Chapter 12. Diving Panoramas

- *In which local direction (or directions) does a local inertial rain observer look to see a given star as she passes coordinate r?*
- *In which direction (or directions) does a shell observer stationary at r and ϕ coordinates look to see the same star?*
- *How does the panorama of the heavens change for the local rain observer as she descends?*
- *Is gravitationally blue-shifted starlight lethal for the rain observer as she approaches the singularity? Is this starlight more dangerous than killer tides?*
- *How close to the singularity will the rain observer survive?*
- *What is the last thing the local rain observer sees?*

CHAPTER 12

Diving Panoramas

Edmund Bertschinger & Edwin F. Taylor *

Tell all the truth but tell it slant –
Success in Circuit lies
Too bright for our infirm Delight
The Truth's superb surprise
As Lightning to the Children eased
With explanation kind
The Truth must dazzle gradually
Or every man be blind –

—Emily Dickinson

12.1 ■ FALLING INTO THE BLACK HOLE

See the same beam in two different directions.

"What's it like to fall into a black hole?"

"What is it like to fall into a black hole?" Our book thus far can be thought of as preparation to answer this question. The simplest possible answer has two parts: "What do I *feel* as I descend?" and "What do I *see* as I descend?" You *feel* tidal accelerations that—sorry about this—"spaghettify" you before you reach the singularity (Query 23, Section 7.9). As you descend, you *see* a changing panorama of the starry heavens, developed in this chapter and narrated in Section 12.7.

Where does a rain observer look to see a given beam?

The preceding Chapter 11 plotted trajectories of starlight in *global* rain coordinates and told us which beams (plural!) connect a given distant star to the *map location* of an observer. But that chapter said nothing about the direction in which that observer looks to see each beam or the beam energy she measures. These are the goals of the present chapter.

Comment 1. The rain diver
To dive—to be a diver—means to free-fall radially inward toward a center of attraction. A diver can drop from rest on—or be hurled radially inward from—any shell, including a shell far from the black hole. Among divers, the rain observer is a special case: a diver that drops from rest far away. In this chapter the word *diver* most often means the *rain diver*.

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

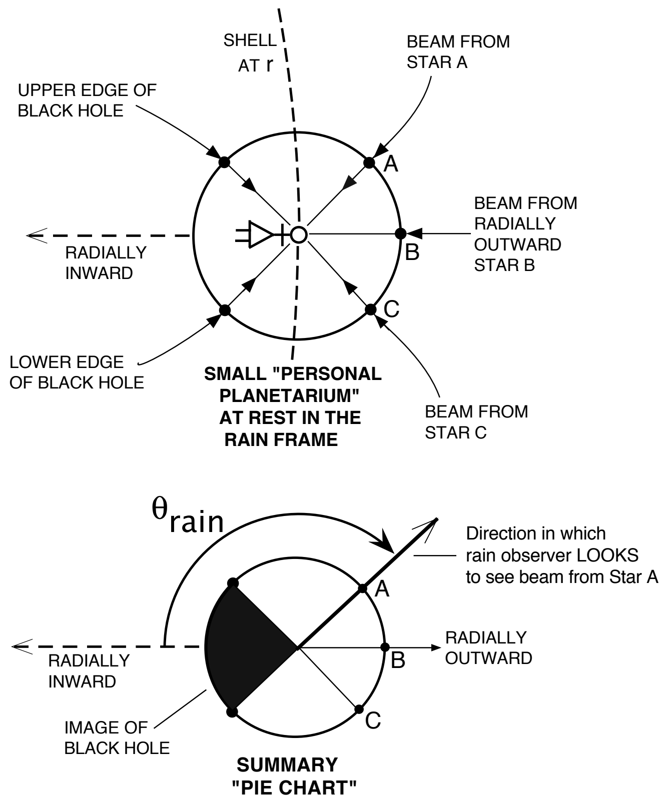

FIGURE 1 *Upper panel:* The **personal planetarium** is a small transparent sphere that encloses—and falls with—the local rain observer. The observer marks on the inside of this sphere the point-images of stars. She also draws a circle around the visual edge of the black hole. *Lower panel:* The **pie chart** summarizes rain observer markings; a black "pie slice" spans the visual image of the black hole. To see a particular beam, the rain observer *looks* in the direction $\theta_{\rm rain}$, which she measures clockwise from the radially inward direction.

12.2 ■ THE PERSONAL PLANETARIUM

Enjoy the view in weightless comfort.

"Personal planetarium" to record local observations

How does the local rain observer view and record starlight beams? One practical answer to this question is the **personal planetarium**: a transparent sphere at rest in the observer's local inertial frame with the observer's eye at its center (upper panel of Figure 1). Light beams run straight with respect to this local inertial frame, as shown in the figure. The observer marks on the inside of the transparent sphere the points of light she sees from stars in all directions; she also draws on the inside of the sphere a circle around the visual edge of the black hole.

We call the lower panel in figure 1 the **pie chart**. The pie chart takes its name from the standard graphical presentation whose black "pie slice" shows the fraction of some quantity as the proportion of the whole. In our pie chart the pie slice shows the range of visual angles covered by the black hole.

On the personal planetarium sphere the observer locates a star with the angle $\theta_{\rm rain}$ between the center of the black hole image and the dot she has placed on the image of that star. For simplicity, we omit the coordinate subscript "obs" for "observer" used in Chapter 11.

Definition: angle $\theta_{\rm rain}$

DEFINITION 1. Angle $\theta_{\rm rain}$

The observer in the personal planetarium looks in the direction $\theta_{\rm rain}$ to see the to see any given star. We define angle $\theta_{\rm rain} = 0$ to be radially inward, from the observer's eye toward the center of the black hole and the positive angle $\theta_{\rm rain}$ to be clockwise from this direction measured in her local rain frame (lower panel, Figure 1).

Full 3-dimensional rain observer's panorama

Can the local rain observer see a star that lies out of the plane of this page? Of course: Every star lies on *some* slice determined by three points: the star, the rain observer's eye, and the map coordinate $r = 0$. To encompass all stars in the heavens, rotate each of the circles in Figure 1 around its horizontal radial line. This rotation turns the pie chart into a sphere and the black hole "pie slice" into a cone. From inside her planetarium, the local rain observer sees the full panorama of stars in the heavens.

?

Objection 1. *You say, "From inside her planetarium, the local rain observer sees the full panorama of stars in the heavens." Why isn't that the end of the story? What more does the rain observer need to know?*

!

If she is satisfied to describe a general view of the heavens, that is sufficient for her. However, she may want to know, for example, where to look to see Alpha Centauri, one of Earth's nearest neighbors, as she plunges past $r = 4M$. The present chapter tells her the angle $\theta_{\rm rain}$ in which she looks to see the star located at global angle ϕ_∞. Finding $\theta_{\rm rain}$ of a star is a two-step process: The present chapter says in what local direction $\theta_{\rm rain}$ the rain observer looks to see a beam with a given value of b. The analysis from Chapter 11 then tells us the global angle ϕ_∞ of the star that emits the beam with that value of b. Angle $\theta_{\rm rain}$ depends on the observer's instantaneous global coordinate r, so varies with r as she descends. *Result:* a changing panorama, described in Section 12.8.

12.3 ■ RAIN FRAME VIEW OF LIGHT BEAMS

"I spy with my little eye . . . "

From stone to light

As she falls past r, the local rain observer sees a beam from a distant star at the angle $\theta_{\rm rain}$ with respect to the radially inward direction. We want an expression for this observation angle as a function of r and the b-value of that light beam. Chapter 11 defines b as the ratio $b \equiv L/E$. Equation (35) in Section 7.5 gives the expression for the map energy of a stone:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau} \qquad \text{(stone)} \qquad (1)$$

The expression for map angular momentum of a stone comes from equation (10) in Section 8.2:

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} \qquad \text{(stone)} \qquad (2)$$

b in global rain coordinates

Divide both sides of (2) by the corresponding sides of (1) and divide numerator and denominator of the result by dT. The symbol m cancels on the left side to yield the ratio $b \equiv L/E$ for light:

$$b \equiv \frac{L}{E} = \frac{r\left(\dfrac{rd\phi}{dT}\right)}{\left(1 - \dfrac{2M}{r}\right) - \left(\dfrac{2M}{r}\right)^{1/2}\dfrac{dr}{dT}} \qquad \text{(light)} \qquad (3)$$

Equation (3) expresses b in rain coordinates. But the local planetarium observer measures visual angles in her local inertial rain frame. Box 4 in Section 7.5 expressed local rain frame coordinates in global coordinates:

$$\Delta t_{\text{rain}} \equiv \Delta T \qquad (4)$$

$$\Delta y_{\text{rain}} \equiv \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2}\Delta T \qquad (5)$$

$$\Delta x_{\text{rain}} \equiv \bar{r}\Delta\phi \qquad (6)$$

Light moves with speed unity in the local inertial rain frame, $\Delta s_{\text{rain}}/\Delta t_{\text{rain}} = 1$, so the Pythagorean Theorem provides labels for legs of the right triangle in Figure 2:

$$\Delta x_{\text{rain}}^2 + \Delta y_{\text{rain}}^2 = \Delta s_{\text{rain}}^2 = \Delta t_{\text{rain}}^2 \qquad \text{(light, in local inertial rain frame)} \quad (7)$$

