

# Schedule of this course

February 6.	lecture 01 (today)	Basics
February 13.	lecture 02	Algorithms 1
February 20.	lecture 03	Decoherence 1
February 27.	lecture 04	Decoherence 2
March 6.	lecture 05	Decoherence 3
March 13.	lecture 06	Algorithms 2
March 20.		
----- Tavaszi szünet -----		
March 27.	lecture 07	Algorithms 3
April 3.	lecture 08	Algorithms 4
April 10.	lecture 09	Algorithms 5
April 17.	lecture 10	Quantum Error Correction 1
April 24.	lecture 11	Quantum Error Correction 2
May 1.		
Munka ünnepe		
May 8.	lecture 12	Bell inequalities
May 15.		<b>May 22</b>

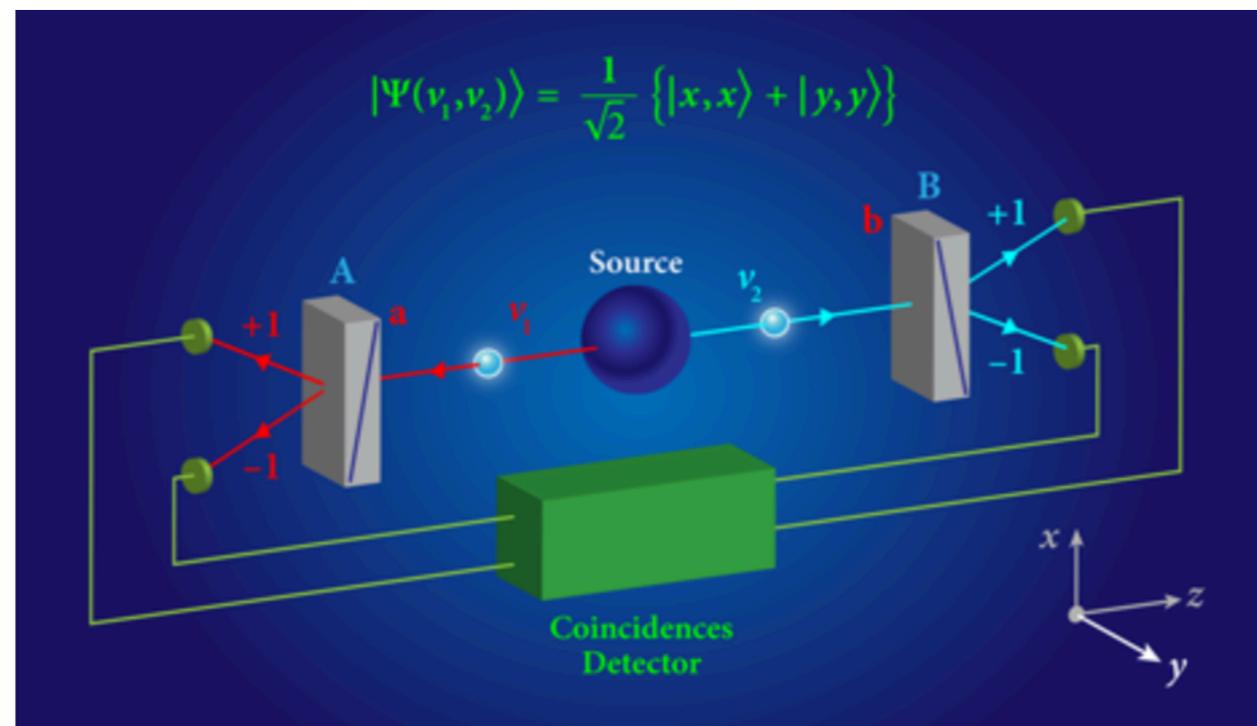
# Quantum Information Processing

Budapest University of Technology and Economics  
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Lecture 12

Bell and CHSH Inequalities - and their violation



# **Bell inequalities and their violation**

The essence of this lecture: Bell inequalities are derived for classical (or classical probabilistic) models. The violation of these provides a "no-go result" that draws an important distinction between quantum mechanics and the world as described by classical mechanics, particularly concerning quantum entanglement where two or more particles in a quantum state continue to be mutually dependent.

