Oscillation of Mass-spring System

The harmonic oscillation is one of the most basic physical phenomena. If the amplitude is small enough, vibrations, oscillations can be modeled with harmonic oscillation. The differential equation of harmonic oscillation frequently appears not only in classical physics (mechanics, electrotechnique) but also in optics, quantum physics, solid state physics.

I. THEORETICAL BACKGROUND

A. Undamped oscillation

If an elastic force acts on an object of mass m, the equation of motion is $ma = -Dx$, where D is the spring constant, x is the displacement of the object measured from the equilibrium position, and a is the acceleration.

The solution of the equation of motion is

$$x = A\sin(\omega_0 t + \alpha), \tag{1}$$

where A is the amplitude, α is an initial phase,

$$\omega_0 = \sqrt{\frac{D}{m}}, \tag{2}$$

angular frequency of the undamped system ($\omega_0 = 2\pi f_0$, where f_0 is the frequency). The speed of the harmonic oscillation is

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = A\omega_0 \cos(\omega_0 t + \alpha), \tag{3}$$

where $A\omega_0$ is the maximal speed, or speed-amplitude.

B. Damped oscillation

Forces causing the damping are usually proportional to the speed, so the equation of motion takes the form of $ma = -Dx - kv$, which can be transformed into

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\delta\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0, \tag{4}$$

by introducing $\delta = k/2m$, the damping factor (k is related to the friction) and using eq. (2).

The solution of the differential equation in case of $\omega_0^2 \geq \delta^2$ results in an oscillation with decreasing amplitude:

$$x = A\mathrm{e}^{-\delta t}\sin(\omega' t + \alpha).$$

The angular frequency of the oscillation is:

$$\omega' = \sqrt{\omega_0^2 - \delta^2}. \tag{5}$$

There are different quantities to describe the reduction of the amplitude. The *damping quotient* is the ratio of two consecutive peaks in the same direction: $K = x_n/x_{n+1} = \exp(\delta T)$, where $T = 2\pi/\omega'$. Another quantity is the logarithm of K, the *logarithmic decrement*:

$$\Lambda = \ln K = \delta T. \tag{6}$$

C. Driven oscillation

Applying a periodically oscillating force on mass m, with a motor and an excenter, after the transient part a stationary oscillation forms of which frequency is equal to the frequency of the external force, while the amplitude depends on the force, the spring constant, the mass, the damping and the exciting frequency. In this case the equation of motion is $ma = -Dx - kv + F_0\sin(\omega t)$. With the above introduced notations this can be transformed into the following differential equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\delta\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{F_0}{m}\sin(\omega t), \tag{7}$$

where F_0 is the maximal value of the exciting force. The solution of the equation is:

$$\begin{aligned} x = {} & A\mathrm{e}^{-\delta t}\sin(\omega' t + \alpha) \\ & + \frac{F_0}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\delta^2\omega^2}}\sin(\omega t + \varphi). \end{aligned} \tag{8}$$

The first term is the previously introduced, decaying oscillation, and the second term is the stationary solution. Here phase φ does not depend on the initiation of the measurement, but it is the phase difference compared to the excitation. The amplitude of the stationary solution has a maximum at

$$\omega_{\max} = \sqrt{\omega_0^2 - 2\delta^2} \tag{9}$$

frequency, and the phase is:

$$\tan\varphi = \frac{2\delta\omega}{\omega_0^2 - \omega^2}. \tag{10}$$

To describe the energy relations in case of driven oscillation, one can introduce the *quality factor*, which is the ratio of the energy dissipated during one period $\langle W\rangle$ and the average stored energy $\langle P\rangle$:

$$Q = 2\pi\frac{\langle W\rangle}{T\langle P\rangle} = \frac{\omega_0}{2\delta}. \tag{11}$$

Higher Q indicates a lower rate of energy loss relative to the stored energy of the resonator; the oscillations die out more slowly.

II. THE MEASUREMENT SETUP

FIG. 1. Setup of the measurement

FIG. 2. The driving wheel

The measurement setup is shown in FIG. 1 The excenter (driving wheel) can be found on the electronic unit, which is the bottom part of the setup.

The amplitude of the driving force can be changed with the position of the amplitude-rod on the driving wheel by changing the distance of the center of the wheel and the fixing point of the cord (see FIG. 2). The other end of the cord, after going through two pulleys at the top of the setup, is connected to the examined spring. The other end of the spring is fixed to a scaled aluminum rod, with a mass of 50 g.

The mass of the oscillating object can be increased by putting a 50 g brass disc to the top end of the aluminum rod. A guide is found at the middle of the column, which prevents the rod to swing sideways. The rod should be positioned in the rectangle shaped hole of the guide.

