# Chapter 9. Orbiting the Black Hole



- *As I approach a black hole from far away, how can I put my spaceship into a circular orbit?*
- *How can I transfer from one circular orbit to another one?*
- *Why am I uncomfortable in some orbits near a black hole?*
- *Can I enter a circular orbit without firing a rocket?*
- *How do I move a probe from a circular orbit inward across the event horizon?*

CHAPTER

# 9 Orbiting the Black Hole

**Edmund Bertschinger & Edwin F. Taylor** *

*I want to know how God created this world. I am not interested in this or that phenomenon, in the spectrum of this or that element. I want to know his thoughts. The rest are details.*

*******

*What really interests me is whether God could have created the world any differently; in other words, whether the requirement of logical simplicity admits a margin of freedom.*

—Albert Einstein

## 9.1 ■ OBSERVE THE BLACK HOLE FROM A SEQUENCE OF CIRCULAR ORBITS

*The sequence of orbits in our exploration plan*

Observe the black hole from circular orbits.

Chapter 8 introduced circular orbits of a free stone around a black hole. The present chapter describes how the captain of an approaching spaceship can insert it into a circular orbit, then transfer to progressively smaller circular orbits in order to get closer looks at the black hole. Our exploration program includes several maneuvers:

Exploration program

**EXPLORATION PROGRAM FOR THE BLACK HOLE**

Step 1. Insert the approaching spaceship into a stable circular orbit at $r = 20M$.

Step 2. Transfer a probe from this initial orbit to the innermost stable circular orbit at $r_{\mathrm{ISCO}} = 6M$.

Step 3. Transfer the probe from the ISCO to an *unstable* circular orbit at $r = 4M$.

Step 4. Tip the probe off the unstable circular orbit at $r = 4M$ so that it spirals inward across the event horizon.

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

To describe this sequence of orbits, use equations from previous chapters, summarized here in global rain coordinates, $T, r, \phi$. Both the unpowered spaceship and the unpowered probe move in the same way as a free stone.

Free motion

**GENERAL FREE MOTION OF UNPOWERED SPACESHIP OR PROBE**

$$\frac{E}{m} \equiv \left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau} \qquad \text{(free: (35) in Sec. 7.5)} \quad (1)$$

$$\frac{L}{m} \equiv r^2\frac{d\phi}{d\tau} \qquad \text{(free: (10) in Sec. 8.2)} \quad (2)$$

$$\frac{E_{\text{shell}}}{m} = \frac{1}{\left(1 - v_{\text{shell}}^2\right)^{1/2}} = \frac{E/m}{\left(1 - \dfrac{2M}{r}\right)^{1/2}} \qquad \text{(free: (17) in Sec. 6.3)} \quad (3)$$

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(\frac{V_{\text{L}}(r)}{m}\right)^2 \qquad \text{(free: (21) in Sec. 8.4)} \quad (4)$$

$$\left(\frac{V_{\text{L}}(r)}{m}\right)^2 \equiv \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2r^2}\right) \qquad \text{(free: (20) in Sec. 8.4)} \quad (5)$$

Motion in a circular orbit

**CIRCULAR-ORBIT MOTION OF UNPOWERED SPACESHIP OR PROBE ($r > 3M$)**

$$\left(\frac{L}{m}\right)^2 = \frac{Mr^2}{r - 3M} \qquad \text{(circle: (28) in Sec. 8.5)} \quad (6)$$

$$r = \frac{L^2}{2m^2M}\left[1 \pm \left(1 - \frac{12m^2M^2}{L^2}\right)^{1/2}\right] \qquad \text{(circle: (27) in Sec. 8.5)} \quad (7)$$

$$\frac{E}{m} = \frac{r - 2M}{\left[r\left(r - 3M\right)\right]^{1/2}} \qquad \text{(circle: (34) in Sec. 8.5)} \quad (8)$$

$$\frac{E_{\text{shell}}}{m} = \left(\frac{r - 2M}{r - 3M}\right)^{1/2} \qquad \text{(circle: (35) in Sec. 8.5)} \quad (9)$$

$$v_{\text{shell}}^2 = \frac{M}{r - 2M} \qquad \text{(circle: (33) in Sec. 8.5)} \quad (10)$$

Figure 1 previews some kinds of orbits we discuss in this chapter.

## 9.2 ■ INSERT THE APPROACHING SPACESHIP INTO A CIRCULAR ORBIT

*Approach from far away and enter a circular orbit.*

Insert into a circular orbit.

How does the captain insert her approaching spaceship into an initial circular orbit from which to observe the black hole? Here's one possible method: While still far from the black hole, the captain uses speed- and direction-changing

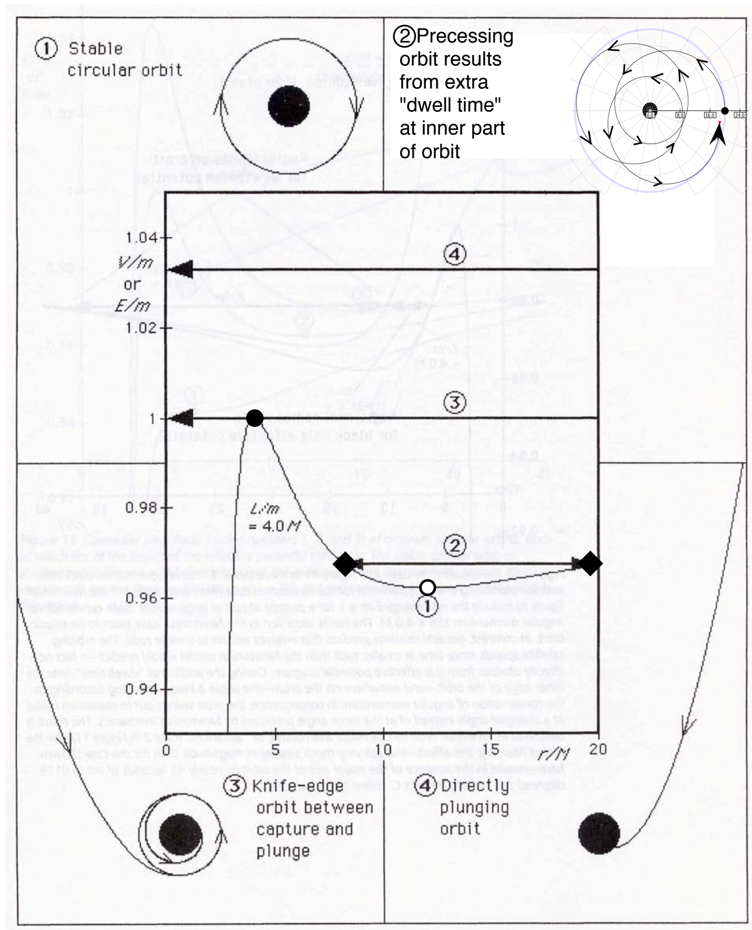


**FIGURE 1** Preview: Some kinds of orbits discussed in this chapter, shown here for a single value of map angular momentum $L/m$ but several different values of map energy $E/m$. A glance at the central plot allows us to make quick predictions about the motion of a stone that orbits or is captured by a black hole. Four different energies numbered on this plot correspond to orbits that appear in the four outer corners of the figure. Adapted from Misner, Thorne, and Wheeler.

rocket thrusts to put the spaceship into a free-fall insertion orbit whose minimum $r$-value matches that of the desired circular orbit (Figure 2). At that minimum, when the spaceship moves tangentially for an instant, the captain fires a rocket to slow down the spaceship to the tangential speed of the stable circular orbit at that $r$.

Insertion orbit

With what values of map $E/m$ and $L/m$ will an unpowered spaceship approaching from far away end up moving tangentially for an instant at the desired $r$-coordinate? To find out, substitute (5) into (4), set $dr/d\tau = 0$, and solve the resulting equation for $L/m$:

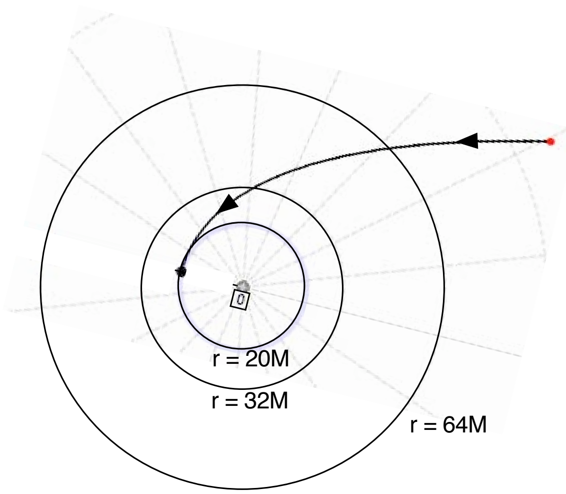


**FIGURE 2** Insertion orbit for unpowered spaceship that approaches from far away. At the instant of tangential motion at $r = 20M$, the spaceship fires a tangential rocket thrust to reduce the locally-measured shell velocity to that for a circular orbit (Figure 3).

$$\boxed{\frac{L}{m} = \pm r^2 \left[\frac{(E/m)^2}{1-(2M/r)} - 1\right]^{1/2} \qquad \text{(tangential motion)} \qquad (11)}$$

The $\pm$ sign in (11) distinguishes between two possible directions of motion at the $r$-value in equation (11). We choose positive angular momentum—that is, in the counterclockwise direction of increasing $\phi$. Equation (11) is valid whe $dr/d\tau = 0$, including turning points of all orbits as well as everywhere along a circular orbit.

Choose circular orbit at $r = 20M$.

The captain chooses her circular orbit at $r = 20M$. While still far from the black hole, she maneuvers the incoming spaceship to move with arbitrarily-chosen map energy $E/m = 1.001$ and the positive value of $L/m$ that results from equation (11)—both entered in Table 1. Then she turns off the rockets and lets the spaceship coast. Figure 2 shows the resulting orbit, which corresponds to the incoming horizontal arrow at $E/m = 1.001$ in Figure 3.

**Definitions:** Subscripts in Table 1

**DEFINITION 1. Subscripts in Table 1**

Here are definitions of the subscripts in the left-hand column of Table 1. All definitions describe the motion of a free stone or unpowered spaceship or unpowered probe.