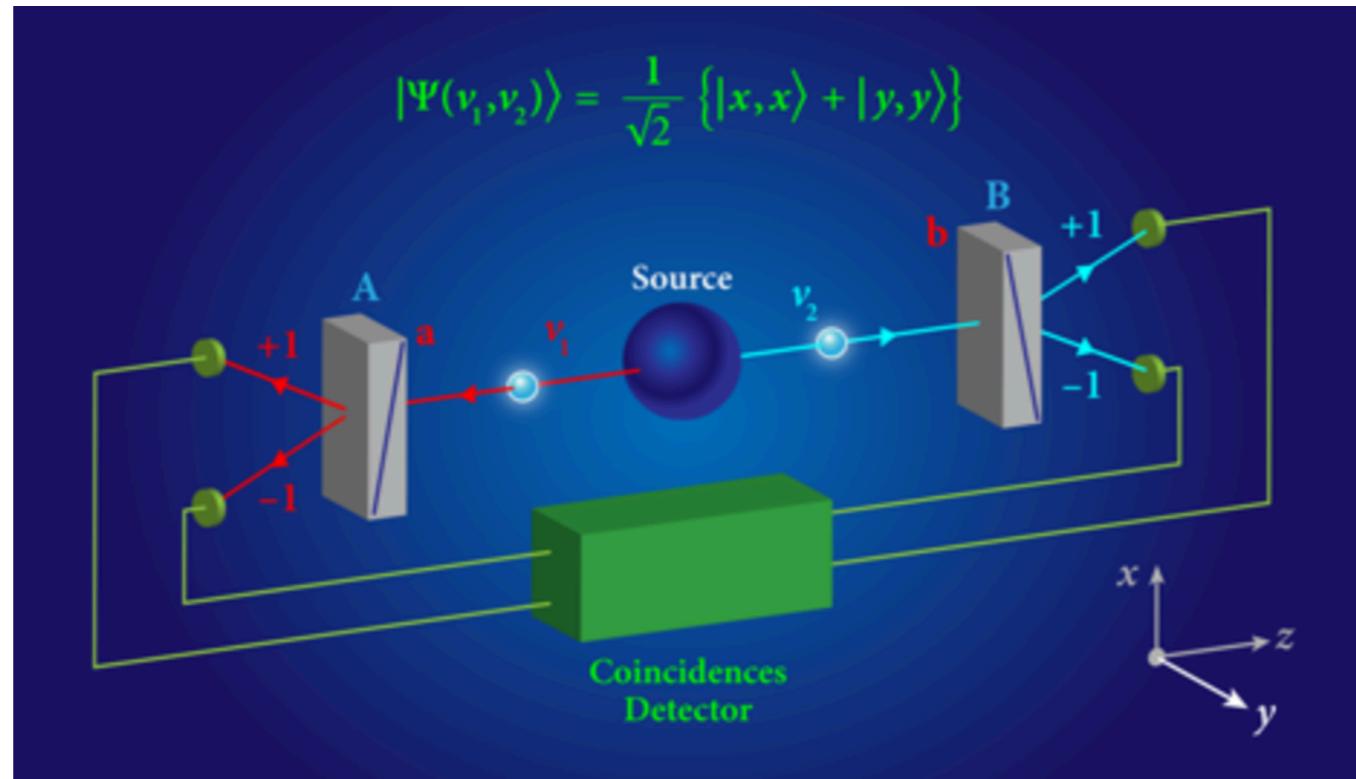
The conclusion one can draw: No physical theory of local hidden variables can ever reproduce all of the predictions of quantum mechanics.

## **The Clauser-Horne-Shimony-Holt inequality.**

The most popular type of Bell inequality is not John Bell's original inequality but the Clauser-Horne-Shimony-Holt (CHSH) inequality.

John Clauser, Michael Horne, Abner Shimony, and Richard Holt derived this inequality in 1969, which, as with John Bell's original inequality (1964) is a constraint on the statistics of "coincidences" in a Bell test experiment which is necessarily true if there exist underlying classical description of the set-up or in more technical terms there is a local hidden variable description. This constraint can, on the other hand, be violated by quantum mechanics.