Find rain observer angle to see light with b

At what angle θ_{rain} does the rain observer look in her local frame in order to see the incoming beam with impact parameter b? The incoming beam in Figure 2 represents any one of the multiple beams arriving at the rain observer from a single star. In Figure 2 the symbols A, B, and C stand for the (positive) lengths of the sides of the right triangle. In contrast, the inward-moving light has negative components of motion radially and tangentially in the figure. The local frame time lapse Δt_{rain} is positive along the worldline. Expressing these results in local rain coordinates (4) through (6) leads to the following expressions for sine and cosine of the angle θ_{rain}:

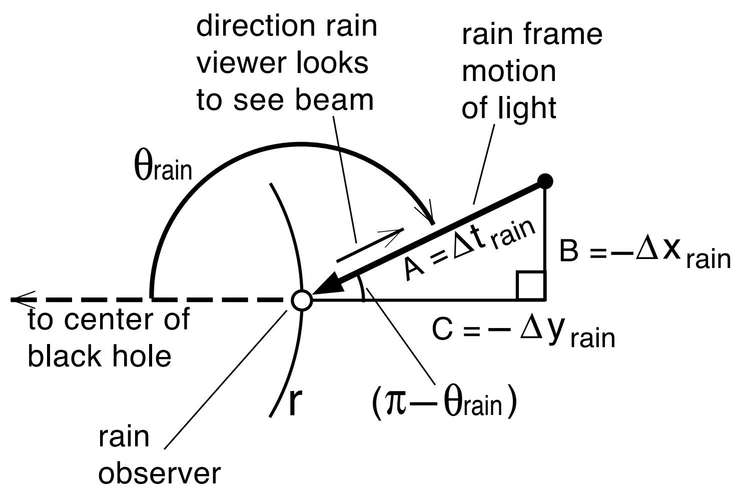

FIGURE 2 The beam from a distant star reaches the local rain observer as she dives inward past the shell at r. She measures the observation angle $\theta_{\rm rain}$ clockwise with respect to the radially inward direction. Letters A, B, and C are the (positive) lengths of the legs of the right triangle in the local rain frame. As the light approaches the observer, it has a clockwise tangential component in the local rain frame so $\Delta x_{\rm rain}$ is negative (and $-\Delta x_{\rm rain}$ is positive, equal to side B, as shown). The light also has a radially inward component, so $\Delta y_{\rm rain}$ is also negative (and $-\Delta y_{\rm rain}$ is positive, equal to C as shown). The hypotenuse is $\Delta s_{\rm rain} = \Delta t_{\rm rain}$ from (7) is positive and lies along the worldline of the beam $\Delta t_{\rm rain}$.

$$\sin\theta_{\rm rain} = \sin(\pi - \theta_{\rm rain}) \tag{8}$$

$$= \lim_{A\to 0}\frac{B}{A} = \lim_{\Delta t_{\rm rain}\to 0}\frac{-\Delta x_{\rm rain}}{\Delta t_{\rm rain}} = \lim_{\Delta T\to 0}\frac{-\bar{r}\Delta\phi}{\Delta T} \qquad \text{so that}$$

$$\sin\theta_{\rm rain} = -\frac{r\,d\phi}{dT} \tag{9}$$

A similar procedure leads to an expression for $\cos\theta_{\rm rain}$:

$$\cos\theta_{\rm rain} = -\cos(\pi - \theta_{\rm rain}) \tag{10}$$

$$= -\lim_{A\to 0}\frac{C}{A} = -\lim_{\Delta t_{\rm rain}\to 0}\frac{-\Delta y_{\rm rain}}{\Delta t_{\rm rain}} = \lim_{\Delta T\to 0}\frac{\Delta r}{\Delta T} + \left(\frac{2M}{r}\right)^{1/2}$$

$$= \frac{dr}{dT} + \left(\frac{2M}{r}\right)^{1/2} \qquad \text{so that}$$

$$\frac{dr}{dT} = \cos\theta_{\rm rain} - \left(\frac{2M}{r}\right)^{1/2} \tag{11}$$

Substitute expressions (9) and (11) into (3), square both sides of the result, then eliminate the remaining sine squared with the identity $\sin^2\theta = 1 - \cos^2\theta$. Rearrange the result to yield the following quadratic equation in $\cos\theta_{\rm rain}$:

$$\left(1+\frac{b^2}{r^2}\frac{2M}{r}\right)\cos^2\theta_{\text{rain}} - 2\frac{b^2}{r^2}\left(\frac{2M}{r}\right)^{1/2}\cos\theta_{\text{rain}} - \left(1-\frac{b^2}{r^2}\right) = 0 \qquad (12)$$

Rain observer angle to see light with b

Solve the quadratic equation (12) to find an expression for $\cos\theta_{\text{rain}}$:

$$\cos\theta_{\text{rain}} = \frac{\frac{b^2}{r^2}\left(\frac{2M}{r}\right)^{1/2} \pm F(b,r)}{1+\frac{b^2}{r^2}\frac{2M}{r}} \qquad \text{(light)} \qquad (13)$$

Sign in numerator for global motion of light: $+$ for $dr > 0$, $-$ for $dr < 0$

where, from equation (16) in Section 11.2,

$$F(b,r) \equiv \left[1 - \frac{b^2}{r^2}\left(1-\frac{2M}{r}\right)\right]^{1/2} \qquad \text{(light)} \qquad (14)$$

QUERY 1. Equations for $\cos\theta_{\text{rain}}$ (*Optional*)

A. Use equations (3), (9), and (11) to derive quadratic equation (12) for $\cos\theta_{\text{rain}}$.
B. Solve quadratic equation (12) to derive (13) for $\cos\theta_{\text{rain}}$.

There is an ambiguity in (13) because $\cos\theta = \cos(-\theta)$. To remove this ambiguity, substitute (9) and (11) into (3), then use (13) to substitute for $\cos\theta_{\text{rain}}$. After considerable manipulation, the result is:

$$\sin\theta_{\text{rain}} = \frac{b}{r}\left[\frac{-1 \pm \left(\frac{2M}{r}\right)^{1/2} F(b,r)}{1+\frac{b^2}{r^2}\frac{2M}{r}}\right] \qquad \text{(light)} \qquad (15)$$

which provides a stand-by correction of sign in (13). As in that equation, the plus sign is for $dr > 0$ and the minus sign for $dr < 0$.

QUERY 2. Derivation of $\sin\theta_{\text{rain}}$ (*Optional*)

Carry out the derivation of the expression for $\sin\theta_{\text{rain}}$ in (15).