**insert:** for free motion from far away to instantaneous tangential motion at $r$

**circle:** for free motion in a circular orbit at $r$

**transfer:** for free motion that is instantaneously tangential at both values of $r$

**shell:** for values measured in the local inertial frame at $r$

**TABLE 1** Numerical values at $r = 20M$ and $r_{\text{ISCO}} = 6M$

| Values of | $r = 20M$ | $r_{\text{ISCO}} = 6M$ |
|---|---|---|
| $(L/m)_{\text{insert}}$ | $6.733\ 036\ 31M$ | ——— |
| $(E/m)_{\text{insert}}$ | 1.001 | ——— |
| $v_{\text{x,shell,insert}}$ | 0.319 056 897 | ——— |
| $(L/m)_{\text{circle}}$ | $4.850\ 712\ 50M$ | $3.464\ 101\ 62M$ |
| $(E/m)_{\text{circle}}$ | 0.976 187 060 | 0.942 809 042 |
| $v_{\text{x,shell,circle}}$ | 0.235 702 260 | 0.5 |
| $(L/m)_{\text{transfer}}$ | $3.787\ 166\ 42M$ | $3.787\ 166\ 42M$ |
| $(E/m)_{\text{transfer}}$ | 0.965 541 773 | 0.965 541 773 |
| $v_{\text{x,shell,transfer}}$ | 0.186 052 102 | 0.266 880 257 |

NOTE: All shell velocities in this table are tangential, in the positive shell $x$-direction.

**Comment 1. Significant digits**

Long numbers in tables

In this chapter we analyze several unstable (knife-edge) circular orbits. Interactive software, such as GRorbits, requires accurate inputs to display the orbit of an unpowered probe that stays in an unstable circular orbit for more than one revolution. To avoid clutter, we put numbers with many significant digits into tables.

**Comment 2. Long subscripts**

In Table 1 the symbols $v_{\text{x,shell,insert}}$, $v_{\text{x,shell,circle}}$, and $v_{\text{x,shell,transfer}}$ have long, ungainly subscripts. We need long subscripts to fully describe these velocity components: that they are $x$-components measured in a local shell frame and whether they describe insertion speed into a circular orbit, speed in that circular orbit, or transfer between circular orbits.

**Comment 3. Impulse rocket thrusts**

Impulse rocket thrusts

We assume that each change in vehicle speed results from a quick rocket thrust, an impulse. In practice there is no hurry; some efficient rocket engines provide low thrust, which carries the vehicle through a series of intermediate orbits. To analyze the outcome of a slow burn complicates calculations and does not add to our understanding. So our vehicles use quick rocket thrusts to transfer from one orbit to another.

**Comment 4. Which direction is the "rocket thrust"?**

What is the meaning of the phrase *outward rocket thrust*? The rocket fires in one direction; the probe or spaceship that carries the rocket changes speed in the opposite direction. We define *outward rocket thrust* to mean that the rocket burn tends to move the rocket to larger $r$. Similarly, the *inward rocket thrust* tends to move the rocket to smaller $r$.

Insert into circular orbit

When the spaceship moves tangentially for an instant at $r = 20M$, the spaceship fires a tangential rocket thrust to put it into the stable circular orbit at that $r$. What change in tangential velocity must this rocket thrust provide? Tangential velocity in *which* frame? Our policy: make every measurement in a local inertial frame; for that purpose, choose the local *shell* frame. Box 2 in Section 7.4 gives shell frame coordinates from which we derive shell components of velocity. For reasons that will become apparent, we start with

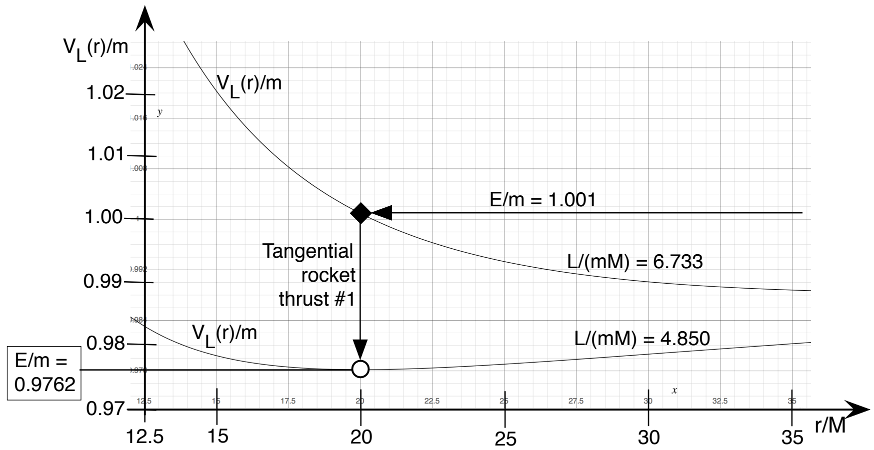


**FIGURE 3** At the instant when the incoming spaceship moves tangentially at the radial turning point $r = 20M$ (Figure 2), it fires tangential rocket thrust #1 that changes its map energy and map angular momentum to insert it into a stable circular orbit.

definitions of $dt_{\text{shell}}/d\tau$, $dy_{\text{shell}}/d\tau$, and $dx_{\text{shell}}/d\tau$, each with wristwatch time differential $d\tau$ in the denominator.

$$\frac{dt_{\text{shell}}}{d\tau} = \lim_{\Delta\tau\to 0}\frac{\Delta t_{\text{shell}}}{\Delta\tau} \tag{12}$$

$$= \left(1-\frac{2M}{r}\right)^{-1/2}\left[\left(1-\frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau}\right] \tag{13}$$

$$= \left(1-\frac{2M}{r}\right)^{-1/2}\frac{E}{m} \tag{14}$$

The last step uses equation (1). Similarly:

$$\frac{dy_{\text{shell}}}{d\tau} = \lim_{\Delta\tau\to 0}\frac{\Delta y_{\text{shell}}}{\Delta\tau} = \left(1-\frac{2M}{r}\right)^{-1/2}\frac{dr}{d\tau} \tag{15}$$

To find an expression for $dr/d\tau$ in this equation, combine equations (4) and (5):

$$\frac{dr}{d\tau} = \pm\left[\left(\frac{E}{m}\right)^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right)\right]^{1/2} \tag{16}$$

And finally:

$$\frac{dx_{\text{shell}}}{d\tau} = \lim_{\Delta\tau\to 0}\frac{\Delta x_{\text{shell}}}{\Delta\tau} = r\frac{d\phi}{d\tau} = \frac{L}{mr} \tag{17}$$

Shell velocity components

The last step uses equation (2). To complete the derivation of shell velocity components, note, for example, that $v_{\text{y,shell}} = (dy_{\text{shell}}/d\tau)(d\tau/dt_{\text{shell}})$, so from (15) and (14):

$$v_{\mathrm{y,shell}} = \frac{dr/d\tau}{E/m} = \pm\left[1 - \left(\frac{E}{m}\right)^{-2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2r^2}\right)\right]^{1/2} \qquad (18)$$

$$v_{\mathrm{x,shell}} = \left(1 - \frac{2M}{r}\right)^{1/2}\frac{L}{rE} \qquad (19)$$

Use the first two entries in Table 1 plus equation (19) to calculate the value of $v_{\mathrm{x,shell,insert}}$ at $r = 20M$ (where the shell $y$-component $v_{\mathrm{y,shell,insert}} = 0$) and check the result in the third line of Table 1.

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**QUERY 1. Tangential shell velocity in a circular orbit**

A. What is the tangential shell velocity of the spaceship in the circular orbit at $r$? Combine equations (6) and (8) to find an expression for $L/E$ and substitute the result into (19):

$$v_{\mathrm{shell,circle}} = \left(\frac{M}{r}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1/2} \qquad \text{(circular orbit, } r > 3M\text{)} \qquad (20)$$

B. Show that your derivation is not valid unless $r > 3M$.

C. Use (20) to calculate a value for $v_{\mathrm{shell,circle}}$ at $r = 20M$. Check your answer with the entry in Table 1.

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Table 1 tells us that the shell frame velocity $v_{\mathrm{x,shell,insert}}$ of the spaceship in its insertion orbit is greater than its shell frame velocity $v_{\mathrm{x,shell,circle}}$ in the circular orbit. Therefore a rocket thrust must bring the spaceship's shell velocity down to that of the circular orbit.

Use velocity addition laws.

Einstein shouts, "Look out! To calculate the needed change in spaceship velocity to be provided by the rocket thrust, you do *not* use the difference between $v_{\mathrm{x,shell,insert}}$ and $v_{\mathrm{x,shell,circle}}$." Why not? Because in special relativity (which rules in every local inertial frame), velocities do not simply add or subtract.

In what local inertial frame can we measure directly the change in velocity provided by the rocket thrust? That would be the local inertial frame in which the spaceship is initially at rest just before the thrust. Just before the rocket thrust, the spaceship moves at velocity $v_{\mathrm{x,shell,insert}}$ in the shell frame. We call the local inertial frame in which the spaceship is at rest the **instantaneous initial rest frame** or IIRF.

**Definition:** Instanteous Initial Rest Frame (IIRF)

**DEFINITION 2. Instantaneous Initial Rest Frame (IIRF)**

The instantaneous initial rest frame (IIRF) is the local inertial frame in which a rocket is at rest just before it fires a rocket thrust to change its velocity with respect to that frame. We use the subscript IIRF to indicate

**TABLE 2** Rocket Thrusts in Instantaneous Initial Rest Frames (IIRF)

| Thrust | at $r =$ | $\Delta v_{\mathrm{IIRF}}$ component | Description |
|---|---|---|---|
| #1 | $20M$ | $\Delta v_{\mathrm{x,IIRF1}} = -0.090\ 132\ 846\ 2$ | into circular orbit |
| #2 | $20M$ | $\Delta v_{\mathrm{x,IIRF2}} = -0.051\ 927\ 321\ 7$ | into transfer orbit |
| #3 | $6M$ | $\Delta v_{\mathrm{x,IIRF3}} = -0.269\ 017\ 469$ | into ISCO |
| #4 | $6M$ | $\Delta v_{\mathrm{x,IIRF4}} = 0.060\ 908\ 153\ 8$ | into transfer orbit |
| #4 | $6M$ | $\Delta v_{\mathrm{y,IIRF4}} = -\ 0.228\ 989\ 795$ | into transfer orbit |

NOTE: After thrust #4, the probe coasts into the unstable circular orbit at $r = 4M$.

quantities in this rest frame, as in the symbols $\Delta v_{\mathrm{x,IIRF}}$ and $\Delta v_{\mathrm{y,IIRF}}$ for the change in velocity components in the IIRF frame caused by that rocket impulse. We describe four different IIRF thrusts, listed with an additional number 1 through 4 added to the subscript (Table 2).

IIRF1 transfer velocity change

Special relativity addition of velocities gives us our first, tangential, IIRF rocket-thrust change $\Delta v_{\mathrm{x,IIRF1}}$ with the number 1 added to the subscript. This rocket thrust must reduce the shell speed of the spaceship. From equation (54) of Section 1.13,

$$\begin{aligned}\Delta v_{\mathrm{x,IIRF1}} &= \frac{v_{\mathrm{x,shell,circle}} - v_{\mathrm{x,shell,insert}}}{1 - v_{\mathrm{x,shell,insert}} v_{\mathrm{x,shell,circle}}} \\ &= -0.090\ 132\ 846\ 2 \qquad \text{(into circular orbit at } r = 20M\text{)}\end{aligned} \tag{21}$$

Put this numerical value into Table 2. This rocket-thrust velocity change ($-27$ 021 kilometers/second) inserts the incoming spaceship into the circular orbit at $r = 20M$.