# The CHSH Inequality - set-up and assumptions classically



There are two  $a$  and  $a'$  detector settings on the side of Alice, and  $b$  and  $b'$  settings on the side of Bob. The four combinations being tested in separate subexperiments.

## The CHSH Inequality - set-up and assumptions classically

When measuring the value of a state with detector set-up  $a$  and  $a'$  the measured values of Alice can only be  $A(a) = \pm 1$  and  $A(a') = \pm 1$ , and similarly for Bob  $B(b) = \pm 1$  and  $B(b') = \pm 1$ . In a classical set-up it follows for the considered one has either

$$A(a) + A(a') = \pm 2 \text{ and } A(a) - A(a') = 0, \text{ or}$$
$$A(a) + A(a') = 0 \text{ and } A(a) - A(a') = \pm 2.$$

Therefore we have that

$$C = (A(a) + A(a'))B(b) + (A(a) - A(a'))B(b') = \pm 2$$

# The CHSH Inequality - set-up and assumptions classically

What happens if we collect measurement data from a statistical mixture of classical deterministic states. Then the joint expectation value of  $A(a) B(b)$  type of measurements are of the form:

$$E(a, b) = \int \underline{A}(a, \lambda) \underline{B}(b, \lambda) \rho(\lambda) d\lambda$$

# The CHSH Inequality - set-up and assumptions classically

Taking absolute values of both sides, and applying the [triangle inequality](#) to the right-hand side, we obtain

$$|E(a, b) - E(a, b')| \leq \left| \int \underline{A}(a, \lambda) \underline{B}(b, \lambda) [1 \pm \underline{A}(a', \lambda) \underline{B}(b', \lambda)] \rho(\lambda) d\lambda \right| + \left| \int \underline{A}(a, \lambda) \underline{B}(b', \lambda) [1 \pm \underline{A}(a', \lambda) \underline{B}(b, \lambda)] \rho(\lambda) d\lambda \right|$$

We use the fact that  $[1 \pm \underline{A}(a', \lambda) \underline{B}(b', \lambda)] \rho(\lambda)$  and  $[1 \pm \underline{A}(a', \lambda) \underline{B}(b, \lambda)] \rho(\lambda)$  are both non-negative to rewrite the right-hand side of this as

$$\int |\underline{A}(a, \lambda) \underline{B}(b, \lambda)| |[1 \pm \underline{A}(a', \lambda) \underline{B}(b', \lambda)] \rho(\lambda) d\lambda| + \int |\underline{A}(a, \lambda) \underline{B}(b', \lambda)| |[1 \pm \underline{A}(a', \lambda) \underline{B}(b, \lambda)] \rho(\lambda) d\lambda|$$

By (4), this must be less than or equal to

$$\int [1 \pm \underline{A}(a', \lambda) \underline{B}(b', \lambda)] \rho(\lambda) d\lambda + \int [1 \pm \underline{A}(a', \lambda) \underline{B}(b, \lambda)] \rho(\lambda) d\lambda$$

which, using the fact that the integral of  $\rho(\lambda)$  is 1, is equal to

$$2 \pm \left[ \int \underline{A}(a', \lambda) \underline{B}(b', \lambda) \rho(\lambda) d\lambda + \int \underline{A}(a', \lambda) \underline{B}(b, \lambda) \rho(\lambda) d\lambda \right]$$

which is equal to  $2 \pm [E(a', b') + E(a', b)]$ .

Putting this together with the left-hand side, we have:

$$|E(a, b) - E(a, b')| \leq 2 \pm [E(a', b') + E(a', b)]$$

which means that the left-hand side is less than or equal to both  $2 + [E(a', b') + E(a', b)]$  and  $2 - [E(a', b') + E(a', b)]$ . That is:

$$|E(a, b) - E(a, b')| \leq 2 - |E(a', b') + E(a', b)|$$

from which we obtain

$$2 \geq |E(a, b) - E(a, b')| + |E(a', b') + E(a', b)| \geq |E(a, b) - E(a, b') + E(a', b') + E(a', b)|$$

(by the [triangle inequality](#) again), which is the CHSH inequality.