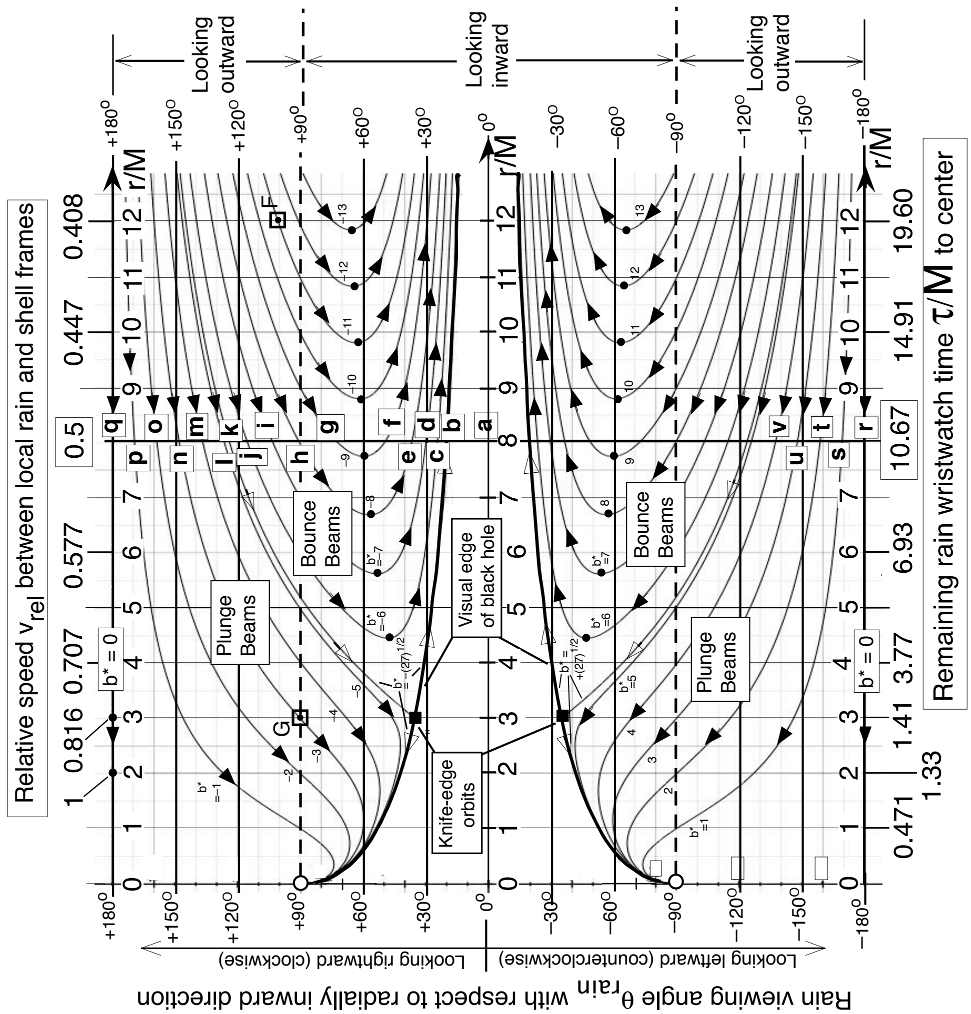

FIGURE 3 Angle $\theta_{\rm rain}$ at which the local rain observer, passing the global location $(r, \phi = 0)$, looks to see starlight beams for several values of b/M, equation (13). To save space, we set $b^* \equiv b/M$. Arrows on the curve for each beam tell whether that beam is incoming or outgoing. A little black dot marks a turning point, the r-coordinate at which an incoming beam reverses dr to become an outgoing beam. Upper and lower three-branch curves with open arrowheads represent light with impact parameter $\pm b_{\rm critical}/M = \pm(27)^{1/2}$. Two little black squares at $r = 3M$ represent circular knife-edge orbits of these critical beams on the tangential light sphere. As she descends, the rain observer sees the visual edge of the black hole at angles shown by the heavy curve. Reminder Figure 4 recalls the meaning of labels "Plunge Beams" and "Bounce Beams" in this figure. The text uses the lowercase labels **a** through **v** to explain details of this figure.

Plot rain observer angle to see light with b

Figure 3 plots equation (13) for several values of $b^* \equiv b/M$; it tells us in which directions $\theta_{\rm rain}$ the local rain observer looks to see starlight beams with various impact parameters b. You can think of the vertical axis as unrolling the polar angle of the local rain frame.

Comment 2. Mirror image of upper and lower parts of Figure 3
Equation (13) is a function of b^2, so cannot distinguish between positive and negative values of b, which leads to the mirror symmetry of Figure 3 above an below the horizontal $\theta_{\text{rain}} = 0$ axis.

Numbers along the top of Figure 3 show shell speeds of the rain frame at various r-coordinates. You can check these numbers with equation (23) in Section 6.4:

$$v_{\text{rel}} = \left(\frac{2M}{r}\right)^{1/2} \qquad \text{(shell speed of rain diver, } r \geq 2M) \tag{16}$$

Numbers along the bottom of Figure 3 tell the remaining wristwatch time τ the rain observer has before arriving at the singularity, from equation (2) in Section 7.2:

$$\tau[r \rightarrow 0] = \frac{2^{1/2}M}{3}\left(\frac{r}{M}\right)^{3/2} \qquad \text{(rain diver } \tau \text{ from } r \text{ to center)} \tag{17}$$

Figure 4 reminds us of the meaning of labels in Figure 3.

Figure 3 carries an immense amount of information. We list here a few examples:

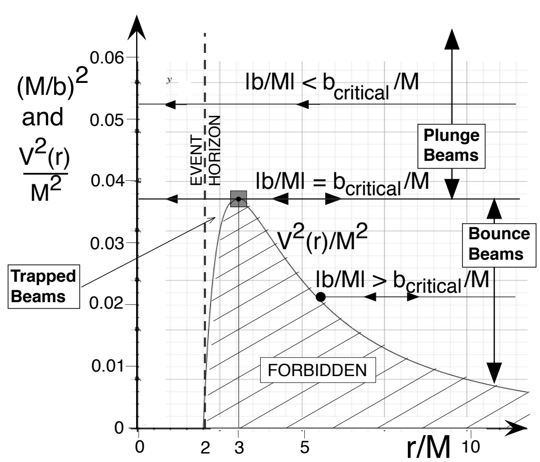

FIGURE 4 Reminder figure of the meaning of labels in Figure 3 for starlight beams—from Figure 3 in Section 11.3. There are no trapped starlight beams.

WHAT THE RAIN VIEWER SEES AS SHE PASSES $r = 8M$

1. **Visual Edge of the Black Hole**
   Two heavy curves straddle the r-axis in Figure 3; they form the outline of a trumpet. As the falling rain observer passes r, she sees ahead of her a black circle whose edges lie between angles $\pm\theta_{\rm rain}$ given by these two heavy curves. As she descends further, the image of the black hole grows until, at $r = 0$, the black hole covers the entire forward hemisphere from $\theta_{\rm rain} = -90°$ to $\theta_{\rm rain} = +90°$.

Does a starlight beam escape?

2. **Dive or Escape?**
   Figure 3 shows that the starlight beam with $|b| < b_{\rm critical}$ (plunge beam) is incoming, with $dr < 0$ along its entire length. In contrast, the starlight beam with $|b| > b_{\rm critical}$ (bounce beam) is initially incoming, then reaches a turning point after which it becomes outgoing and escapes. Every starlight beam that escapes has a turning point, marked with the black dot in Figure 4.

Where does $r = 8M$ shell observer look to see beams with different values of b?

3. **Direction in which the Rain Viewer Looks to See a Star**
   Here is a detailed account of what the local rain observer sees as she falls past $r = 8M$: In Figure 3, look at points labeled with lower-case letters **a** through **v** on the vertical line at $r = 8M$. At point **a** on the r-axis, the rain observer looks radially inward at the center of the black hole, where she sees no starlight. When she looks somewhat to the right (point **b**), she sees the visual edge of the black hole at $\theta_{\rm rain} \approx +23°$. This image is brought to her by the outgoing beam with $b = -b_{\rm critical}$. Farther to her right, at points **c, d, e,** and **f** she sees outgoing beams with $b/M = -6, -7, -8$ and -9, respectively. She sees beams **a** through **g** by looking *inward*—that is, at angles $0 \leq \theta_{\rm rain} < 90°$, as labeled on the right side of the figure. At point **h** she sees the beam with $b/M = -8$ at $\theta_{\rm rain} = 90°$.

   Comment 3. Look inward to see a beam coming from behind?
   At point **g** in Figure 3, the rain viewer sees *incoming beam* $b/M = -9$ ahead of her at approximate angle $70°$. *Question:* How can she look *inward* to see a beam that the plot shows is coming from behind her? *Answer:* Aberration (Section 12.9). In Query 6 you explain this paradox for $r = 8M$.

   To see beams **i** through **q**, the local rain observer looks *outward*, at rain frame angles $90° < \theta_{\rm rain} \leq 180°$, in which directions she sees incoming beams with values of b from about $b/M = -9$ to $b/M = 0$.

   Point **q** at the top of the diagram, for which $\theta_{\rm rain} = +180°$, represents the radially outward direction, and is the same as point **r** at the bottom of the diagram, for which $\theta_{\rm rain} = -180°$. Points **s** through **v** represent directions in which the rain observer looks to the left of the radially inward direction to see beams with $b/M = +1$ through $b/M = +4$ as she turns her gaze back toward the center of the black hole back at point **a**.

Important: The rain observer sees all of these beams *simultaneously* in her local frame as she falls inward past $r = 8M$.

4. **Three-Dimensional Panorama**
   Where does the local rain observer look to see beams from a star that does not lie in the plane of Figure 3? We know the answer to this question: Rotate Figure 3 around the central r-axis until the candidate star lies on the resulting global symmetry plane that contains the star, the observer's eye, and $r = 0$. You can use this result to construct the three-dimensional rain observer's view of every star in the heavens as she passes every r-coordinate.

3-dimensional panorama

Figure 3 also reveals something complex but fascinating: Look at incoming Plunge Beams at the top and bottom of Figure 3. As long as she remains outside $r = 3M$, the rain observer sees Plunge Beams move steadily to smaller visual angles θ_{rain}, even while the visual edges of the black hole are moving steadily to larger visual angles (heavy "trumpet" lines). The rain observer sees *only* Plunge Beams after she passes inward through $r = 3M$; after that she watches images of remote stars that emitted these Plunge Beams swing inward to a minimum visual angle, then back out again to final angles $\theta_{\text{rain}} = \pm 90°$. In other words, after the rain observer passes inward through $r = 3M$, she sees all the multiple images of stars in the heavens swing forward to meet the expanding edge of the black hole, then remain at this edge as the black hole continues to grow visually larger.

Star images swing forward, then back to $\theta_{\text{rain}} = \pm 90^o$.

12.4 ■ CONNECT STAR MAP ANGLE TO RAIN VIEWING ANGLE.

The rain observer views the heavens

Can we now predict the sequence of panoramas of stars enjoyed by the rain observer as she descends? Figure 3 is powerful: It tells us where each rain observer looks to see a starlight beam with any given value of the impact parameter b/M. This anchors the receiving end of the beam at the rain observer. Now we need to anchor the sending end of the same beam at the star—that is, to find the map angle ϕ_∞ of the star that emits this beam.

Want relation between ϕ_∞ and θ_{rain}.