?

**Objection 1.** *Wait! The two velocities, $v_{\mathrm{x,shell,circle}}$ and $v_{\mathrm{x,shell,insert}}$ are measured in the same local inertial shell frame. The difference in $x$-components is the measured difference in $x$-components; why confuse things with complicated equation (21)?*

!

Remember in special relativity the *law of addition of velocities* between two inertial frames in relative motion (Part A of Exercise 17 , Section 1.13)? Equation (21) could be called the *law of subtraction of velocities*—Part B of that earlier exercise. The complication of equation (21) does not require general relativity.

?

**Objection 2.** *Wow, that is quite a long vertical line in Figure 3. How fast does the probe move along that line? That quick transition must violate the light-speed limit!*

!

No, the probe does not change any global coordinate, $T$, $r$, or $\phi$, as it traverses the (idealized) vertical line. That transition results from a rocket thrust; it simply changes $L$ and $E$ almost instantaneously (Comment 3).

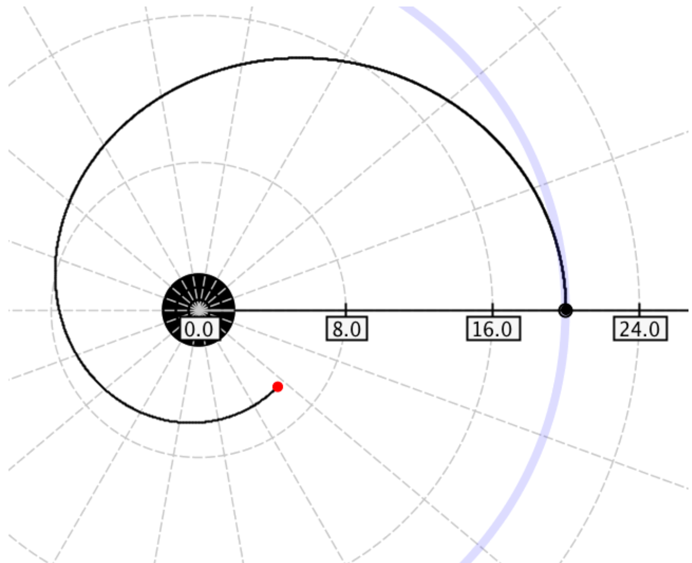


**FIGURE 4** Transfer orbit in which the unpowered probe coasts from tangential motion at $r_{\mathrm{A}} = 20M$ to tangential motion at $r_{\mathrm{ISCO}} = 6M$. Figure 5 shows the effective potential for this transfer and change in tangential speed required to put the probe into this transfer orbit.

**?**

**Objection 3.** *Your analysis of insertion into a circular orbit takes no account of mass loss due to required rocket thrusts. Whenever spaceship mass changes, its map energy and map angular momentum also change.*

**!**

Right you are. However, constants of motion in our equations are map energy and map angular momentum *per unit mass.* Map energy $E/m$ and map angular momentum $L/(mM)$ are unitless. Therefore the initial mass of the spaceship (before a rocket thrust) and the final spaceship mass (after the rocket thrust) do not affect these equations.

## 9.3 ■ TRANSFER TO THE ISCO

*Get closer*

The spaceship completes observations from the stable circular orbit at $r = 20M$ and its captain wants to make further observations from a smaller circular orbit—still outside the event horizon. To take the entire spaceship to this smaller orbit requires a large amount of rocket fuel; instead the captain launches a small probe toward the smaller orbit.

Transfer to circular orbit at $r_{\mathrm{ISCO}} = 6M$.

What $r$-value shall we choose for the inner circular orbit? Be bold! Take the probe all the way down to the so-called Innermost Stable Circular Orbit at $r_{\mathrm{ISCO}} = 6M$ (Definition 6, Section 8.5).

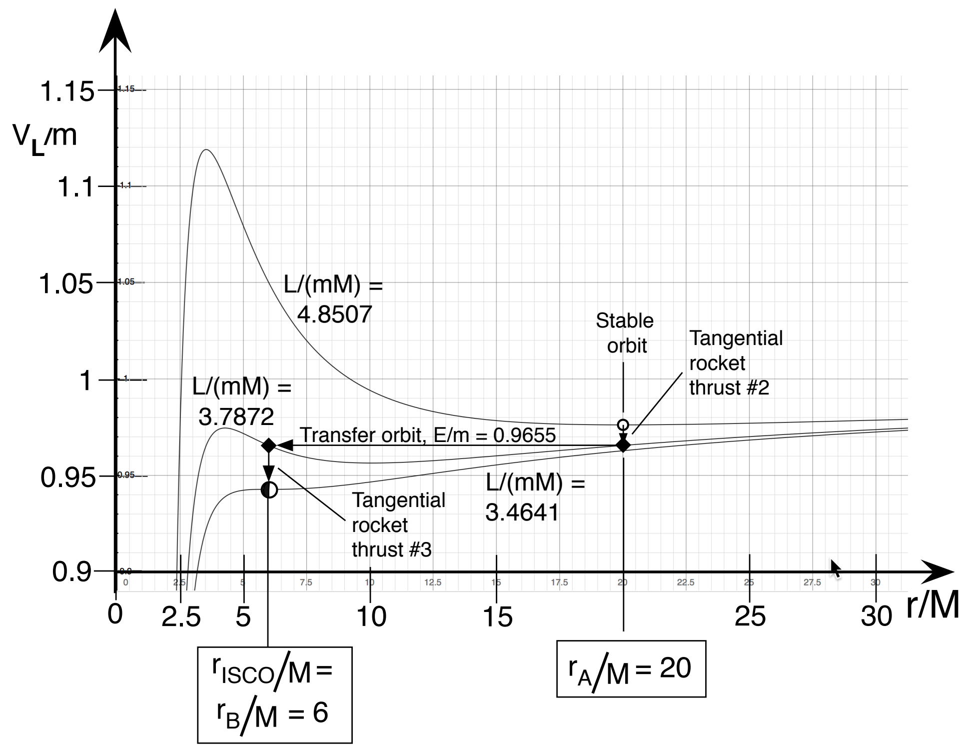


**FIGURE 5** Transfer orbit between sequential tangential rocket thrusts #2 and #3. This maneuver moves the probe from the stable circular orbit at $r = 20M$ to the half-stable ISCO at $r_{\rm ISCO} = 6M$. Figure 4 plots this transfer orbit on the $[r, \phi]$ slice.

**Comment 5. ISCO as a limiting case**
The ISCO is hazardous because it's "half stable" and may lead to a death spiral inward through the event horizon. To prevent this, the inner circular orbit $r$-value should be slightly greater than $r_{\rm ISCO}$ to make it fully stable. In what follows we ignore this necessary small $r$-adjustment.

Figure 4 shows a transfer orbit, tangential at both $r_{\rm A} = 20M$ and $r_{\rm B} = r_{\rm ISCO} = 6M$. Recall that these radii are called **radial turning points,** because at both $r$-values $dr/d\tau = 0$, so the orbiter instantaneously sweeps around only tangentially. Figure 5 displays the corresponding map energy on the effective potential plot.

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**QUERY 2. Profile of transfer orbit**
In 1925 Walter Hohmann described a transfer orbit between two planetary orbits around our Sun as "half an ellipse." Half an ellipse would have maxima of $r_{\rm A}$ and $r_{\rm B}$ on opposite sides of the center of attraction. The orbit plot in Figure 4 does not look like half an ellipse. Why is this different from Hohmann's prediction?

We seek a transfer orbit between the specified <u>A</u>bove circular orbit at $r_{\mathrm{A}}/M$ and the <u>B</u>elow circular orbit at $r_{\mathrm{B}}/M$; Figure 5 shows this transfer. In equation (4), $dr/d\tau = 0$ at the two turning points $r_{\mathrm{A}}/M$ and $r_{\mathrm{B}}/M$, which yields:

$$\left(\frac{E}{m}\right)^2 = \left(\frac{V_{\mathrm{L}}(r_{\mathrm{A}})}{m}\right)^2 = \left(\frac{V_{\mathrm{L}}(r_{\mathrm{B}})}{m}\right)^2 \qquad \text{(at turning points)} \qquad (22)$$

Transfer orbit map $L$ and $E$

Look first at the right equality in (22), in which the square of the effective potential (5) has the same value at two different $r$. Write down this equality and solve the resulting equation for $(L/m)^2$. The result is equation (23). Next look at the left equality in (22), in which the square of the map energy $(E/m)^2$ is equal to the square of the effective potential at either $r$. Write down this equality and solve the resulting equation for $(E/m)^2$. The result is equation (24).

$$\boxed{\begin{aligned} \left(\frac{L}{m}\right)^2_{\text{transfer}} &= \frac{2Mr_{\mathrm{A}}^2 r_{\mathrm{B}}^2(r_{\mathrm{A}} - r_{\mathrm{B}})}{r_{\mathrm{A}}^3(r_{\mathrm{B}} - 2M) - r_{\mathrm{B}}^3(r_{\mathrm{A}} - 2M)} \qquad \text{(between circular orbits)} \quad (23) \\ \left(\frac{E}{m}\right)^2_{\text{transfer}} &= \frac{(r_{\mathrm{A}} - 2M)(r_{\mathrm{B}} - 2M)(r_{\mathrm{A}}^2 - r_{\mathrm{B}}^2)}{r_{\mathrm{A}}^3(r_{\mathrm{B}} - 2M) - r_{\mathrm{B}}^3(r_{\mathrm{A}} - 2M)} \qquad \text{(between circular orbits)} \quad (24) \end{aligned}}$$

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**QUERY 3. Transfer either way**

Show that equations (23) and (24) are both symmetrical in $r_{\mathrm{A}}$ and $r_{\mathrm{B}}$. In other words, show that the same values of $(L/m)_{\text{transfer}}$ and $(E/m)_{\text{transfer}}$ apply, irrespective of the direction of transfer between the circular orbits. Is this result obvious?

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Substitute values $r_{\mathrm{A}} = 20M$ and $r_{\mathrm{B}} = r_{\mathrm{ISCO}} = 6M$ into equations (23) and (24); enter resulting values of $L/m$ and $E/m$ into Table 1. Then equations (18) and (20) give us values of $v_{\mathrm{x,shell,transfer}}$ and $v_{\mathrm{x,shell,circle}}$. These results allow us to compute the rocket thrust needed to put the probe into the transfer orbit. This is our second, also tangential, instantaneous initial rest frame IIRF thrust (Definition 2) with the number 2 added to the subscript, $\Delta v_{\mathrm{x,IIRF2}}$.