# Violation of the CHSH Inequality in Quantum Mechanics

Suppose we have prepared the state:

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

We choose the detection observables  $a, a', b,$  and  $b'$  as

$$\begin{aligned} Q &= Z_1 & S &= \frac{-Z_2 - X_2}{\sqrt{2}} \\ R &= X_1 & T &= \frac{Z_2 - X_2}{\sqrt{2}}. \end{aligned}$$

# Violation of the CHSH Inequality in Quantum Mechanics

Simple calculations show that the average values for these observables are:

$$\langle QS \rangle = \frac{1}{\sqrt{2}}; \quad \langle RS \rangle = \frac{1}{\sqrt{2}}; \quad \langle RT \rangle = \frac{1}{\sqrt{2}}; \quad \langle QT \rangle = -\frac{1}{\sqrt{2}}.$$

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Thus

$$\langle QS \rangle = \frac{1}{\sqrt{2}}; \langle RS \rangle = \frac{1}{\sqrt{2}}; \langle RT \rangle = \frac{1}{\sqrt{2}}; \langle QT \rangle = -\frac{1}{\sqrt{2}}.$$

We violate the CHSH inequality!

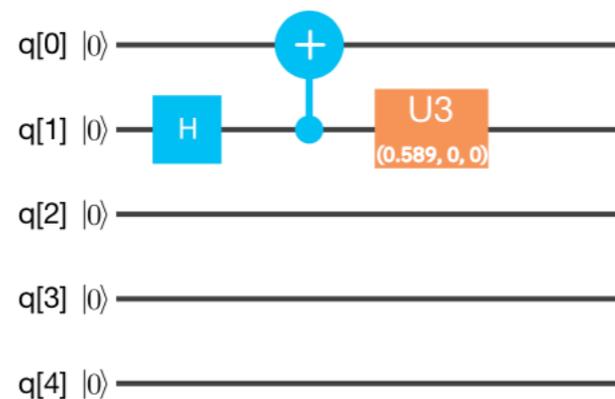
# Bell test experiments - and loop holes

## Notable experiments

- 4.1 Freedman and Clauser (1972)
- 4.2 Aspect et al. (1982)
- 4.3 Tittel et al. (1998)
- 4.4 Weihs et al. (1998): experiment under "strict Einstein locality" conditions
- 4.5 Pan et al. (2000) experiment on the GHZ state
- 4.6 Rowe et al. (2001): the first to close the detection loophole
- 4.7 Gröblacher et al. (2007) test of Leggett-type non-local realist theories
- 4.8 Salart et al. (2008): separation in a Bell Test
- 4.9 Ansmann et al. (2009): overcoming the detection loophole in solid state
- 4.10 Giustina et al. (2013), Larsson et al (2014): overcoming the detection loophole for photons
- 4.11 Christensen et al. (2013): overcoming the detection loophole for photons
- 4.12 Hensen et al., Giustina et al., Shalm et al. (2015): "loophole-free" Bell tests
- 4.13 Schmied et al. (2016): Detection of Bell correlations in a many-body system
- 4.14 Handsteiner et al. (2017): "Cosmic Bell Test" - Measurement Settings from Milky Way Stars
- 4.15 Rosenfeld et al. (2017): "Event-Ready" Bell test with entangled atoms and closed detection and locality loopholes
- 4.16 The BIG Bell Test Collaboration (2018): "Challenging local realism with human choices"
- 4.17 Rauch et al (2018): measurement settings from distant quasars

# Exercise

1. Create a two qubit gate using the simple circuit below. Calculate by hand which state it is. Then test the CHSH inequality using the local X and Z observables for both of the qubits.



The definition U3

$$\begin{pmatrix} \cos(\theta/2) & -e^{i\lambda} \sin(\theta/2) \\ e^{i\phi} \sin(\theta/2) & e^{i\lambda+i\phi} \cos(\theta/2) \end{pmatrix}$$