To anchor both ends of each beam, we use our graphical relations among values of b/M, θ_{rain}, and ϕ_∞ for an observer located at $(r, \phi = 0)$. Figure 3 shows the rain angle θ_{rain} at which a local rain observer looks to see a beam with given b-value. Figures 8 and 10 in Section 11.8 show the relation between the impact parameter b and the map angle ϕ_∞ to the star. Taken together, the figures in these two chapters (and their generating equations) solve our problem: They provide the two-step procedure to go between ϕ_∞ and θ_{rain}. To begin, we focus on the **primary image**—the image due to the primary beam, the most direct beam from star to observer. But the following procedure is valid for any beam whose b-value connects a star to a rain observer.

Focus on primary beam.

FROM STAR MAP ANGLE ϕ_∞ TO RAIN VIEWING ANGLE $\theta_{\rm rain}$

From ϕ_∞, find $\theta_{\rm rain}$

At what angle $\theta_{\rm rain}$ does a local rain observer located at map coordinates $(r, \phi = 0)$ look to see the primary image of a star at given map angle ϕ_∞? Here is the two-step procedure:

Step A. Figures 8 and 10 in Section 11.8 tell us the b-value of the primary beam that connects the star at map angle ϕ_∞ to this observer's location and whether that beam is incoming or outgoing.

Step B. For that b-value—and knowledge of whether the beam is incoming or outgoing—Figure 3 gives the local viewing angle $\theta_{\rm rain}$ in which the rain observer looks to see his primary image of that star.

Comment 4. Reminder: Two angles, ϕ_∞ and $\theta_{\rm rain}$
We measure the map angle ϕ_∞ to a star—as we measure all map angles—*counterclockwise* with respect to the radially *outward* direction. In contrast—and for our own convenience—we measure the rain observing angle $\theta_{\rm rain}$ *clockwise* from the radially *inward* direction.

From $\theta_{\rm rain}$, find ϕ_∞.

We can also run this process backward, from rain observation angle $\theta_{\rm rain}$ to map angle ϕ_∞. The rain observer sees a star at rain angle $\theta_{\rm rain}$. Find the map angle ϕ_∞ to that star using Step B above, followed by Step A: From the value of $\theta_{\rm rain}$, Figure 3 tells us the b-value of the beam and whether it is incoming or outgoing. From this information, Figures 8 and 10 in Section 11.8 give us the map angle ϕ_∞ of the star from which this beam comes.

Automate rain panorama plots.

Comment 5. Automate rain panorama plots.
To plot panoramas of the rain viewer, we read numbers from curves in figures, which yield only approximate values. Nothing stops us from converting the data in these figures (and the equations from which they come) into look-up tables or mathematical functions directly used by a computer. Then from the map angle ϕ_∞ to every star in the heavens, the computer automatically projects onto the inside of the personal planetarium (Figure 1) the visual panorama seen by the rain observer.

The following Queries and Sample Problems provide examples and practice connecting star map angle ϕ_∞ with the angle $\theta_{\rm rain}$ in which a local rain observer looks to see that star.

QUERY 3. Sequential changes in rain viewing angles of different stars
A rain viewer first looks at a given star when she is far from the black hole; later she looks at the same star as she hurtles in turn past each of the r-values $r/M = 12$, 5, 2, 1, and just before she reaches the singularity. Do the following Items twice: once using plots in the figures, and second *optional* using equations (13) and (14).

Find the viewing angle $\theta_{\rm rain}$ at each r-coordinate for the star at each of the following map angles. At each r, find the value b/M of the beam that the rain observer sees from that star.

Sample Problems 1. What can the local rain observer at $r = 6M$ see?

In the following cases the local rain observer is passing one of our standard locations, $(r, \phi = 0)$.

A. What is the range of b-values of starlight beams that the rain observer at $r = 6M$ can see? When this observer looks inward, $0 \le |\theta_{\rm rain}| < 90°$, what is the range of b-values of beams that she can see? **SOLUTION:** In Figure 3, look at the vertical line at $r = 6M$. Beams with b-values in the range $0 \le |b/M| \le\approx 7.3$, either incoming or outgoing, cross that vertical line. These are the beams that she can see. Beams that the rain observer can see looking inward, $-90° < \theta_{\rm rain} < +90°$, have b-values in the range $b_{\rm critical}/M \le |b/M| \le\approx 7.3$.

B. In Part A, the largest value of $|b|$ for light seen by a rain observer at r occurs for a beam whose turning point is at that r-coordinate. What is that maximum value of $|b|$ for $r = 6M$? **SOLUTION:** Use equation (36) in Section 11.4. For $r_{\rm tp} = 6$, the answer is $b/M = \pm 7.348$, the correct value compared with the approximate value we read off the plot in Figure 3.

C. At what rain angle $\theta_{\rm rain}$ will the rain viewer passing $r = 6M$ look to see a star at map angle $\phi_\infty = 180°$, exactly on the opposite side of the black hole from her? **SOLUTION:** Begin with Figure 10, Section 11.7. Look at the intersection of the vertical line at $r = 6M$ and top and bottom horizontal lines at $\phi_\infty = \pm 180°$. The b values of these beams is $b/M \approx \pm 6.6$; the figure tells us that these are outgoing beams. The plus or minus refers to beams that come around opposite sides of the black hole. Now return to Figure 3 and find the intersection of vertical line $r = 6M$ with outgoing beams whose impact parameters are $b/M = \pm 6.6$. These intersections correspond to $\theta_{\rm rain} \approx \pm 110°$. These angles are greater than $\pm 90°$, so the rain observer looks somewhat benind her to see the star on the opposite side of the black hole.

D. The local rain observer passing $r = 6M$ sees a star at angle $\theta_{\rm rain} = +74°$. What is the map angle ϕ_∞ to that star? Is this beam incoming or outgoing? **SOLUTION:** In Figure 3 the point $(r = 6M, \theta_{\rm rain} = +74°)$ lies on the curve for the incoming beam with $b/M = -7$. Now go to Figure 10, Section 11.7 to find the intersection of $r = 6M$ with the incoming beam with $b/M = -7$. This occurs at the map angle $\phi_\infty \approx +90°$.

E. Can the rain observer at $r = 6M$ see a star that lies outside of the plane of Figure 3, for example? **SOLUTION:** Sure: just rotate every relevant figure around its horizontal axis until the desired star lies in the resulting plane, then carry out the analysis as before.

A. The star at $\phi_\infty = 30°$
B. The star at $\phi_\infty = 90°$
C. The star at $\phi_\infty = 150°$

QUERY 4. Given ϕ_∞, find $\theta_{\rm rain}$.
Each of the following items lists the map angle ϕ_∞ to a star and the r-coordinate of a rain observer at map coordinates $(r, \phi = 0)$ who looks at that star. In each case find the rain angle $\theta_{\rm rain}$ (with respect to the radially inward direction) at which the local rain observer looks to see that star.

A. $\phi_\infty = +30°$, $r = 6M$
B. $\phi_\infty = -120°$, $r = 10M$
C. $\phi_\infty = +90°$, $r = 2.5M$
D. $\phi_\infty = -180°$, $r = 12M$
E. The rain observer is at the turning point of the beam that has $b/M = -7$. In what rain direction $\theta_{\rm rain}$ does she look to see that star? At what map angle ϕ_∞ is the star that she sees?

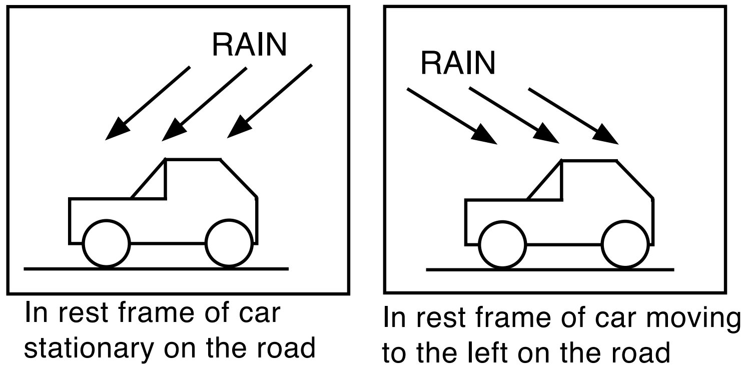

FIGURE 5 Aberration of rain as an analogy of the aberration of light. In the left panel no rain falls directly on the windshield; in the right panel the driver sees the rain coming from a forward direction.

QUERY 5. Given θ_{rain}, find ϕ_∞.
Each of the following items lists the r-coordinate of a rain observer at map coordinates $(r, \phi = 0)$ and the rain angle θ_{rain} at which she looks to see a given star. In each case find the map angle ϕ_∞ of that star.

A. $r = 4M,\ \theta_{\text{rain}} = -115°$
B. $r = 10M,\ \theta_{\text{rain}} = +80°$
C. $r = 2.5M,\ \theta_{\text{rain}} = -145°$
D. $r = 6M,\ \theta_{\text{rain}} = +155°$
E. The rain observer sees the visual edge of the black hole at $\theta_{\text{rain}} = +70°$. What is the map angle of the star that he sees at this visual edge?