IIRF2 transfer velocity change

$$\begin{aligned} \Delta v_{\mathrm{x,IIRF2}} &= \frac{v_{\mathrm{x,shell,transfer}} - v_{\mathrm{x,shell,circle}}}{1 - v_{\mathrm{x,shell,transfer}} v_{\mathrm{x,shell,circle}}} \qquad &&\text{(into transfer orbit} \qquad (25) \\ &= -0.051\ 927\ 321\ 7 &&\text{from } r = 20M \text{ to } r_{\mathrm{ISCO}}) \end{aligned}$$

Enter this numerical result into Table 2. This rocket-thrust velocity change ($-15\ 567$ kilometers/second) inserts the probe into a transition orbit that carries it from tangential motion at $r = 20M$ down to tangential motion at $r_{\mathrm{ISCO}} = 6M$.

?

**Objection 4.** *You talk about moving into a circular orbit and transferring between orbits. But what will our orbiting observers* ***see****? You have told us nothing about what they see as they look around.*

!

Guilty as charged! Section 7.7 showed only what a raindrop diver sees radially inward and radially outward as she plunges to the center of the black hole. Beyond that, we have made no predictions whatsoever about what any observer sees. For example: In what local frame direction must an observer look to see a particular star? What must we know to make such predictions? Chapters 13 answers these questions. The cosmic trip planner must read beyond the present chapter!

IIRF3 transfer velocity change

When the probe reaches $r_{\rm ISCO} = 6M$, it travels tangentially for an instant at shell velocity $v_{\rm x,shell,transfer}$. Then a third insertion rocket thrust changes this shell velocity to $v_{\rm x,shell,circle}$ for the circular orbit at $r_{\rm ISCO}$. Table 1 has values of both of these velocities. What insertion rocket thrust does this? As before, it is a tangential thrust in the instantaneous inertial rocket frame IIRF (Definition 2), with the number 3 added to the subscript, $\Delta v_{\rm x,IIRF3}$.

$$\begin{aligned}\Delta v_{\rm x,IIRF3} &= \frac{v_{\rm x,shell,transfer} - v_{\rm x,shell,circle}}{1 - v_{\rm x,shell,transfer}v_{\rm x,shell,circle}} \\ &= -0.269\ 017\ 469 \qquad (\text{into circular orbit at } r_{\rm ISCO} = 6M)\end{aligned} \tag{26}$$

Enter the numerical result in Table 2. This rocket-thrust velocity change (−86 494 kilometers/second) inserts the probe into the circular orbit at $r_{\rm ISCO} = 6M$.

## 9.4 ■ TRANSFER TO AN UNSTABLE CIRCULAR ORBIT

*Put the probe at risk!*

Transfer to unstable orbit at $r = 4M$

Thus far we have inserted our spaceship into a stable circular orbit at $r = 20M$, then transferred a probe down to the half-stable circular orbit at $r_{\rm ISCO} = 6M$. Now the spaceship captain wants to make observations even closer to the black hole. She decides to transfer the probe from $r_{\rm ISCO} = 6M$ to the unstable circular orbit at $r = 4M$, a maneuver shown in Figures 6 and 7.

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**QUERY 4. Unstable circular orbit at $r = 4M$**

A. Show that the unstable circular orbit at $r = 4M$ has map angular momentum $L/m = 4M$.
B. Show that the unstable circular orbit at $r = 4M$ has map energy $E/m = 1$.
C. Make an argument that the transfer orbit from $r = 6M$ to $r = 4M$ in Figures 6 and 7 must have the same values of map energy and map angular momentum given in the first two items of this Query.

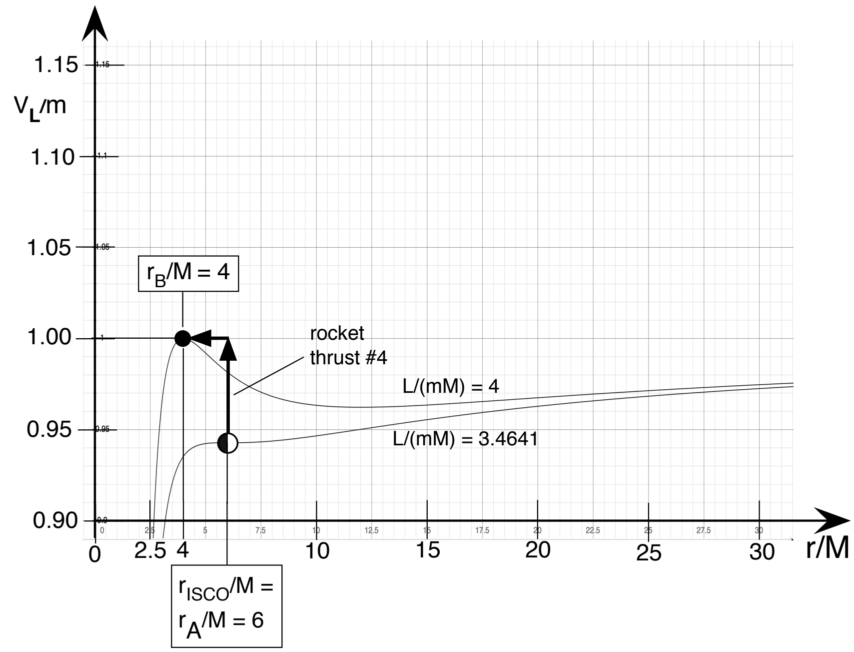


**FIGURE 6** Probe transfer orbit between half-stable orbit at $r_{\rm ISCO} = 6M$ and unstable circular orbit at $r = 4M$. See Figure 7.

D. Verify the bottom right hand entry in Table 3, namely that at $r = 4M$, $v_{\rm x,shell,circle} = v_{\rm x,shell,transfer} = |v_{\rm shell,transfer}|$

---

Transfer orbits have radial turning points where $E/m = V_{\rm L}(r)$. Usually these turning points are not at an extremum of the effective potential, so they are not at $r$-values of circular orbits. In this case, however, we need a rocket thrust to *create* the extremum for a circular orbit at that $r$-value.

No rocket thrust needed for insertion into unstable orbit.

At a maximum of the effective potential, the turning point occurs at the $r$-value of the circular orbit, so we need no rocket thrust to put the probe into that circular orbit. Figure 6 shows this special case: The probe moves to smaller $r$ along the horizontal arrow in Figure 6. As it does so it reaches the effective potential maximum at $r = 4M$ where it automatically enters the unstable circular orbit at that $r$-value. So we need only a single rocket thrust at $r = 6M$ to change map energy and map angular momentum to that of the circular orbit at $r = 4M$ (Figure 7).

**?**

**Objection 5.** *Once the rocket thrust #4 shoots the probe upward in Figure 6 to map energy $E/m = 1$, why should the probe go left in that figure, to smaller $r$? Why doesn't it go right, to larger $r$?*

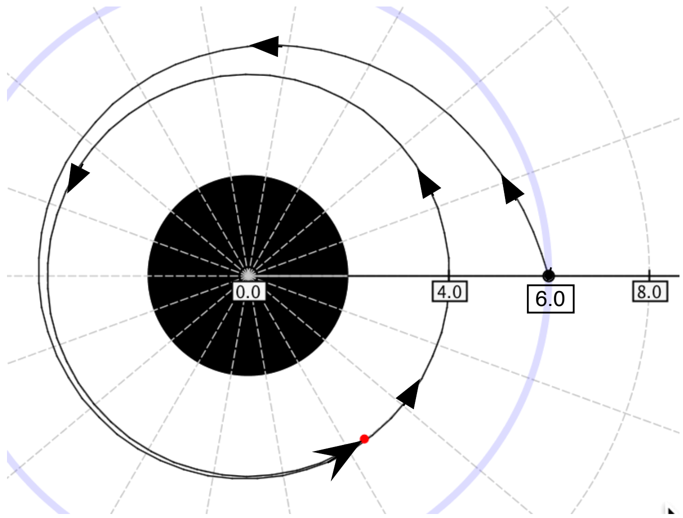


**FIGURE 7** Transfer orbit from $r_{\rm ISCO} = 6M$ to the unstable circular orbit at $r = 4M$ (Figure 6). This requires a velocity $v_{\rm shell,transfer}$ inward from $90^\circ$ by 19.471 degrees, with shell velocity components and magnitude given in Table 3.

**!** Figure 7 and Table 3 show the answer: The rocket thrust is not tangential but has an inward $r$-component.

Need two thrust components for transfer orbit

Query 4 already tells us the map values $E/m = 1$ and $L/m = 4M$ of the leftward horizontal arrow in Figure 6. Because the rocket thrust is not tangential, we need to apply the full set of equations (18) and (19) to find the shell components of the velocity in the transfer orbit. Enter these results for $v_{\rm y,shell,transfer}$ and $v_{\rm x,shell,transfer}$ in Table 3.

To start this transfer from $r_{\rm ISCO}$ we use the fourth rocket thrust measured in the instantaneous initial rest frame. This thrust requires two components, which we call $\Delta v_{\rm x,IIRF4}$ and $\Delta v_{\rm y,IIRF4}$, with the number 4 added to the subscript. In this case we must adapt both velocity addition equations (54) in Section 1.13.

$$\Delta v_{\rm x,IIRF4} = \frac{v_{\rm x,shell,transfer} - v_{\rm x,shell,circle}}{1 - v_{\rm x,shell,circle}v_{\rm x,shell,transfer}} \quad \text{(into the transfer orbit...} \qquad (27)$$

$$\Delta v_{\rm y,IIRF4} = \frac{v_{\rm y,shell,transfer}}{\gamma_{\rm x,shell,circle}(1 - v_{\rm x,shell,circle}v_{\rm x,shell,transfer})} \quad \text{...from } r = 6M \qquad (28)$$

$$\text{where} \quad \gamma_{\rm x,shell,circle} = (1 - v^2_{\rm x,shell,circle})^{-1/2} \qquad \text{...to } r = 4M) \qquad (29)$$

Substitute into these equations from $r = r_{\rm ISCO} = 6M$ values in Tables 1 and 3 and enter the resulting components into Table 2. This rocket thrust, which corresponds to the vertical arrow in Figure 6, causes a velocity change of magnitude, $|\Delta v_{\rm IIRF4}| = 0.236\ 951\ 745 = 71\ 036$ kilometers/second.

