, the integral for wristwatch time becomes

$$\begin{aligned} \frac{\tau}{M} &= 2^{1/2} \left(\frac{r_0}{M}\right)^{3/2} \int_0^{\pi/2} \cos^2 \psi d\psi \\ &= 2^{1/2} \left(\frac{r_0}{M}\right)^{3/2} \left[\frac{\psi}{2} + \frac{\sin 2\psi}{4} \right] \Big|_0^{\pi/2} \end{aligned} \quad (48)$$

668 Both sides of (48) are unitless. Complete the formal solution. For a black hole
 669 20 times the mass of our Sun, how many seconds of wristwatch time elapse
 670 between the drop from rest just outside the event horizon to the singularity?

671 **6. Release a stone from rest**

672 You release a stone from rest on a shell of map coordinate r_0 .

- 673 A. Derive an expression for $|dr/dt|$ of the stone as a function of r . Show
 674 that when the stone drops from rest far away, $|dr/dt|$ reduces to the
 675 expression (22) for a raindrop. Find the r -value at which map speed is
 676 *maximum* and the expression for that maximum map speed. Verify that
 677 in the limit in which the stone is dropped from rest far away, these
 678 expressions reduce to those found in Exercise 6.2 for the raindrop.
- 679 B. Derive an expression for the *shell velocity* of the stone as a function of
 680 r . Show that in the limit in which the stone drops from rest far away,
 681 the shell velocity reduces to the expression (19) for a raindrop.

Section 6.8 Exercises **6-27**

682 C. Sketch graphs of shell speed *vs.* r similar to Figure 2 for the following
683 values of r_0 :

684 (a) $r_0/M = 10$

685 (b) $r_0/M = 6$

686 (c) $r_0/M = 3$

687 **7. Hurl a stone inward from far away**

688 You hurl a stone radially inward with speed v_{far} from a remote location. (At a
689 remote r where spacetime is flat, $|dr/dt|$ equals shell speed.)

690 A. Derive an expression for dr/dt of the stone as a function of r . Show
691 that when you launch the stone from rest, dr/dt reduces to the
692 expression (22) for a raindrop. Find the value of r at which $|dr/dt|$ is
693 *maximum* and the expression for $|dr/dt|$. Verify that in the limit in
694 which the stone is dropped from rest far away, these expressions reduce
695 to those found in Exercise 6.2 for the raindrop.

696 B. Derive an expression for the *shell velocity* of the stone as a function of
697 r . Show that in the limit in which the stone drops from rest far away,
698 the shell velocity reduces to the expression (19) for a raindrop.

699 C. Sketch graphs of shell speed *vs.* r similar to Figure 2 for the following
700 values of v_{far} :

701 (a) $v_{\text{far}} = 0.20$

702 (b) $v_{\text{far}} = 0.60$

703 (c) $v_{\text{far}} = 0.90$

704 **8. All Possible Shell Speeds**

705 Think of a shell observer at any $r > 2M$. Consider the following three launch
706 methods for a stone that passes him moving radially inward: (a) released at
707 rest from a shell at $r_0 \geq r$, (b) released from rest far away, and (c) hurled
708 radially inward from far away with initial speed $0 < |v_{\text{far}}| < 1$. Show that,
709 taken together, these three methods can result in all possible speeds
710 $0 \leq |v_{\text{shell}}| < 1$ measured by this shell observer at $r > 2M$.

711 **9. Only One Shell Speed—with the Value One—at the Event Horizon**

712 Show that the three kinds of radial launch of a stone described in Exercise 8
713 yield the *same* shell speed, namely $|v_{\text{shell}}| = 1$, as a limiting case when the
714 stone moves inward across the event horizon. Your result shows that at the
715 event horizon (as a limiting case): (a) You cannot make the shell-observed
716 speed of a stone *greater* than that of light, no matter how fast you hurl it
717 inward from far away. (b) You cannot make the shell-observed speed of the
718 stone *less* than that of light, no matter how close to the event horizon you
719 release it from rest.

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720 **10. Energy from garbage using a black hole**

721 Define an **advanced civilization** as one that can carry out any engineering
 722 task not forbidden by the laws of physics. An advanced civilization wants to
 723 use a black hole as an energy source. Most useful is a “live” black hole, one
 724 that spins (Chapters 17 through 21), with rotation energy available for use.
 725 Unfortunately the nonrotating black hole that we study in this chapter is
 726 “dead:” no energy can be extracted from it (except for entirely negligible
 727 Hawking radiation, Box 5). Instead, our advanced civilization uses the dead
 728 (nonspinning) black hole to convert garbage to useful energy, as you analyze in
 729 this exercise.

730 A bag of garbage of mass m drops from rest at a power station located at
 731 r_0 , onto a shell at r ; a machine at the lower r brings the garbage to rest and
 732 converts all of the *shell kinetic energy* into a light flash. Express all energies
 733 requested below as fractions of the mass m of the garbage.

- 734 A. What is the energy of the light flash measured on the shell where it is
 735 emitted?
- 736 B. The machine now directs the resulting flash of light radially outward.
 737 What is the energy of this flash as it arrives back at the power station?
- 738 C. Now the conversion machine at r releases the garbage so that it falls
 739 into the black hole. What is the increase ΔM in the mass of the black
 740 hole? What is its increase in mass if the conversion machine is
 741 located—as a limiting case—exactly at the event horizon?
- 742 D. Find an expression for the efficiency of the resulting energy conversion,
 743 that is (output energy at the power station)/(input garbage mass m) as
 744 a function of the converter r and the r_0 of the power station. What is
 745 the efficiency when the power station is far from the black hole,
 746 $r_0 \rightarrow \infty$, and the conversion machine is on the shell at $r = 3M$?
 747 (Except for matter-antimatter collisions, the efficiency of
 748 mass-to-energy conversions in nuclear reactions on Earth is never
 749 greater than a fraction of one percent.)
- 750 E. *Optional:* Check the conservation of *map* energy in all of the processes
 751 analyzed in this exercise.

752 **Comment 2. Decrease disorder with a black hole vacuum cleaner?**

753 Suppose that the neighborhood of a black hole is strewn with garbage. We tidy
 754 up the vicinity by dumping the garbage into the black hole. This cleanup reduces
 755 disorder in the surroundings of the black hole. But wait! Powerful principles of
 756 thermodynamics and statistical mechanics demand that the disorder—technical
 757 name: **entropy**—of an isolated system (in this case, garbage plus black hole)
 758 cannot decrease. Therefore the disorder of the black hole itself must increase
 759 when we dump disordered garbage into it. Jacob Bekenstein and Stephen
 760 Hawking quantified this argument to define a measure of the entropy of a black
 761 hole, which turns out to be proportional to the Euclidean-calculated spherical
 762 “area” of the event horizon. See Kip S. Thorne, *Black Holes and Time Warps*,
 763 pages 422–448.

764 **11. Temperature of a Black Hole**

- 765 A Use equation (32) to find the temperature, when viewed from far away,
766 of a black hole of mass five times the mass of our Sun.
- 767 B. What is the mass of a black hole whose temperature, viewed from far
768 away, is 1800 degrees Kelvin (the melting temperature of iron)?
769 Express your answer as a fraction or multiple of the mass of Earth.
770 (Equation (32) tells us that “smaller is hotter,” which leads to
771 increased emission by a smaller black hole and therefore shorter life. If
772 this analysis is correct, small black holes created in the Big Bang must
773 have evaporated by now.)

6.9 ■ REFERENCES

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