12.5 ■ ABERRATION

Rain on the windshield

Aberration: spectacular consequences

Aberration is the difference in direction in which light moves as observed in overlapping inertial frames in relative motion. Aberration has spectacular consequences for what the local rain observer sees as she approaches and crosses the black hole's event horizon. Figure 5 shows an analogy: Rain that falls on a stationary and on a fast-moving car comes from different directions as viewed by a rider in the car. Light moves differently than rain, but the general idea is the same.

We deal here with different viewing directions in overlapping local inertial frames, so special relativity suffices for this analysis. Exercise 18 in Chapter 1 derived expressions for aberration between laboratory and rocket frames in special relativity. We need to modify these equations in four ways:

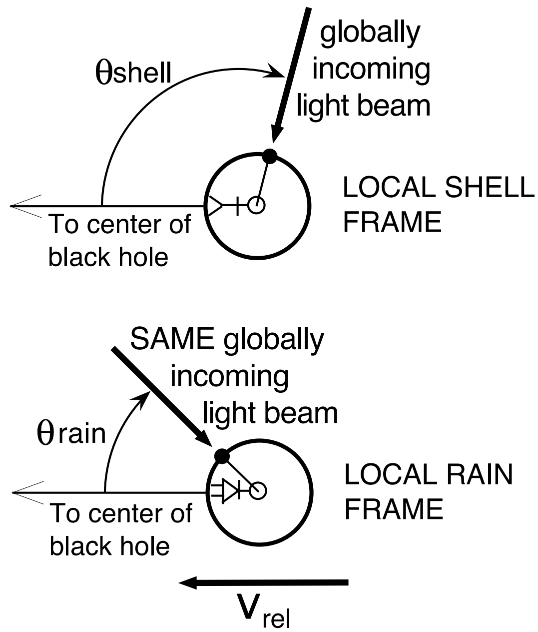

FIGURE 6 Example of light aberration for shell and local rain observers from equation (18). The shell observer at $r = 3M$ looks at the angle $\theta_{\text{shell}} = 105^\circ$ to see the beam from a star. The rain observer who passes this shell sees the beam at $\theta_{\text{rain}} = 45^\circ$.

Modify special relativity aberration equation for local rain and shell frames.

1. Choose the shell frame (outside the event horizon) to be the laboratory frame and the local rain frame to be the rocket frame.
2. The direction of relative motion is along the common Δy_{frame} line instead of along the common Δx_{frame} line in Chapter 1.
3. The local rain frame moves in the negative y_{shell} direction, so v_{rel} in the aberration equations must be replaced by $-v_{\text{rel}}$. Equation (16) gives $v_{\text{rel}} = (2M/r)^{1/2}$.
4. The original special relativity aberration equations describe the direction (angle ψ) in which light *moves*. In contrast, our angles θ_{shell} and θ_{rain} refer to the direction in which the observer *looks* to see the beam, which is the opposite direction. Because of this, $\cos\psi$ in Chapter 1 becomes $-\cos\theta$ in the present chapter.

Modify Chapter 1 aberration equations.

When all these changes are made, the aberration equation (54) in exercise 18 of Chapter 1 becomes:

$$\cos\theta_{\text{shell}} = \frac{\cos\theta_{\text{rain}} + \left(\dfrac{2M}{r}\right)^{1/2}}{1 + \left(\dfrac{2M}{r}\right)^{1/2}\cos\theta_{\text{rain}}} \qquad \text{(light)} \qquad (18)$$

QUERY 6. Resolve the paradox in Comment 3 First of all, note that the question posed in Comment 3 in Section 12.3 is bogus. The beam "comes from behind" only in global coordinates; but we must not trust global coordinates to tell us about measurements or observations. Instead, you can use our equations to show that the descending local rain observer at $r = 8M$ sees the incoming beam with $b/M = -9$ at the inward angle $\theta_{\text{rain}} \approx 72°$, as follows:

A. Substitute the data from point **g** into equation (14) to show that $F(b, r) = F(-9M, 8M) = 0.225$.

B. Plug the results of Item A plus $r = 8M$ and $b/M = -9$ into equation (13) to show that $\cos\theta_{\text{rain}} = 0.310$.

C. From Item B, show that $\theta_{\text{rain}} = 72°$. Does this result match the vertical location of point **g** in Figure 3?

QUERY 7. Rain view at the event horizon.

A. When the rain observer passes through $r = 2M$, at what angle θ_{rain} does she see the edge of the black hole?

B. Can the rain observer use her panorama of stars to detect the moment when she crosses the event horizon?

C. Does your answer to Item B violate our iron rule that a diver cannot detect when she crosses the horizon?

12.6 ■ RAIN FRAME ENERGY OF STARLIGHT

Falling through gravitationally blue-shifted light.

Rain observer in danger from starlight?

Section 7.9 predicted that as she approaches the singularity, tidal forces will end the experience of the rain diver during a fraction of a second, as measured on her wristwatch—independent of black hole mass. But starlight increases its frequency, and hence its locally-measured energy, as it falls toward the black hole. Will the rain observer receive a lethal dose of high-energy starlight before she reaches the singularity? To engage this question, we analyze the energy of light E_{rain} measured in the local rain frame compared to its map energy E. Equation (1) gives the map energy E of a stone in global rain coordinates. The special relativity expression for rain frame energy with the substitution $\Delta t_{\text{rain}} \equiv \Delta T$ from (4) yields

$$\frac{E_{\text{rain}}}{m} = \lim_{\Delta\tau\to 0}\left(\frac{\Delta t_{\text{rain}}}{\Delta\tau}\right) = \lim_{\Delta\tau\to 0}\left(\frac{\Delta T}{\Delta\tau}\right) = \frac{dT}{d\tau} \qquad \text{(stone)} \qquad (19)$$

Modify this expression to describe light. Substitute $dT/d\tau$ from (19) into (1) and solve for $E_{\rm rain}$:

$$E_{\rm rain} = \left(1 - \frac{2M}{r}\right)^{-1} E \left[1 + \left(\frac{2M}{r}\right)^{1/2} \frac{m}{E}\frac{dr}{d\tau}\right] \qquad \text{(stone)} \qquad (20)$$

Recall equation (15) in Section 8.3:

$$\frac{dr}{d\tau} = \pm\left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2r^2}\right)\right]^{1/2} \qquad \text{(stone)} \qquad (21)$$

Use equation (21) to replace the expression $(m/E)(dr/d\tau)$ just inside the right-hand square bracket in (20):

$$\frac{m}{E}\frac{dr}{d\tau} = \pm\left[1 - \left(1 - \frac{2M}{r}\right)\left(\frac{m^2}{E^2} + \frac{L^2}{E^2r^2}\right)\right]^{1/2} \qquad \text{(stone)} \qquad (22)$$

This equation is for a stone. Turn it into an equation for light by going to the limit of small mass and high speed: $m \to 0$ and $L/E \to b$. Plug the result into (20) and divide through by E, which then becomes:

$$\frac{E_{\rm rain}}{E} = \frac{1 \pm \left(\dfrac{2M}{r}\right)^{1/2}\left[1 - \dfrac{b^2}{r^2}\left(1 - \dfrac{2M}{r}\right)\right]^{1/2}}{1 - \dfrac{2M}{r}} \qquad \text{(light)} \qquad (23)$$

Use (14) to write this as:

$$\frac{E_{\rm rain}}{E} = \frac{1 \pm \left(\dfrac{2M}{r}\right)^{1/2} F(b,r)}{1 - \dfrac{2M}{r}} \qquad \text{(light)} \qquad (24)$$

Numerator: minus sign for incoming starlight, plus sign for outgoing light

where equation (14) defines $F(b,r)$. In (23)—and therefore in (24)—the impact parameter b is squared; therefore the beam has the same map energy whether it moves clockwise or counterclockwise around the black hole, as we expect.

In the exercises you derive an expression $E_{\rm rain}/E$ for the stone.

Rain-frame energies of beams

Figure 7 plots results of equation (24). We expect the rain frame energy of an *incoming* beam to depend on two competing effects: the gravitational blue shift (increase in local frame energy) of the falling light, reduced for the diving observer by her inward motion. The result is the Doppler downshift in energy of the light viewed by this local rain observer—compared to the same light