**TABLE 3** Numerical values for transfer from $r_{\rm ISCO} = 6M$ to $r = 4M$

| Values of | $r_{\rm ISCO} = 6M$ | $r = 4M$ |
|---|---|---|
| $(L/m)_{\rm transfer}$ | $4M$ | $4M$ |
| $(E/m)_{\rm transfer}$ | 1 | 1 |
| $v_{\rm x,shell,transfer}$ | 0.544 331 054 | 0.707 106 781 |
| $v_{\rm y,shell,transfer}$ | −0.192 450 090 | 0 |
| $\lvert v_{\rm shell,transfer}\rvert$ | 0.577 350 269 | 0.707 106 781 |
| $\theta_{\rm x,shell}$ | −19.471 220 6° | 0 |
| $v_{\rm x,shell,circle}$ | 0.500 000 000 | 0.707 106 781 |

Good-bye probe!

Our probe coasts to the unstable circular orbit at $r = 4M$, an effective potential peak close to the black hole. After it completes measurements there, the captain decides to dispose of the probe. To do this, she commands the probe to fire a tiny inward rocket thrust to tip it off the effective potential peak and send it spiraling inward across the event horizon. Good job!

Section 9.5 applies some of what we have learned to analyze Larry Niven's short story “Neutron Star.”

## 9.5 ■ “NEUTRON STAR” BY LARRY NIVEN

*Close to a neutron star? Look out!*

Why did earlier explorers die?

Larry Niven's science fiction short story “Neutron Star” describes the trip by spaceship pilot Beowulf Schaeffer to discover why two earlier pilots died while orbiting a neutron star. Sponsors of Beowulf's trip are aliens called puppeteers, who manufacture spaceship hulls that are utterly indestructable and—so they claim—impenetrable. Naturally, the death of two pilots in an “impenetrable” puppeteer spaceship hull has reduced sales. The puppeteers want to know what deadly force has managed to enter their high-tech hulls.

Passage through closest approach

As Beowulf approaches the neutron star, the long axis of his spaceship inexorably orients along a radial line to the star (Why?). Beowulf suddenly realizes that he must position himself at the point in the spaceship where at least one part of his body feels no gravity in order to be in free-fall motion around the neutron star. Here is Niven's description of his passage through the $r$-coordinate of closest approach:

*“Giants' hands gripped . . .”*

> *My time was up. A red disk leapt up at me; the ship swung around me; I gasped and shut my eyes tight. Giants' hands gripped my arms and legs and head, gently but with great firmness, and tried to pull me in two. In that moment it came to me that Peter Laskin had died like this. He'd made the same guesses I had, and he'd tried to hide in the access tube. But he'd slipped . . . as I was slipping . . . From the control room came a multiple shriek of tearing metal. I tried to dig my feet into the hard tube walls. Somehow they held.*

Close-call survival

According to Niven's story, Beowulf is (barely!) able to cling to the point of zero local gravity, though the skin on his extremities is injured. After returning to base, he reports to the puppeteers that the deaths of earlier explorers were due to their slipping from this gravity zero point and falling to the front (or back) of the spaceship.

**?**

**Objection 6.** *What in (or out of) this world is happening to Beowulf? His orbit around the neutron star is similar to those we use to insert our spaceship into a circular orbit. Why is Beowulf in danger, and why did earlier explorers die?*

**!**

"All politics is local," said politician Tip O'Neill. A monster may lurk at opposite ends of your spaceship. In "Neutron Star" the monster is *tidal acceleration*, which can be lethal.

Killer tides

Tidal acceleration is nothing new for us. Section 7.9 introduced it for the radial fall into the black hole, and in the present chapter Section 9.7, Appendix: Killer Tides, gives expressions for radial and tangential tidal accelerations. This information allows us to answer the question, "Can Beowulf Schaffer survive his transit past the neutron star?"

Survival?

We need numerical values from "Neutron Star" in order to apply tidal acceleration expressions from Section 9.7. Larry Niven tells us that (a) the neutron star's mass is 1.3 times the mass of our Sun, (b) the minimum $r$-coordinate of approach is approximately 10.5 kilometers, so that $r_{\text{min}} \approx 5.5M$. (The neutron star is also spinning, but too slowly to have a significant effect on Beowulf's global orbit or local safety.)

---

**QUERY 5. Einstein predicts Beowulf Schaeffer's fate**

Use the parameters in the preceding paragraph to find out whether or not Beowulf Schaeffer survives tidal accelerations during his encounter with the neutron star. Assume that the distant speed of approach to the neutron star is nonrelativistic, so that $E/m \approx 1$.

A. Use (3) to determine $v_{\text{shell}}$ at the closest approach $r_{\text{min}}$.

B. By what multiple is the radial tidal effect (in the local spaceship $\Delta y_{\text{ship}}$ direction) larger than the Newtonian prediction?

C. At the moment of closest approach to the neutron star, Beowulf Schaeffer extends his arm one meter radially inward. What happens to him next?

D. Give a definitive answer to the question, "Can Beowulf Schaeffer survive the trip described in "Neutron Star"? (When our class sent numerical results to Larry Niven, he replied, "Thank you for the calculations. I'm not sure how I will use them, but thanks anyway.")

E. If you conclude that Beowulf cannot survive the "Neutron Star" trip, find an $r$-coordinate of closest approach to the neutron star at which Beowulf Schaeffer can survive. State your criteria for *survival.* On the way to this result, give a specific numerical value for $\Delta g/\Delta y_{\text{ship}}$ that, in your estimate, is survivable.

**QUERY 6. Blackmail**

*Discussion question:* Beowulf Schaefer blackmails the secretive puppeteers by threatening to reveal that they come from a moonless world. How does he know that?

**QUERY 7.** *Optional:* **Swimming in spacetime?**

A massive mother ship is in a circular orbit with its long dimension tangential with respect to the black hole. Astronauts inside extend a mechanical arm radially inward toward the black hole. The "hand" on this arm experiences a radially inward force.

A. Can such a maneuver be used to change the orbit of the mother ship?

B. Can similar maneuvers provide a method for balancing a spaceship in a circular knife-edge orbit without using rockets?

C. Using repeated "calisthenics," can a freely-floating astronaut "swim" around the mother ship? (See "Swimming in Spacetime" in the references.)

D. Do such maneuvers violate the laws of conservation of map energy or map angular momentum?

E. Do similar maneuvers work in flat spacetime?

## 9.6 ■ A COMFORTABLE CIRCULAR ORBIT

*How close to the black hole?*

Meaning of "comfortable"?

Up to this point, our description of circular orbits has a serious flaw: We do not answer the question, "What is the minimum $r$-value of a circular orbit in which the astronaut will be comfortable?" Our answer to this question has three parts:

- Part I. What are the tidal accelerations in a circular orbit of given $r$-coordinate? To answer this question, we consult Section 9.7, Appendix: Killer Tides.
- Part II. What is the maximum tidal acceleration for which a human is comfortable?
- Part III. What is the minimum $r$-coordinate of a circular orbit (Part I) for which a human is comfortable (Part II)?

Instead of choosing an orbit that is comfortable for a human, we can replace the human with a probe hardened to withstand hundreds or thousands of times the tidal accelerations that would injure or kill a person.

### Part I: Tidal acceleration in circular orbit

In order to apply tidal equations (46) through (48) to a circular orbit, we need the square of the tangential shell velocity in (10).

Tidal acceleration in circular orbit

Think of an astronaut in a circular orbit with the long axis of his body oriented along the radial direction. His height is larger than his width, so we carry out our calculations for the radial tidal component only, knowing that the other components will be smaller. Half his height provides a value for $\Delta y_{\text{local}}$ in equation (46). Substitute (10) into (46) and rearrange so the right side of the equation contains only expressions in $r$.

$$\Delta g_{\text{local},y} \approx \frac{M}{\bar{r}^3}\left(\frac{2\bar{r}-3M}{\bar{r}-3M}\right)\Delta y_{\text{local}} \qquad \text{(circular orbit)} \tag{30}$$

### Part II: Define human comfort.

Tidal acceleration for human comfort

How large a tidal acceleration is comfortable for a human being? The answer is different for people of different heights. Here we treat our human astronaut gently, using the definition employed in Section 7.9 under the assumption that he is oriented along a radial line, with head above feet. Then with his stomach in free fall, the astronaut remains comfortable if his head is accelerated upward with the acceleration it would experience on Earth—call it $g_{\text{E}}$—and his feet are accelerated downward with the same magnitude of Earth acceleration.

Assume the astronaut is approximately two meters tall, so his measured distance between head and stomach is one meter, the same as the separation between stomach and feet. Then $\Delta y_{\text{local}} = 1\,\text{meter}$ in equation (30).

### Part III: Minimum-$r$ circular orbit for human comfort

Minimum $r$ for comfort?

The acceleration $g_{\text{E}}$ at Earth's surface has the numerical value $g_{\text{E}} = 1.09\times10^{-16}$ meter$^{-1}$ (inside the front cover). We want to insert $g_{\text{E}}$ into (30) when the circling astronaut's "half height" is $\Delta y_{\text{local}} = 1$ meter:

$$g_{\text{E}} = \Delta g_{\text{local},y} \approx \frac{M}{\bar{r}^3_{\text{comfort}}}\left(\frac{2\bar{r}_{\text{comfort}}-3M}{\bar{r}_{\text{comfort}}-3M}\right)\times 1\,\text{meter} \quad \text{(human comfort limit)} \tag{31}$$

$$g_{\text{E}} \approx \frac{M^{-2}}{\left(\bar{r}_{\text{comfort}}/M\right)^3}\left(\frac{2\bar{r}_{\text{comfort}}/M-3}{\bar{r}_{\text{comfort}}/M-3}\right)\times 1\,\text{meter} \tag{32}$$

In this equation, $\bar{r}_{\text{comfort}}$ refers to the smallest $r$-value of the circular orbit in which the observer is comfortable. Multiply the left and right sides of (32) by $M^2$ and divide by $g_{\text{E}}$. The result is

$$M^2 \approx \frac{1}{\left(\bar{r}_{\text{comfort}}/M\right)^3}\left(\frac{2\bar{r}_{\text{comfort}}/M-3}{\bar{r}_{\text{comfort}}/M-3}\right)\frac{1\,\text{meter}}{g_{\text{E}}} \quad \text{(human comfort limit)} \tag{33}$$

We can rearrange (33) to give the mass of the black hole in number of Suns, $M/M_{\text{Sun}}$, as a function of the minimum $r$-value, $r_{\text{comfort}}$, of the circular orbit in which a human astronaut will be comfortable:

