Schedule of this course

Szerda		
February 6.	lecture 01 (today)	Basics
February 13.	lecture 02	Algorithms 1
February 20.	lecture 03	Decoherence 1
February 27.	lecture 04	Decoherence 2
March 6.	lecture 05	Decoherence 3
March 13.	lecture 06	Algorithms 2
March 20. Tavaszi szünet		
March 27.	lecture 07	Algorithms 3
April 3.	lecture 08	Algorithms 4
April 10.	lecture 09	Algorithms 5
April 17.	lecture 10	Quantum Error Correction 1
April 24.	lecture 11	Quantum Error Correction 2
May 1. Munka ünnepe		
M[illegible]8.		
Ma[illegible]5.		

lecture 12 Bell inequalities

May 22

Quantum Information Processing

Budapest University of Technology and Economics
2019 Spring

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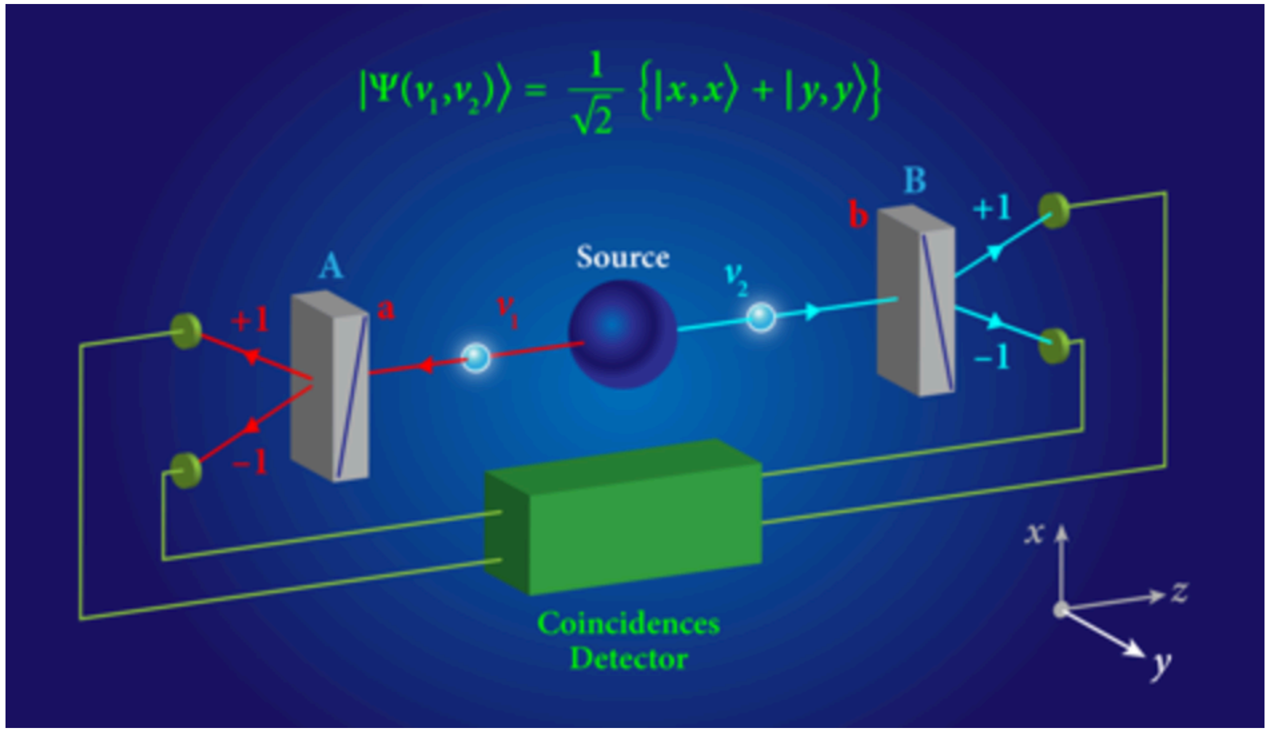
Lecture 12
Bell and CHSH Inequalities - and their violation



Bell inequalities and their violation

The essence of this lecture: Bell inequalities are derived for classical (or classical probabilistic) models. The violation of these provides a "no-go result" that draws an important distinction between quantum mechanics and the world as described by classical mechanics, particularly concerning quantum entanglement where two or more particles in a quantum state continue to be mutually dependent.