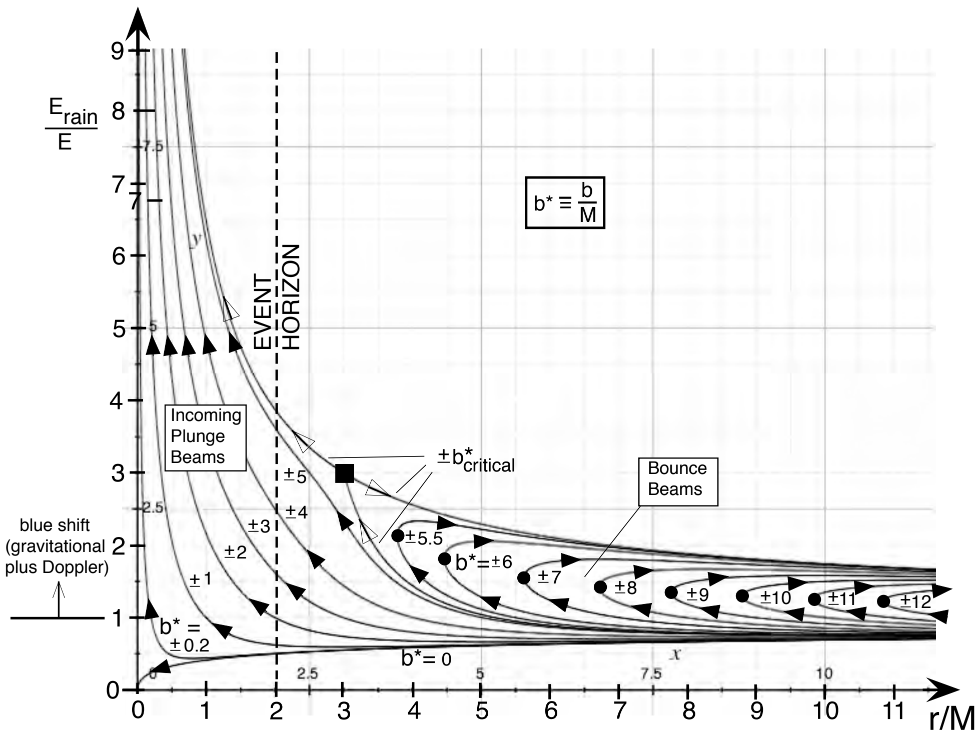

FIGURE 7 Ratio E_{rain}/E of starlight measured by the local rain observer, from equation (24). The curve rising out of each turning point describes an outgoing beam. Beams with $0 \leq |b/M| \leq 5$ are incoming plunge beams.

beam viewed by the local shell observer. In contrast, starlight that has passed its turning point and heads outward again in global coordinates moves opposite to the incoming rain observer, so she will measure its energy to be Doppler up-shifted. Figure 7 shows that these two effects yield a net blue shift for some beams and parts of other beams, and a net red shift for still other beams.

QUERY 8. Rain energy of light at large r.

What happens to the value of E_{rain} as $r \to \infty$? Show that

$$\lim_{r\to\infty} E_{\mathrm{rain}} = E \qquad \text{(light)} \tag{25}$$

QUERY 9. Rain energies of radially-moving starlight.

Find an expression for the rain energy of light with $b = 0$ that moves radially inward (for any value of r) or outward (for $r \geqslant 2M$). Show that in this case equation (24) becomes

$$\frac{E_{\text{rain}}}{E} = \frac{1 \pm \left(\dfrac{2M}{r}\right)^{1/2}}{1 - \dfrac{2M}{r}} = \frac{1 \pm \left(\dfrac{2M}{r}\right)^{1/2}}{\left[1 + \left(\dfrac{2M}{r}\right)^{1/2}\right]\left[1 - \left(\dfrac{2M}{r}\right)^{1/2}\right]} = \frac{1}{1 \mp \left(\dfrac{2M}{r}\right)^{1/2}} \qquad (\text{light, } b = 0) \quad (26)$$

where the upper sign is for outgoingbeams. But starlight with $b = 0$ cannot be outgoing, so:

$$\frac{E_{\text{rain}}}{E} = \frac{1}{1 + \left(\dfrac{2M}{r}\right)^{1/2}} \qquad (\text{starlight, } b = 0) \qquad (27)$$

This is the curve displayed at the bottom of Figure 7.

A. Show that E_{rain}/E has the value 1/2 at $r = 2M$, and that this result is consistent with the $b = 0$ curve in Figure 7.

B. A shell observer remote from a black hole shines radially inward a laser of map energy E_{laser}, measured in his local frame, which is also global E in flat spacetime. A local rain observer moving inward along the same radial line looks radially outward at this laser beam as she descends. Write a short account about the ratio $E_{\text{rain}}/E_{\text{laser}}$ of this laser light that she measures outside the event horizon, when she is at the event horizon, and as she approaches the singularity. If she is given the value of E_{laser}, can she detect when she crosses the event horizon? Could you design an "event horizon alarm" for our black hole explorations? Does your design violate our iron rule that a diver cannot detect when she crosses the event horizon?

QUERY 10. Details of Figure 7

Without equations, provide qualitative explanations of rain frame beam energies in Figure 7.

A. Show that E_{rain} approaches the value E at large r, as demonstrated in (25).

B. Why do beam energies not depend on the sign of b?

C. Why do the outgoing Bounce Beams at any given r have greater rain frame energy than incoming Bounce Beams?

D. For the Plunge Beams, $0 \leq |b| \leq b_{\text{critical}}$ at any given r, why do beams with larger values of $|b|$ have greater rain frame energy than beams with smaller values of $|b|$?

QUERY 11. *Optional:* Trouble at the event horizon?

The denominator $1 - 2M/r$ in (24) goes to zero at the event horizon. Does this mean that at $r = 2M$ starlight has infinite rain frame energy for every value of b? To answer, use our standard approximation (inside the front cover); set $r = 2M(1 + \epsilon)$, where $0 < \epsilon \ll 1$ and verify that E_{rain}/E is finite at the event horizon, provided that the beam is not an outgoing Plunge Beam. Show that your approximation at $r/M = 2$ correctly predicts values of E_{rain}/E for two or three of the curves in Figure 7.

QUERY 12. Killer starlight?

What is the energy of starlight measured by the local rain observer as she approaches the singularity? Let $r/M = \epsilon$, where $0 < \epsilon \ll 1$ and show that for starlight (incoming Plunge Beams: minus sign in (24)):

$$\lim_{r\to 0} \frac{E_{\text{rain}}}{E} = \frac{|b|}{r} \qquad \text{(incoming Plunge Beams)} \tag{28}$$

Lethal starlight is bad, but tides are worse.

In Query 12 you show that close to the singularity the energy of starlight measured by the plunging local rain observer increases as the *inverse first power* of the decreasing r. The other mortal danger to the rain observer comes from tidal accelerations. Section 7.9 showed that the rain observer "ouch time" from first discomfort to arrival at the singularity is two-ninths of a second, independent of the mass of the black hole. Equations (38) through (40) in Section 9.7 tell us that tidal accelerations increase as the *inverse third power* of the decreasing r-coordinate, which is proportionally faster than the inverse first power increase in the rain frame energy of incoming starlight.

Which will finally be lethal for the rain observer: killer starlight or killer tides? Inverse third power tidal acceleration appears to be the winning candidate. Analyzing tidal acceleration is straightforward: its effects are simply mechanical. In contrast, we have trouble predicting results for light: they depend not only on the rain frame energy of the light but also on its intensity and the rain observer's wristwatch exposure time. This book says nothing about the focusing properties of curved spacetime near the black hole—an advanced topic—so we lack the tools to predict the (short-term!) consequences of the rain observer's accumulated exposure to starlight as she descends.

Assume tides are lethal

We have a lot of experience protecting humans against radiation of different wavelengths. Perhaps a specially-designed personal planetarium (Section 12.2) will allow the rain observer to survive killer starlight all the way down to her tidal limit. In contrast, we know nothing that can shield us from tidal effects. In the description of the final fall in Section 12.7 we assume that it is killer tides that prove lethal for the rain diver.

12.7 ■ THE FINAL FALL

Free-fall to the center

We celebrate with the final parade of an all-star cast. Let's follow general relativists Richard Matzner, Tony Rothman, and Bill Unruh (see the references) looking at the starry heavens as we free-fall straight down into a

Time of a movie inside the event horizon

non-spinning black hole so massive, so large that even after crossing the event horizon we have nearly two hours of existence ahead of us—roughly the length of a movie—to behold the whole marvelous ever-changing spectacle. Almost everything we have learned about relativity—both special and general—contributes to our appreciation of this mighty sequence of panoramas.

Panoramas Seen by the Rain Frame Observer

—Adapted from Matzner, Rothman, and Unruh. Some numerical values calculated by Luc Longtin.

Fall into a one-billion-solar-mass black hole.

Imagine a free-fall journey into a billion-solar-mass black hole ($M = 10^9 M_{\text{Sun}} = 1.477 \times 10^9$ kilometers $= 1.6 \times 10^{-4}$ light-years—about one-third of the estimated mass of the black hole at the center of galaxy M87). The map r-coordinate of the event horizon—double the above figure—is about the size of our solar system. We adjust our launch velocity to match the velocity which a rain frame, falling from far away, would have at our shell launch point at $r = 5\,000M$. Our resulting inward shell launch velocity, $v = -(2M/r)^{1/2}$ with respect to a local shell observer, is equal to two percent of the speed of light. We record each stage in the journey by giving both the time-to-crunch on our wristwatch and our current map r.

26 years to the end

The beginning of the journey, 26 years before the end. At this point the black hole is rather unimpressive. There is a small region (about 1 degree across—i.e., twice the size of the Moon seen from Earth) in which the star pattern looks slightly distorted and within it (covering about one-tenth of a degree) a disk of total blackout. Careful examination shows that a few stars nearest the rim of the blacked-out region have second images on the opposite side of the rim. Had these images not been pointed out to us, we probably would have missed the black hole entirely.

300 days

Three hundred days before the end, at $r = 500M$. Some noticeable change has occurred. The dark circular portion of the sky has now grown to one full degree in width.