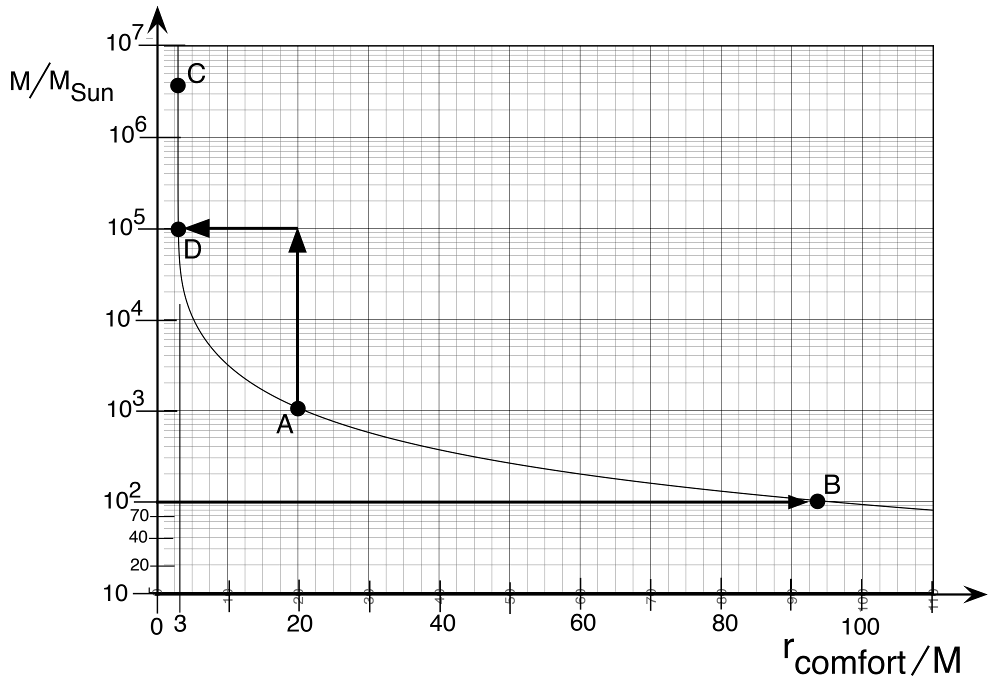


**FIGURE 8** The horizontal axis, $r_{\mathrm{comfort}}/M$, gives the minimum-$r$ circular orbit in which a human will be comfortable. On the vertical axis, $M/M_{\mathrm{Sun}}$ is a number equal to the mass of the black hole in units of the mass of our Sun. Arrows and little filled circles illustrate solutions of Sample Problems 1A through 1D.

$$\frac{M}{M_{\mathrm{Sun}}} = \frac{1}{M_{\mathrm{Sun}}}\left(\frac{1\,\mathrm{meter}}{g_{\mathrm{E}}}\right)^{1/2}\left[\frac{1}{(\bar{r}_{\mathrm{comfort}}/M)^3}\left(\frac{2\bar{r}_{\mathrm{comfort}}/M-3}{\bar{r}_{\mathrm{comfort}}/M-3}\right)\right]^{1/2} \quad (34)$$

$$= 6.47\times 10^4\left[\frac{1}{(\bar{r}_{\mathrm{comfort}}/M)^3}\left(\frac{2\bar{r}_{\mathrm{comfort}}/M-3}{\bar{r}_{\mathrm{comfort}}/M-3}\right)\right]^{1/2} \quad (35)$$

(minimum-$r$ circular orbit for human comfort)

The last step substitutes values of $M_{\mathrm{Sun}}$ and $g_{\mathrm{E}}$ from inside the front cover. Verify that both sides of this equation are unitless. Figure 8 plots the curve of this equation. Sample Problems 1 explain the arrows.

### Sample Problems 1. Minimum-$r$ Circular Orbit for Human Comfort

**PROBLEM 1A**
What is the numerical value of $M/M_{\text{Sun}}$ for which $r_{\text{comfort}}/M = 20$ is the minimum circular orbit in which a human feels comfortable? What is the value of $r_{\text{comfort}}$ in meters?

**SOLUTION 1A**
Figure 8 shows that at $r_{\text{comfort}}/M = 20$, $M/M_{\text{Sun}} \approx 10^3$, indicated by point A in the figure. The value of $r_{\text{comfort}}$ in meters is $r_{\text{comfort}} = 20 \times M \text{ meters} = 20 \times (M/M_{\text{Sun}}) \times M_{\text{Sun}} \text{ meters} \approx 20 \times 10^3 \times 1.48 \times 10^3$ meters $\approx 3 \times 10^7 \text{meters} \approx 3 \times 10^4$ kilometers.

**PROBLEM 1B**
I approach the black hole of mass value $N_{\text{Suns}} = 10^2$. What is the minimum $r_{\text{comfort}}$ of the circular orbit in which I will feel comfortable?

**SOLUTION 1B**
The long horizontal arrow to the right at $N_{\text{Suns}} = 10^2$ in Figure 8 crosses the "comfort curve" at $r_{\text{comfort}}/M \approx 93$, indicated by point B in Figure 8.

**PROBLEM 1C**
I approach the monster black hole in the center of our galaxy, for which $N_{\text{Suns}} \approx 4 \times 10^6$. Assume (incorrectly) that this monster black hole is not spinning. What is the approximate value of $r_{\text{comfort}}$ for this circular orbit?

**SOLUTION 1C**
The number $M/M_{\text{Sun}} = 4.1 \times 10^6$ is point C on the curve in Figure 8. You will be comfortable in an orbit of approximately $r_{\text{comfort}}/M = 3$

**PROBLEM 1D**
The robot satellite released by the spaceship at $r_{\text{comfort}}/M = 20$ in Problem 1A is made small and hardened in various ways to withstand tidal accelerations $10^4$ times as great as that for which a human will be comfortable. What is the value of $r_{\text{comfort}}$ of the circular orbit in which this probe will continue to operate?

**SOLUTION 1D**
Look at equation (34). The black hole remains the same, so the ratio $M/M_{\text{Sun}}$ on the left side remains the same. Therefore the right side must remain the same. When $g_{\text{E}}$ in the denominator on the right side *increases* by a factor of $10^4$, then its square root contribution to the right side *decreases* by the factor $10^2$. To compensate, the square root of the square-bracket expression must *increase* by the factor $10^2$. The vertical arrow in the figure extends upward by this factor of $10^2$. The leftward horizontal arrow finds $r_{\text{conf}}/M$, for the "comfort orbit" of the robot. This $r_{\text{comfort}}/M \approx 3$ for the robot is at almost the minimum $r$-value for an unstable circular orbit.

## 9.7 ■ APPENDIX: KILLER TIDES

*Avoid spaghettification!*

Size of local inertial frame limited by tides.

The dangers experienced by Beowulf and other explorers near a neutron star should not surprise us. Objects near to one another in curved spacetime can experience relative accelerations. Section 1.11 described these "tidal accelerations" that limit the size of a local inertial frame. At locations near to one another on Earth's surface, these relative accelerations are too small for us to notice in everyday life. In contrast, near a neutron star or a black hole relative tidal accelerations at different locations on a single human body can injure or kill. We call such different accelerations **killer tides**.

In principle, you can derive the following tidal accelerations using only basic tools for the motion of a stone: the metric plus the Principle of Maximal Aging. This process, however, is an algebraic nightmare, so we simply quote results obtained with the use of a more advanced general-relativistic formalism.

### TIDES DURING RADIAL MOTION

Radial motion: Newton's tidal accelerations are valid.

Surprise! For the special cases of an observer either at rest in global coordinates near a black hole or moving radially toward or away from it, local tidal effects predicted by general relativity are identical to those predicted by Newton. Write Newton's expression for gravitational acceleration in the radially outward or local $y$-direction due to a point or spherically symmetric source. In unitless coordinates:

$$g_y = -\frac{M}{r^2} \qquad \text{(Newton)} \tag{36}$$

Take the differential of this to measure radial tidal effects and write the result in the approximate form for local frame measurements:

$$\Delta g_{\text{local},y} \approx \frac{2M}{\bar{r}^3}\Delta r \approx \frac{2M}{\bar{r}^3}\Delta y_{\text{local}} \qquad \text{(Newton)} \tag{37}$$

The final step, equating $\Delta r$ to $\Delta y_{\text{local}}$, makes sense only for Newton; in general relativity the relation between global increment $\Delta r$ and local frame increment $\Delta y_{\text{local}}$ depends on the position and motion of the local frame in global coordinates. Nevertheless—surprise again!—the full general relativity analysis also yields the last expression in (37). To show this is difficult. The following boxed three equations tell us the tidal accelerations in the three directions in the inertial frame.

$$\Delta g_{\text{local},y} \approx \frac{2M}{\bar{r}^3}\Delta y_{\text{local}} \tag{38}$$

$$\Delta g_{\text{local},x} \approx -\frac{M}{\bar{r}^3}\Delta x_{\text{local}} \tag{39}$$

$$\Delta g_{\text{local},z} \approx -\frac{M}{\bar{r}^3}\Delta z_{\text{local}} \tag{40}$$

Subscript "local" means *any* local frame at rest or moving radially inward or outward in global rain coordinates.

Spaghettification: radial stretch plus tangential compression

A radially-diving observer suffers not only stretching in the radial direction, but also compression in tangential directions as her descending body funnels into an ever-narrowing local space. Negative signs in (39) and (40) reflect this compression. We give the light-hearted name **spaghettification** to the physical result of these combined stretch and compression tidal effects: lengthwise extension combined with transverse compression. Sample Problem 2 carries out a Newtonian analysis of gravity gradients (tides), whose results turn out to be identical in form to general relativistic results (38) through (40).

Expressions (38) through (40) shrink to become calculus expressions (44) at a point. Every approximate equation in this section can lead to a similar calculus expression. We keep the $\Delta$ notation, however, to remind us that we deal here with a local frame of finite extent.

Now apply equations (38) through (40) to a local *inertial* frame. A liquid drop of nearly incompressible fluid, such as water or mercury, has a surface tension that tends to minimize surface area, which makes the droplet spherical

## Sample Problem 2. Newton's tidal components

Derive expressions similar to (38) through (40) for Newton's case, in the calculus limit.

**SOLUTION:**
This is one of only two places in this book where we use vector expressions and partial derivatives. Represent unit vectors in the $x$, $y$, and $z$ directions by $\hat{x}$, $\hat{y}$, and $\hat{z}$, respectively. Use this notation to write (36) as a vector equation:

$$\mathbf{g} = -\frac{M\,(x\hat{x} + y\hat{y} + z\hat{z})}{(x^2+y^2+z^2)^{3/2}} \qquad \text{(Newton)} \qquad (41)$$

Each component of this vector has the algebraic form:

$$g_q = -\frac{Mq}{(x^2+y^2+z^2)^{3/2}} \qquad (42)$$

where $q$ stands for any coordinate $x$, $y$, or $z$. Take the partial derivates similar to the general relativistic equations (38) through (40). You can show that the results also have the same form for all three components:

$$\frac{\partial g_q}{\partial q} = -\frac{M}{r^3} + \frac{3Mq^2}{r^5} \qquad (q \rightarrow x, y, z) \qquad (43)$$

We want expressions for these partial derivatives at global coordinate $r$ in flat spacetime. Take $y$ to be along the radial direction, so at that point $y = r$, while $x = z = 0$. Equations (43) become:

$$\begin{aligned} \frac{\partial g_x}{\partial x} &= -\frac{M}{r^3} \qquad \text{(Newton)} \qquad (44)\\ \frac{\partial g_y}{\partial y} &= -\frac{M}{r^3} + \frac{3M}{r^3} = +\frac{2M}{r^3}\\ \frac{\partial g_z}{\partial z} &= -\frac{M}{r^3} \end{aligned}$$

Inspection shows that equations (44) have the same form as equations (38) through (40).