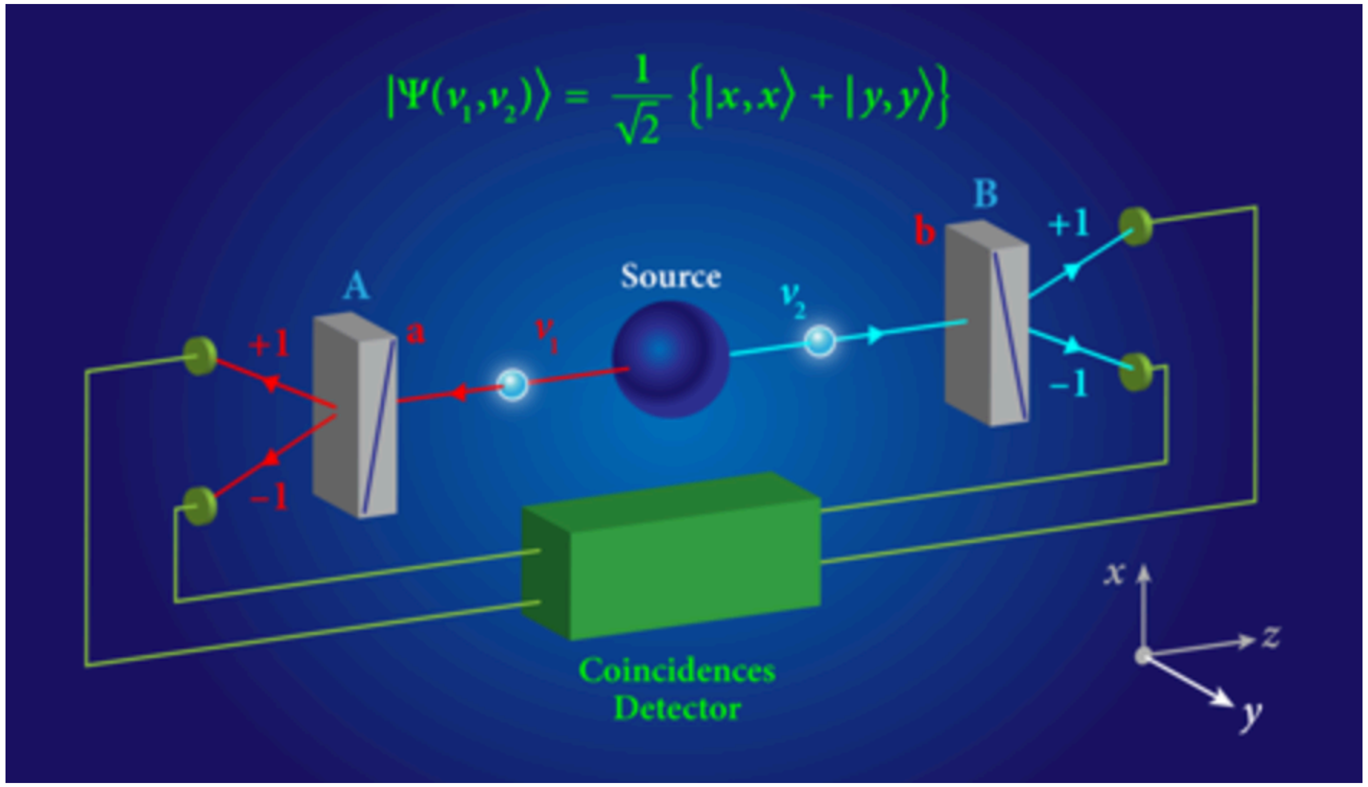
The conclusion one can draw: No physical theory of local hidden variables can ever reproduce all of the predictions of quantum mechanics.

The Clauser-Horne-Shimony-Holt inequality.

The most popular type of Bell inequality is not John Bell’s original inequality but the Clauser-Horne-Shimony-Holt (CHSH) inequality.

John Clauser, Michael Horne, Abner Shimony, and Richard Holt derived this inequality in 1969, which, as with John Bell's original inequality (1964) is a constraint on the statistics of "coincidences" in a Bell test experiment which is necessarily true if there exist underlying classical description of the set-up or in more technical terms there is a local hidden variable description. This constraint can, on the other hand, be violated by quantum mechanics.

The CHSH Inequality -
set-up and assumptions classically



There are two a and a′ detector settings on the side of Alice, and b and b′ settings on the side of Bob. The four combinations being tested in separate subexperiments.

The CHSH Inequality - set-up and assumptions classically

When measuring the value of a state with detector set-up a and a′ the measured values of Alice can only be $A(a)= \pm 1$ and $A(a')= \pm 1$, and similarly for Bob $B(b)= \pm 1$ and $B(b')= \pm 1$. In a classical set-up it follows for the considered one has either

$A(a)+A(a')=\pm 2$ and $A(a)-A(a')=0$, or
$A(a)+A(a')= 0$ and $A(a)-A(a')=\pm 2$.

Therefore we have that

$$C= (A(a)+A(a'))B(b) + (A(a)-A(a'))B(b')= \pm 2$$

The CHSH Inequality - set-up and assumptions classically

What happens if we collect measurement data from a statistical mixture of classical deterministic states. Then the joint expectation value of A(a) B(b) type of measurements are of the form:

$$E(a,b) = \int \underline{A}(a,\lambda)\underline{B}(b,\lambda)\rho(\lambda)d\lambda$$