One week

One week before the end, at $r = 41M$. The image has grown immensely. There is now a pure dark patch ahead with a diameter of about 22 degrees (approximately the size of a dinner plate held at arm's length). The original star images that lay near the direction of the black hole have been pushed away from their original positions by about 15 degrees. Further, between the dark patch itself and these images lies a band of second images of each of these stars. Looking at the edge of this darkness with the aid of a telescope, we can even see faint second images of stars that lie behind us! This light has looped around the black hole on its way to our eye (Figure 9, Section 11.8). From this point on, Doppler shift and gravitational blue shift radically change the observed frequencies of light that originate from different stars.

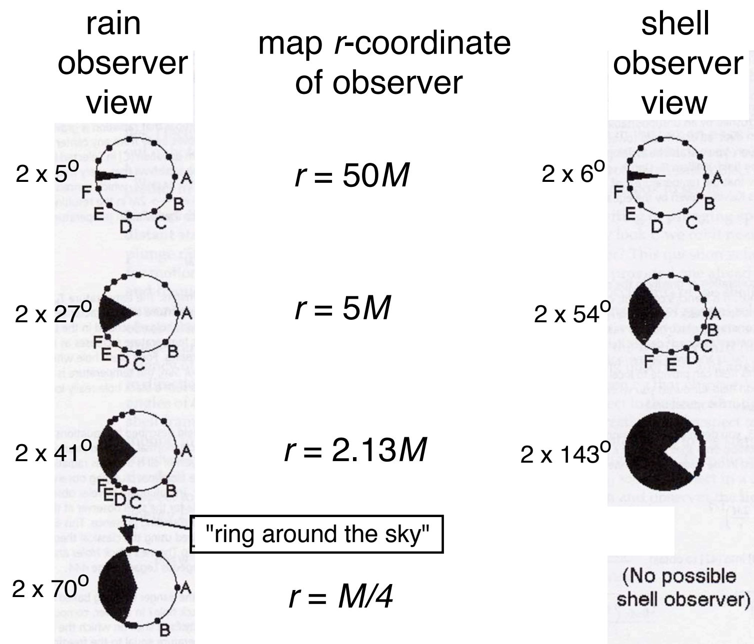

FIGURE 8 Pie charts showing rain viewing directions of stars and the visual edge of the black hole seen in sequence by a rain frame viewer (left-hand column) and by a set of stationary shell observers at different radii (right-hand column). Dots labeled A through F represent directions of stars plotted by rain and shell observers on their personal planetariums. In the final instants of her journey (at smaller radii than shown here), the sky behind the rain observer is black, nearly empty of stars, and the black hole covers the sky ahead of her. Cleaving the forward half of the firmament from the backward half is a bright ring around the sky. This figure does not show multiple images of stars due to one or several orbits of starlight around the black hole. (Figure based on the work of M. Sikora, courtesy of M. Abramowicz.)

12 hours

Twelve hours before the end, at $r = 7M$. A sizeable portion of the sky ahead of us is now black; the diameter of the black hole image covers a 44-degree angle, over 10 percent of the entire visual sphere.

3.3 hours

3.3 hours before the end. As we pass inward through $r = 3M$, we see all the stars in the heavens swing forward to meet the expanding edge of the black hole, then remain at this edge as the black hole continues to grow visually larger.

2 hours

Two hours before the end. We are now at $r = 2.13M$, just outside the event horizon and our speed is 97 percent that of light as measured in the local shell frame that we are passing. Changes in viewing angle (aberrations) are now extremely important. Anything we see after an instant from now will be a secret taken to our grave, because we will no longer be able to send any information out to our surviving colleagues. Although we will be “inside” the black hole, not all of the sky in front of us appears entirely dark. Our high speed causes light beams to arrive at our eyes at extreme forward angles. Even so, a disk subtending a total angle of 82 degrees in front of us is fully black—a substantial fraction of the forward sky.

Secondary images

Behind us we see the stars grow dim and spread out; for us their images are not at rest, but continue to move forward in angle to meet the advancing edge of the black hole. This apparent star motion is again a forward-shift due to our increasing speed. But there is a more noticeable feature of the sky: We can now see second images of all the stars in the sky surrounding the black hole. These images are squeezed into a band about 5 degrees wide around the image of the black hole. These second images are now brighter than were the original stars. Surrounding the ring of second images are the still brighter primary images of stars that lie ahead of us, behind the black hole. The band of light caused by both the primary and secondary images now shines with a brightness ten times that of Earth's normal night sky.

2 minutes

Approximately two minutes before oblivion: $r = M/7$. The black hole now subtends a total angle of 150 degrees from the forward direction—almost the entire forward sky. Behind us star images are getting farther apart and rushing forward in angle. Only 20 percent of star images are left in the sky behind us. In a 10-degree-wide band surrounding the outer edges of the black hole, not only second but also third and some fourth images of the stars are now visible. This band running around the sky now glows 1000 times brighter than the night sky viewed from Earth.

Final seconds

The final seconds. The sky is dark everywhere except in that rapidly thinning band around the black disk. This luminous band—glowing ever brighter—runs completely around the sky perpendicular to our direction of motion. At 3 seconds before oblivion it shines brighter than Earth's Moon. New star images rapidly appear along the inner edge of the shrinking band as higher and higher-order star images become visible from light wrapped many times around the black hole. The stars of the visible Universe seem to brighten and multiply as they compress into a thinner and thinner ring transverse to our direction of motion.

Awesome ring bisects the sky.

Only in the last 2/9 of a second on our wristwatch do tidal forces become strong enough to end our journey and our view of that awesome ring bisecting the sky.

12.8 ■ EXERCISES

1. Impact parameter at a turning point

From equation (29) in Section 11.5, show that $b/M \to \infty$ not only as $r_{\rm tp} \to \infty$ but also as $r_{\rm tp}/M \to 2^+$, where the subscript tp means turning point. Since b/M is finite for values between these two limits, therefore there must be at least one minimum in the b vs. $r_{\rm tp}$ curve. Verify the map location and value of this minimum, shown in Figure 9. Remember that beams for which $r_{\rm tp} < 3M$ cannot represent starlight.

2. Direction of a star seen by the local shell observer.

Exercise 18 in Section 1.13 shows the relation between the directions in which light moves in inertial laboratory and rocket frames. Replace laboratory with local shell frame and rocket with local rain frame. The direction of relative motion is along the local y-axes, and the rain frame moves moves in the negative $\Delta y_{\rm shell}$ direction, so the sign of the relative velocity $v_{\rm rel}$ must be reversed in the special relativity formulas. Then equation (55) in Section 1.13 becomes

$$\cos\psi_{\rm shell} = \frac{\cos\psi_{\rm rain} - v_{\rm rel}}{1 - v_{\rm rel}\cos\psi_{\rm rain}} \qquad (\psi = \text{direction of light motion}) \qquad (29)$$

where $\psi_{\rm shell}$ is the direction the light moves in the shell frame and $\psi_{\rm rain}$ its direction of motion in the rain frame.

A. In the notation of Chapter 1, angles ψ are the directions in which the light *moves*; in the notation of the present chapter angles θ are the angles in which an observer *looks* to see the beam. The cosines of two angles that differ by 180° are the negatives of one another. Show that equation (18) becomes, in the notation of our present chapter:

$$\cos\theta_{\rm shell} = \frac{\cos\theta_{\rm rain} + v_{\rm rel}}{1 + v_{\rm rel}\cos\theta_{\rm rain}} \qquad (\theta = \text{direction viewer looks}) \qquad (30)$$

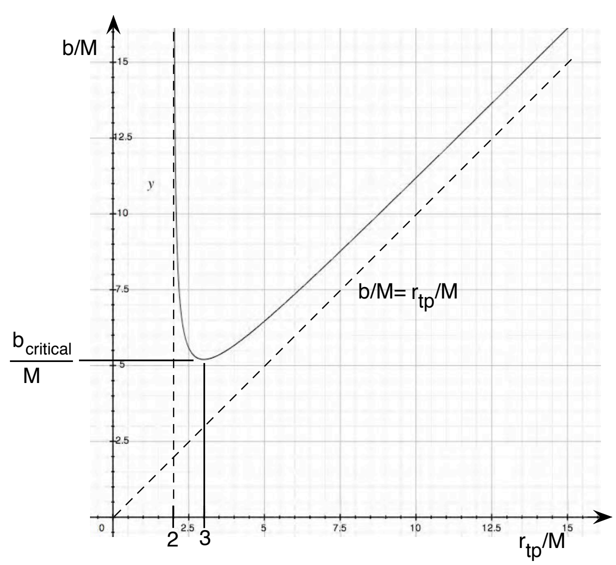

FIGURE 9 Plot of b/M vs. $r_{\rm tp}/M$ from equation (29) in Section 11.5.