All radial speeds give same local tidal accelerations.

in an inertial frame. Equations (38) through (40) show us that for radial motion, the drop will be distorted into the shape of a throat lozenge or smooth potato—technical term: **prolate spheroid**—shown in Figure 9.

Equations (38) through (40) are valid for *all possible* radial speeds—including rest—for example a local inertial frame launched in any of the following ways:

- **Local rain frame:** Local inertial frame dropped from rest far away (Box 4, Section 7.4).
- **Local hail frame:** Local inertial frame hurled radially inward from far away with any initial local shell speed.
- **Local drip frame:** Local inertial frame dropped from rest at any initial $r_0 > 2M$.

Radial free-fall frames

All of these are radially-moving *local free-fall frames* (Section 2.1). Taken together, free-fall frames result in every possible inward or outward radial speed of the radially moving frame as measured by a shell observer at any given average $\bar{r}$. General relativity provides results independent of radial speed in (38) through (40), but the tools developed in this book are not sufficient to explain the reason for this result.

Notice that equations (38) through (40) satisfy the equation

$$\frac{\Delta g_{\text{local},y}}{\Delta y_{\text{local}}} + \frac{\Delta g_{\text{local},x}}{\Delta x_{\text{local}}} + \frac{\Delta g_{\text{local},z}}{\Delta z_{\text{local}}} \approx 0 \qquad (45)$$

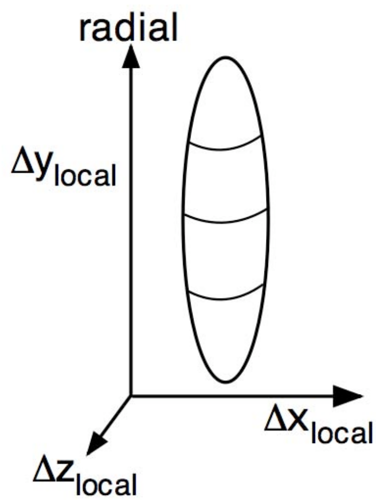


**FIGURE 9** Schematic diagram of tide-induced shape for an incompressible liquid drop with surface tension restoring force, observed in a local inertial frame instantaneously at rest or moving radially with respect to a black hole. From the symmetry of the black hole with respect to radial motion, it follows that the tidal squeeze is symmetric perpendicular to the radial direction. Result: the shape is that of an oblong throat lozenge or smooth potato.

Relation among tidal components

This is a general result for tides analyzed by general relativity. In the calculus limit, the approximate equality in (45) becomes mathematically exact, and applies to partial derivatives in (44).

**Comment 6. Tides preserve volume.**
In the calculus limit, equation (45) expresses a simple and powerful result: The volume of a tiny cloud of free, non-interacting dust particles remains constant as tidal accelerations act on the cloud. This central result is valid even for the far more complicated tidal accelerations near a spinning black hole (Chapter 19).

Tidal effects are continuous across event horizon.

Notice that equations (38) through (40) are continuous across the event horizon at $r/M = 2$. This result provides additional evidence for our repeated claim that an observer falling through the event horizon experiences a steady increase in tidal effects but no sudden jar or jolt there. Indeed, from evidence internal to her local frame the diver cannot tell when she passes radially inward through the event horizon.

### TIDES DURING TANGENTIAL MOTION

Tangential motion: tidal accelerations differ from Newton's.

An observer moving in the $r, \phi$ plane streaks through a local shell frame in the tangential, or $\Delta x_{\text{shell}}$, direction with shell velocity $v_{\text{shell},x}$. In the following equations, only the factor $M/\bar{r}^3$ reminds us of the corresponding Newtonian analysis in equation (37). *For motion along the tangential* $\pm\Delta x_{\text{shell}}$ *directions:*

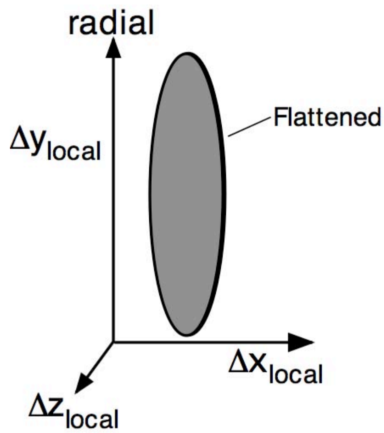


**FIGURE 10** Schematic diagram of tide-induced shape for an incompressible liquid drop with surface tension restoring force, observed in a local *inertial* frame that moves in either direction along a $\Delta x_{\text{shell}}$ tangential line. This figure shows results for high tangential speed $v_{\text{shell},x}$: both the tidal stretch in the $\Delta y_{\text{shell}}$ direction and the tidal squeeze in the $\Delta z_{\text{shell}}$ direction are huge, much greater than the tidal squeeze in the $\Delta x_{\text{local}}$ direction. The resulting shape: a thin ribbon with rounded ends lying in the $\Delta x_{\text{shell}}$, $\Delta y_{\text{shell}}$ plane.

$$\Delta g_{\text{local},y} \approx \left(\frac{1+v^2_{\text{shell},x}/2}{1-v^2_{\text{shell},x}}\right)\frac{2M}{\bar{r}^3}\Delta y_{\text{local}} \tag{46}$$

$$\Delta g_{\text{local},x} \approx -\frac{M}{\bar{r}^3}\Delta x_{\text{local}} \tag{47}$$

$$\Delta g_{\text{local},z} \approx -\left(\frac{1+2v^2_{\text{shell},x}}{1-v^2_{\text{shell},x}}\right)\frac{M}{\bar{r}^3}\Delta z_{\text{local}} \tag{48}$$

Subscript "local" means *any* local frame moving tangentially in either direction in global coordinates.

Limiting cases for tangential motion

Notice that equation (47) is the same as equation (39) for radial motion, while the equations for the other two directions simply multiply the radial results by coefficients that depend on $v^2_{\text{shell},x}$. In the low-speed limit ($v^2_{\text{shell},x} \ll 1$), these equations also reduce to the radial ones (38) and (40). Finally, note that as $v_{\text{shell},x}$ increases toward the speed of light, the $y$ component leads to radical stretching, while the $z$ component leads to much greater tangential compression than that in the $\Delta x_{\text{local}}$ direction.

Expressions (46) through (48) also satisfy the general relation (45) among the local components of gravity gradient, which preserves the volume of a tiny dust cloud moving in the map tangential direction.

For a local *inertial frame*, the result is the tidal distortion of a drop of water or liquid mercury into a flat ribbon with rounded ends, shown in Figure

10 for tangential motion. Equations (46) through (48) are correct for *any* value of $v_{\text{shell},x}$, not just the value of a stone's local shell speed when it is in a circular orbit. For example, a stone that approaches a black hole from far away and returns to far away will travel tangentially at its point of closest approach; these three equations apply at this point.

Section 9.3 applies these results to find the minimum-$r$ circular orbit for human comfort.

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**QUERY 8. Departure from Newton's gravity gradient**

Expressions in parentheses on the right sides of (46) and (48) are a measure of the departure of Einstein's gravity gradients from those predicted by Newton. Temporarily call these expressions **Einstein multipliers**.

A. For what values of $v_{\text{shell},x}$ does the largest of the Einstein multipliers become "significant," which we define as the value 1.1?

B. For what values of $v_{\text{shell},x}$ does the largest of the Einstein multipliers become "large," which we define as the value 10?

C. Exercise 5 in Chapter 1 analyzes the highest energy cosmic ray so far detected, with an energy of $3 \times 10^{20}$ electron volts. Let this cosmic ray be a speeding proton (mass $= 1.63 \times 10^{-27}$ kilogram $= 9.38 \times 10^{8}$ electron-volts) that streaks tangentially past Earth just above its atmosphere, about 100 kilometers above the surface. Estimate the value of the largest Einstein multiplier in this case. Hint: Define $v_{\text{shell},x} \equiv 1 - \delta$, then use our approximation formula from inside the front cover to redefine the Einstein multipliers in terms of $\delta$.

D. The proton is a quantum particle; its "radius" is not a classical quantity. Nevertheless, estimate the tidal stress on the proton cosmic ray of Part C: Assume this proton radius to be $10^{-15}$ meter. What are the tidal accelerations at the surface of the "fastest proton" moving tangentially above Earth's atmosphere?

E. Repeat Part D for the "fastest proton" skimming past the surface of a neutron star with $r/M = 10$ kilometers.

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## 9.8 ■ EXERCISES

### 1. Smallest circular orbit for a hardened probe around the black hole

We harden a probe so that it can withstand $K$ times the maximum comfortable tidal acceleration of a human (Section 9.6). The probe enters a circular orbit around the black hole of mass $M$ in which the tidal acceleration has this maximum. What is the $r$-value of this circular orbit?

### 2. The "Perfect" (Star Trek) Rocket

An advanced civilization develops the "perfect" rocket engine, one that combines matter and antimatter in a controlled way to yield photons

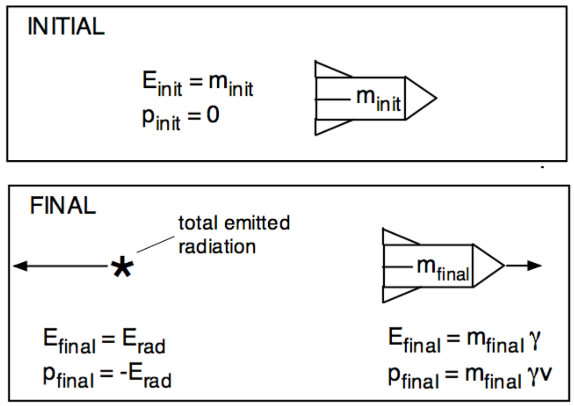


**FIGURE 11** Exercise 2. Diagram showing initial and final states of a "perfect" rocket that emits only radiation.

(high-energy gamma rays), all of which it directs out the rear of the rocket. This is called the "perfect" rocket engine because it has the greatest possible change of velocity in flat spacetime for a given fractional change in mass of the rocket ship. Analyze the perfect rocket using special relativity, including the definition $\gamma \equiv (1 - v^2)^{-1/2}$.