The CHSH Inequality - set-up and assumptions classically

Taking absolute values of both sides, and applying the triangle inequality to the right-hand side, we obtain

$$|E(a,b) - E(a,b')| \leq \left| \int \underline{A}(a,\lambda)\underline{B}(b,\lambda)\left[1 \pm \underline{A}(a',\lambda)\,\underline{B}(b',\lambda)\right]\rho(\lambda)d\lambda \right| + \left| \int \underline{A}(a,\lambda)\underline{B}(b',\lambda)\left[1 \pm \underline{A}(a',\lambda)\,\underline{B}(b,\lambda)\right]\rho(\lambda)d\lambda \right|$$

We use the fact that $\left[1 \pm \underline{A}(a',\lambda)\,\underline{B}(b',\lambda)\right]\rho(\lambda)$ and $\left[1 \pm \underline{A}(a',\lambda)\,\underline{B}(b,\lambda)\right]\rho(\lambda)$ are both non-negative to rewrite the right-hand side of this as

$$\int \left|\underline{A}(a,\lambda)\underline{B}(b,\lambda)\right| \left|\left[1 \pm \underline{A}(a',\lambda)\,\underline{B}(b',\lambda)\right]\rho(\lambda)d\lambda\right| + \int \left|\underline{A}(a,\lambda)\underline{B}(b',\lambda)\right| \left|\left[1 \pm \underline{A}(a',\lambda)\,\underline{B}(b,\lambda)\right]\rho(\lambda)d\lambda\right|$$

By (4), this must be less than or equal to

$$\int \left[1 \pm \underline{A}(a',\lambda)\,\underline{B}(b',\lambda)\right]\rho(\lambda)d\lambda + \int \left[1 \pm \underline{A}(a',\lambda)\,\underline{B}(b,\lambda)\right]\rho(\lambda)d\lambda$$

which, using the fact that the integral of ρ(λ) is 1, is equal to

$$2 \pm \left[\int \underline{A}(a',\lambda)\,\underline{B}(b',\lambda)\,\rho(\lambda)d\lambda + \int \underline{A}(a',\lambda)\,\underline{B}(b,\lambda)\rho(\lambda)d\lambda\right]$$

which is equal to $2 \pm \left[E(a',b') + E(a',b)\right]$.

Putting this together with the left-hand side, we have:

$$|E(a,b) - E(a,b')| \ \leq 2 \ \pm \left[E(a',b') + E(a',b)\right]$$

which means that the left-hand side is less than or equal to both $2 + \left[E(a',b') + E(a',b)\right]$ and $2 - \left[E(a',b') + E(a',b)\right]$. That is:

$$|E(a,b) - E(a,b')| \ \leq \ 2 - |E(a',b') + E(a',b)|$$

from which we obtain

$$2 \ \geq \ |E(a,b) - E(a,b')| + |E(a',b') + E(a',b)| \ \geq \ |E(a,b) - E(a,b') + E(a',b') + E(a',b)|$$

(by the triangle inequality again), which is the CHSH inequality.

Violation of the CHSH Inequality in Quantum Mechanics

Suppose we have prepared the state:

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

We choose the detection observables a,a', b, and b' as

$$Q = Z_1 \qquad S = \frac{-Z_2 - X_2}{\sqrt{2}}$$

$$R = X_1 \qquad T = \frac{Z_2 - X_2}{\sqrt{2}}.$$

Violation of the CHSH Inequality in Quantum Mechanics

Simple calculations show that the average values for these observables are:

$$\langle QS \rangle = \frac{1}{\sqrt{2}};\ \langle RS \rangle = \frac{1}{\sqrt{2}};\ \langle RT \rangle = \frac{1}{\sqrt{2}};\ \langle QT \rangle = -\frac{1}{\sqrt{2}}.$$

Violation of the CHSH Inequality in Quantum Mechanics

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Thus

$$\langle QS \rangle = \frac{1}{\sqrt{2}};\ \langle RS \rangle = \frac{1}{\sqrt{2}};\ \langle RT \rangle = \frac{1}{\sqrt{2}};\ \langle QT \rangle = -\frac{1}{\sqrt{2}}.$$

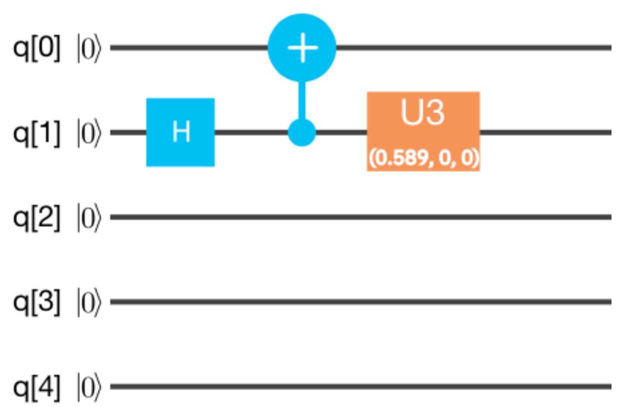
We violate the CHSH inequality!

Bell test experiments - and loop holes

Notable experiments

Exercise

1. Create a two quit gate using the simple circuit below. Calculate by hand which state it is. Then test the CHSH inequality using the local X and Z observables for both of the qubits.



The definition U3

$$\begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i\lambda+i\phi}\cos(\theta/2) \end{pmatrix}$$