The motion of the rod can be recorded with an ultrasonic sensor, which is placed under the rod, on the table. To reduce the noise in the measurement a thin copper disc is glued to the bottom of the aluminum rod, this way the ultrasound is reflected from a higher surface. The measured data is transferred to a computer via USB port, and the Logger Lite 1.4 software visualizes it. The usage of the program can be studied in few minutes.

The pin fixed to the driving axis crosses the beam of an optogate, producing an electric impulse in every turns, which also can be recorded by the computer via an USB port. Comparing this signal to motion of the rod, the phase difference of the driven oscillation φ can be determined.

Properly setting the setup, the damping, caused by the drag, and friction of different components, is rather small. To introduce a tunable damping, a U shaped holder is placed under the guide, which contains two disc shaped magnets (see FIG. 3). The aluminum rod is moving in between these magnets. Due to the magnetic field and the motion, eddy currents are formed in the rod, and their Joule-heat dissipation causes the damping of the oscillation. Reducing the distance between the magnets, the strength of the magnetic field is increasing, and so the dissipation and the damping.

FIG. 3. Magnets for tunable damping

A. Preparing the setup

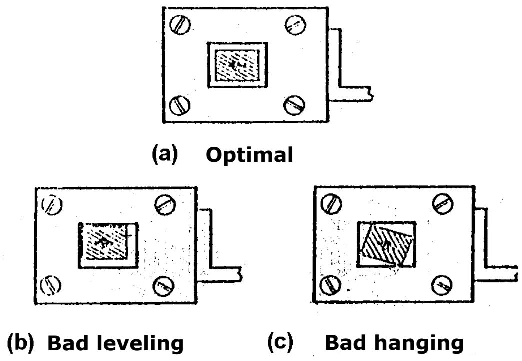

FIG. 4. Different postions of the rod viewed from top

- With optimal setting the aluminum rod does not touch the guide on any side, and the edges of the rod are parallel with the edges of the guide (see FIG. 4a. FIG. 4b, and FIG. 4c show bad settings. In case of FIG. 4b the measurement setup is not horizontal, which can be compensated by adjusting the height of legs of the setup. In case of FIG. 4c the hanging of the rod should be adjusted (by rotating it).
- Changing the length of the cord, the relative position of the aluminum rod and the guide can be set.

III. MEASUREMENT TASKS

1. *Spring constant*

   Set the length of cord in a way that the lowest mark on the rod is at the same height as the top of the guide. Put a copper cylinder of 25 g on the spring and measure its extension. Put a second and then a third weight and measure the extensions again. Provide the spring constant (averaging and linear fitting).

2. *Undamped oscillation*

   Remove the U shaped holder with the magnets. Adjust the setup according to FIG. 4 and section II. A. Set the length of the cord in a way that the guide is at the middle of the scale. Put the ultrasonic sensor below the aluminum rod. Remove the hook from the bottom of the rod, and stick the copper disk to its place.

   Pull the rod 3 cm below its equilibrium and release it. At the same time start the data collection on the computer. Repeat the measurement with 50 g of extra weight. Repeat the measurement 3 times for every setting. Set the length of the measurement so at least 4-5 oscillations could fit in the time window in order to give the time period more accurately. Save the data in the designated folder.

   There are two ways to save the data. In case of *Save As...* the data can be read by Excel, and contains two columns, the time and the distance. In case of *Export As* the data will be saved as a .csv file, which contains all for quantities (time, distance, speed, acceleration), but it requires some work to be treatable in Excel. **Please bring a USB stick to take the files.**

   Determine the frequency of the oscillation and compare it with that calculated from the results of Task 1 by using eq. (2).

3. *Damped oscillation*

   Restore the U shaped holder with magnets to its place and repeat the previous measurements (2 different weights: rod, and rod+50 g weight) with 2 different damping. For weaker damping the magnets should be further from each other, for stronger damping screw the magnets closer, but they should not touch the rod. Record the distance of the magnets.

   Pull the rod 3 cm below its equilibrium and release it. At the same time start the data collection on the computer. Repeat the measurement 3 times at every setting. Calculate the damping constant δ from the amplitude reduction and determine the oscillation frequency. Compare the frequency with that calculated by eq. (5). Calculate the Q factor of the oscillator.

4. *Driven oscillation*

   Set one of the the variable output of the power supply the following way: turn the Voltage switch to zero (left end) and the Current switch to middle position. Connect the motor to this output. Connect the optogate to the fix 5V output. Be careful with the polarity: **Red:** $+$, **Black:** $-$. Set the variable output to 0.5 V and push gently the excenter to start the rotation. Slowly increasing the voltage find the resonance frequency. If the amplitude is too large at resonance, reduce the excitation force, by changing the position of the excenter. Use the damping from the previous measurement. Measure at 7 different excitation frequency (below and above of the resonance) and plot the amplitude as a function of the oscillation frequency.