A. Write down the energy and momentum conservation laws using Figure 11.
B. Combine the conservations laws, show that $\gamma v = \left(\gamma^2 - 1\right)^{1/2}$, and derive the equation for the *mass ratio*:

$$\frac{m_{\text{init}}}{m_{\text{final}}} = \gamma + \left(\gamma^2 - 1\right)^{1/2} \quad \text{(photon rocket, flat spacetime)} \qquad (49)$$

where $m_{\text{init}}$ is the initial mass of the rocket ship.
C. Find the mass ratio for $\gamma = 10$
D. Show that the result of Part C is an example of the approximation

$$\frac{m_{\text{init}}}{m_{\text{final}}} \approx 2\gamma \quad (\text{when } \gamma^2 \gg 1) \quad \text{(photon rocket, flat spacetime)} \quad (50)$$

### 3. Newton's Tangential Tidal Displacement Near Earth.

Brave Monica Sefner "walks the plank" at the top of the 828-meter-tall Dubai Tower, Burj Khalifa (Figure 12), on which she moves horizontally outward to a point that clears the base of the tower. Then she steps off the plank attached to a bungee cord and falls freely for 600 meters, at which point the cord "takes hold" and slows her to a stop before she reaches the ground. As she leaves the plank, Monica stretches out her arms and releases from rest two marbles


**FIGURE 12** Exercise 3. DubaiTower, 828 meters high.

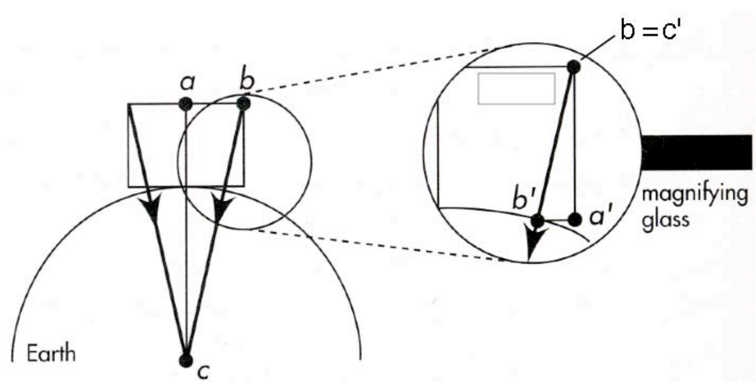


**FIGURE 13** Exercise 3. Construction to analyze tangential tidal acceleration of radially falling marbles in Newton's mechanics. Not to scale, and with gross differences in relative scale of different parts of the diagram.

initially 2 meters apart horizontally. Just before the end of her 600-meter free fall, how much will the measured separation between these marbles have decreased? Will Monica be able to measure this decrease in separation? To answer these questions, use the following method of similar triangles (Figure 13) or your own method.

> Assume that the air neither slows down nor deflects either marble from its straight-line course. Then each marble falls from rest toward the center of Earth, as indicated by arrows in Figure 13. Solve the problem using the ratio of sides of similar triangles $abc$ and $a'b'c'$. These triangles

are upside down with respect to one another, but they are similar because their respective sides are parallel. We know the lengths of some of these sides (some greatly exaggerated in the figure): Side $b'c' = 600$ meters; side $bc$ is effectively equal to the $r$-coordinate of Earth; side $ab = 1$ meters equals half of the original separation of the marbles; side $a'b'$ equals *half the change* in their separation after a drop of 600 meters.

A. Use the ratio of sides of similar triangles to find the "half change" in separation as the two marbles fall 600 meters. From this result, find the entire change in separation between the marbles.

B. Suppose that, as she steps off the plank, Monica releases the two marbles from rest with a *vertical* separation of 2 meters. From Newton's equations (36) and (37), find the increase in separation of two marbles after they fall 600 meters, under the assumption that the marbles fall in a vacuum.)

C. Re-derive your result of Part A using the simpler Part B plus equation (45).

### 4. Measure your global radial coordinate $r$ near a black hole?

You are the captain of a spaceship with rockets blasting as you descend slowly toward a black hole along a radial line. In effect, you stand for a minute on each shell, then step downward sequentially to the next shell below. From earlier observations you know the value of the black hole mass $M$ and would like to measure your map $r$-coordinate in order to be sure you are not near the event horizon.

A. Describe how you can determine $r$ from the initial acceleration of a test particle as you descend.

B. Oops! Is there a paradox here? You have measured a map quantity, $r$, using observations on a local shell. Isn't that illegal?

### 5. Spaceship approach at relativistic speed

The present chapter assumes that the approaching spaceship moves slowly—not at relativistic speed—with respect to the black hole, so that $E/m \approx 1$. But the captain of the approaching spaceship does not want to waste valuable rocket fuel to slow down in order to apply the analysis of this chapter. She decides not to reduce the large value of her map energy $E/m$ (with respect to the black hole) and instead to use her main thrusters to adjust the value of her map angular momentum $L/(mM)$ so that she moves directly to a knife-edge orbit. If the rocket thrust that increase $L/m$ also increases $E/m$, no problem: Just use the final value of $E/m$ in what follows.

A. For a large value of map energy $E/m \gg 1$, the $r$-value of the knife-edge orbit is only slightly greater than 3M. Set $r/M = 3(1+\delta)$ in (8). Show that:

$$\frac{E}{m} \approx \frac{1}{3\delta^{1/2}} \qquad (E/m \gg 1,\ \text{knife-edge orbit}) \tag{51}$$

so that for the given large value of $E/m$,

$$\delta^{1/2} \approx \frac{m}{3E} \qquad (E/m \gg 1,\ \text{knife-edge orbit}) \tag{52}$$

B. Show that for this case, equation (6) for the knife-edge orbit becomes:

$$\frac{L}{mM} \approx \left(\frac{3}{\delta}\right)^{1/2} = 3^{3/2}\frac{E}{m} \qquad (E/m \gg 1,\ \text{knife-edge orbit}) \tag{53}$$

C. When observations are complete, how does the commander move away from the black hole? Give a general description of this maneuver; don't sweat the details.

### 6. Swoop Orbit

Figure 14 shows the effective potential for a so-called **swoop orbit** of a stone whose map energy $E/m$ is slightly smaller than that of the effective potential peak at small $r$-value.

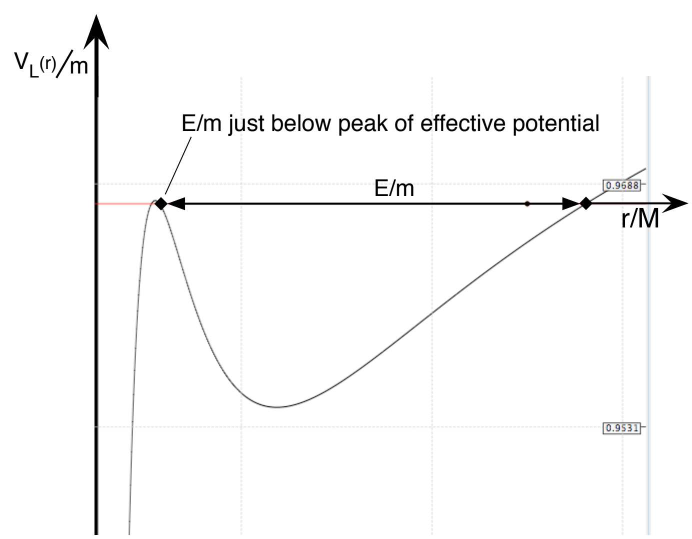


**FIGURE 14** Exercise 6: Effective potential for the **swoop orbit** of a stone with map energy $E/m$ just below the (left-hand peak) of the effective potential.

A. Make a rough sketch of the swoop orbit on the $[r, \phi]$ slice. *Optional:* Use interactive softward GRorbits to create and print this swoop orbit.

Luc Longtin is a junior engineer at the Space Agency. He claims that with a small rocket thrust he can put the entire incoming spaceship into a swoop orbit that oscillates between $r = 4M$ and $r = 100M$. This will allow direct observations from the spaceship at $r$-values between these two limits, completely eliminating the need for probes.

The Space Agency rejects Luc's plan as too risky. Luc invites you, the Chief Engineer, to a bar where he tries to convince you to that the Space Agency should reverse its decision and use his plan. Luc lays out his proposal as follows:

B. Luc begins, "Look at the effective potential for $L/(mM) = 4$ in Figure 6. The inner peak of this effective potential is at $r = 4M$ with $E/m = 1$ and the spaceship approaches from far away with $E/m = 1 + \epsilon$, where $\epsilon = 0.001$. My plan is that when the spaceship reaches, say $r = 20$, it uses a tiny rocket thrust to flip its map energy to $E/m = 1 - \epsilon$ without changing its angular momentum (so the effective potential does not change). Let engineers worry about details of that thrust; just look at the result. The spaceship enters a swoop orbit that bounces off the effective potential peak just outside $r = 4M$. At that bounce, $dr/d\tau = 0$, so equation (17) in Section 8.4 becomes"

$$\frac{dr}{d\tau} = 0 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right) \qquad (54)$$

$$0 = (1-\epsilon)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{16M^2}{r^2}\right) \qquad (55)$$

$$0 = 32\left(\frac{M}{r}\right)^3 - 16\left(\frac{M}{r}\right)^2 + 2\left(\frac{M}{r}\right) - \left[1 - (1-\epsilon)^2\right] \qquad (56)$$

Fill in the steps between (55) and (56).

C. Luc continues, "We set up equation (56) for the bounce point near $r = 4M$. But this equation has only global map quantities in it, so is also correct for the bounce point at the large $r$-value at the outward end of the swoop orbit. At this large $r$-value, the first term on the right of (56) is small compared to the other terms, so neglect this first term. What remains is a quadratic in the small quantity $M/r$. Solve this quadratic to show that the only acceptable solution for large $r/M$ is $M/r = \epsilon$ or $r = M/\epsilon = 100M$ for the right-hand bounce point of the swoop orbit."

Verify Luc's calculations.

C. Luc concludes, "So a very small rocket thrust installs the entire incoming spaceship in a swoop orbit that moves in and out between

$r = 100M$ and an $r$-value slightly greater than $r = 4M$. No need for those silly probes. Astronauts can make observations in this orbit as long as they want as they move in and out. When they finish, a small rocket thrust similar to that described in Item B (during the outgoing portion of its orbit) flips the spaceship map energy back to $E/m = 1 + \epsilon$, so the spaceship escapes the black hole."

Do you agree with this part of Luc's plan?

Will you recommend Luc's program to the Space Agency?

## 9.9 ■ REFERENCES

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Exercise 3 on tidal displacement near Earth and Figure 11 adapted from Edwin F. Taylor and John Archibald Wheeler, *Spacetime Physics*, second edition, 1990, W. H. Freeman, New York, page 46.

Dubai Tower image from Skidmore, Owings and Merrill at `http://www.emporis.com/application/?nav=building&lng=3&id=182168*`

Most of the orbit plots in this chapter were made using the interactive software GRorbits.