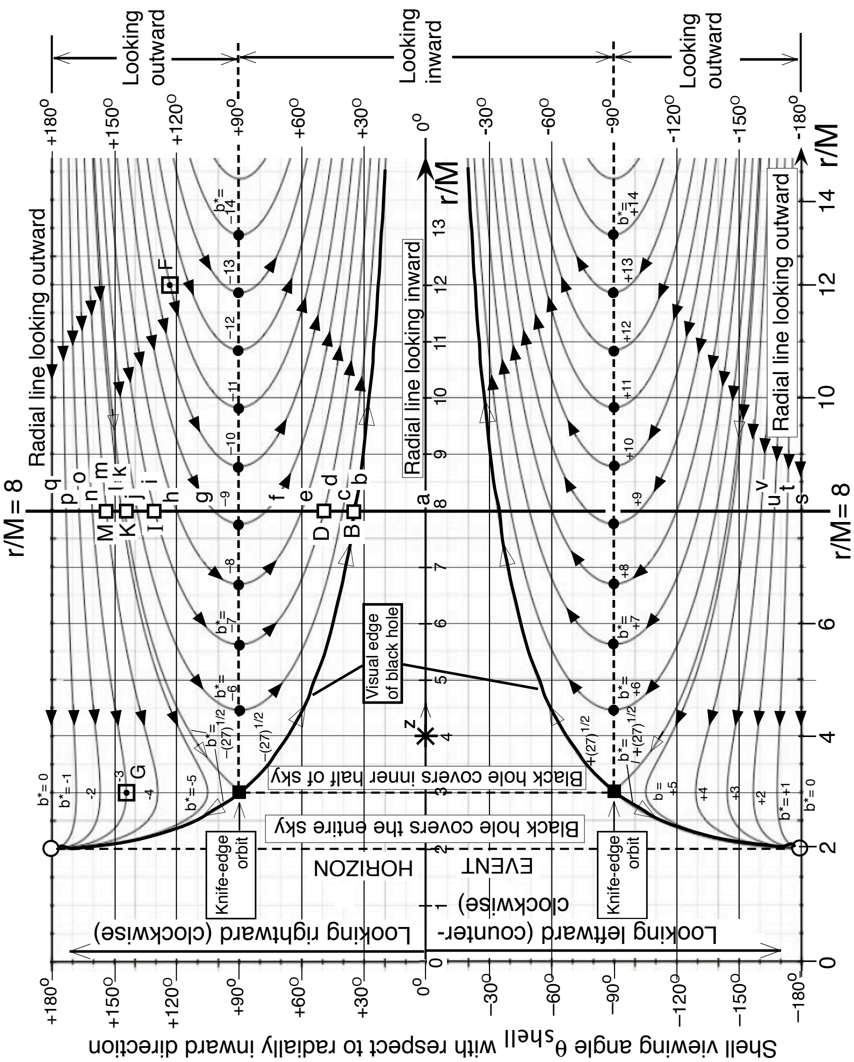

FIGURE 10 Angle $\theta_{\rm shell}$ at which the shell observer located at global coordinates $(r, \phi = 0)$ looks to see starlight beam with values of $b^* \equiv b/M$. Arrows on each curve tell us whether that beam is incoming or outgoing. A black dot marks a turning point, the r-coordinate at which an incoming beam reverses its dr to become an outgoing beam. Upper and lower three-branch curves with open arrowheads represent light with impact parameter $b/M = \pm b_{\rm critical}/M = \pm(27)^{1/2}$. Two little black squares at $r = 3M$ represent circular knife-edge orbits of these critical beams on the tangential light sphere. When viewed by starlight, the shell observer sees the black hole at shell angles inside the heavy curve, which represents its visual edge. The shell observer near the event horizon looks radially outward to see the entire heavens contracted to a narrow cone (Figure 8).

B. Why can't equations (18) and (30) be used inside the event horizon?

C. A shell observer at a given r and a local rain observer who passes through that map location both view the same beam. Items (a) through (c) below give the value of b and r in each case, and whether the beam is incoming or outgoing. For each case, find $\theta_{\rm rain}$ from Figure 3; use (30) with $v_{\rm rel}$ from (16) to convert to shell angle $\theta_{\rm shell}$; then check your result in Figure 10.

(a) Outgoing beam with $b/M = -12$ observed at $r = 12M$.

(b) Incoming beam with $b/M = -7$ observed at $r = 6M$.

(c) Incoming beam with $b/M = -4$ observed at $r = 3M$.

D. Look at the list "What the Rain Viewer Sees as She Passes $r = 8M$" in Section 12.3. Use the lowercase bold letters on the $r = 8M$ vertical line in Figure 10 to write a similar analysis of what the local shell observer at $r = 8M$ sees.

3. Expression $E_{\rm rain}/E$ for a stone.

Section 12.6 derives the expression $E_{\rm rain}/E$ for *light*. Derive the same expression for a *stone*.

4. Direction of a star seen by an orbiting observer

In what direction does the observer in circular orbit look to see the same beam? Special relativity can answer this question, because it requires a simple aberration transformation—similar to (30)—first from $\theta_{\rm rain}$ to $\theta_{\rm shell}$ and then from $\theta_{\rm shell}$ to $\theta_{\rm orbiter}$. Equation (16) gives the relative speed in the radial direction between the rain diver and the shell observer, while equation (31) in Section 8.5 gives the relative speed in the tangential direction between shell and orbiting observers: $v_{\rm rel} = v_{\rm shell} = (r/M - 2)^{-1/2}$. The resulting transformations, although messy, use nothing but algebra and trigonometry. The results are plotted in Figure 11, which is similar to Figure 3.

WE HAVE NOT FINISHED THIS EXERCISE. PLEASE ASK SEVERAL NON-TRIVIAL QUESTIONS THAT CAN BE ANSWERED USING FIGURE 11. THEN ANSWER YOUR OWN QUESTIONS.

12.9 ■ REFERENCES

Initial quote: Emily Dickinson, poem number 143, version A, about 1860, *The Poems of Emily Dickinson, Variorum Edition*, Edited by R. W. Franklin, Cambridge Massachusetts, The Belknap Press of Harvard University, 1998, Volume I, page 183.

Description of final dive (Section 12.7) and Figure 8 are adapted from Richard Matzner, Tony Rothman, and Bill Unruh, "Grand Illusions: Further Conversations on the Edge of Spacetime," in *Frontiers of Modern Physics: New Perspectives on Cosmology, Relativity, Black Holes and Extraterrestrial Intelligence*, edited by Tony Rothman, Dover Publications, Inc., New York,

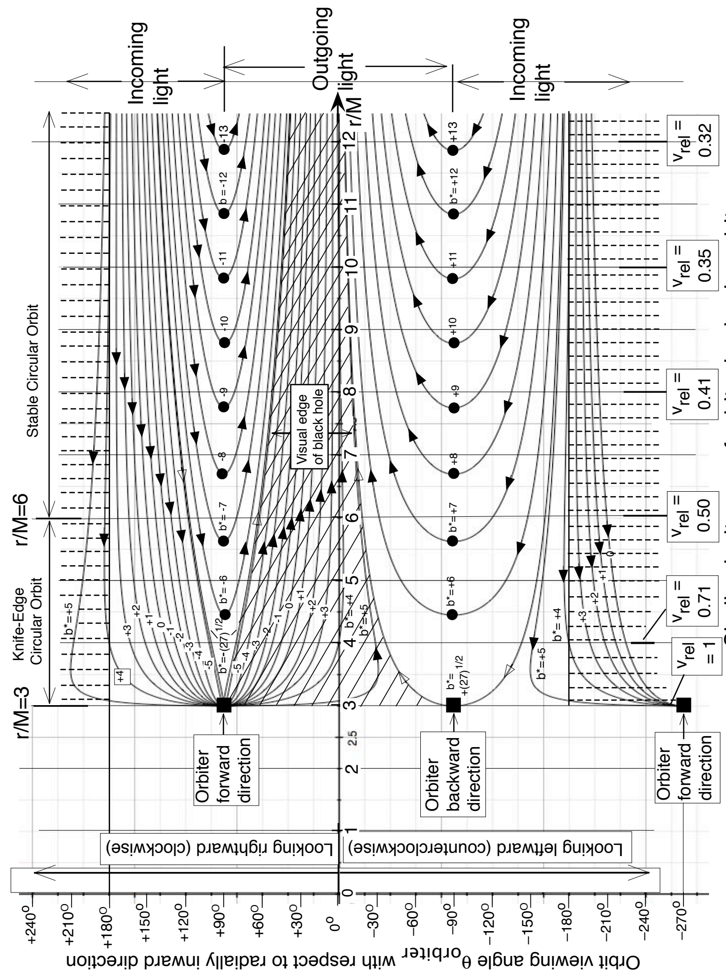

FIGURE 11 Observation angles $\theta_{\rm orbiter}$ at which an orbiter looks to see beams with different impact parameters $b^* = b/M$. The edges of the diagonally shaded region are the orbiter's visual edges of the black hole. Values of $v_{\rm rel}$ along the bottom are the relative velocities of the orbiter with respect to the local shell frame. In the limiting case of the orbit at $r = 3M$ (moving at the speed of light in the shell frame), the inner half of the sky is black for the orbiter. Upper and lower regions shaded by vertical dashed lines include some of the curves between $\theta_{\rm orbiter} = -180°$ and $\theta_{\rm orbiter} = +180°$.

1985, pages 69–73. Luc Longtin provided corrections for "times before oblivion" in in the Section The Final Fall and calculated numbers for Figure 8.