Chapter 10
Advance of Mercury's Perihelion

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- *What does "advance of the perihelion" mean?*
- *You say Newton does not predict any advance of Mercury's perihelion in the absence of other planets. Why not?*
- *The advance of Mercury's perihelion is tiny. So why should we care?*
- *Why pick out Mercury? Doesn't the perihelion of every planet change with Earth-time?*
- *You are always shouting at me to say whose time measures various motions. Why are you so sloppy about time in analyzing Mercury's orbit?*

CHAPTER 10 Advance of Mercury's Perihelion

Edmund Bertschinger & Edwin F. Taylor *

This discovery was, I believe, by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him. He had to be right. "For a few days, I was beside myself with joyous excitement." Later, he told Fokker that his discovery had given him palpitations of the heart. What he told de Haas is even more profoundly significant: when he saw that his calculations agreed with the unexplained astronomical observations, he had the feeling that something actually snapped in him.

—Abraham Pais

10.1 ■ JOYOUS EXCITEMENT

Tiny effect; large significance.

"Perihelion precession"?

What discovery sent Einstein into "joyous excitement" in November 1915? It was his calculation showing that his brand new (not quite completed) theory of general relativity gave the correct value for one detail of the orbit of the planet Mercury that had not been previously explained, an effect with the technical name **precession of Mercury's perihelion**.

Newton: Sun-Mercury perihelion fixed.

Mercury (and every other planet) circulates around the Sun in a not-quite-circular orbit. In this orbit it oscillates in and out radially while it circles tangentially. A full Newtonian analysis predicts an elliptical orbit. Newton tells us that if we consider only the interaction between Mercury and the Sun, then the time for one 360-degree trip around the Sun is *exactly* the same as the time for one in-and-out radial oscillation. Therefore the orbital point closest to the Sun, the so-called **perihelion,** stays in the same place; the elliptical orbit does not shift around with each revolution—according to Newton. You will begin by verifying his nonrelativistic prediction for the simple Sun-Mercury system.

However, observation shows that Mercury's orbit does indeed change. The perihelion moves forward in the direction of rotation of Mercury; it *advances*

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*
 Latest drafts at dropsite exploringblackholes.com.

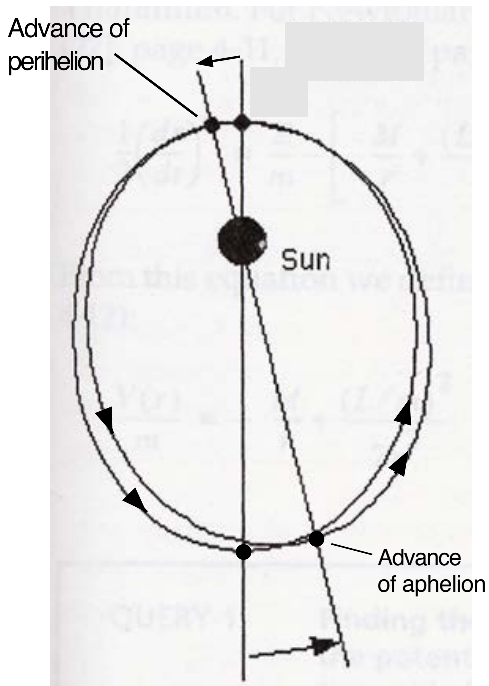

FIGURE 1 Exaggerated view of the advance, during one century, of Mercury's perihelion (and aphelion). The figure shows two elliptical orbits. One of these orbits is the one that Mercury traces over and over again in the year, say, 1900. The other is the elliptical orbit that Mercury traces over and over again in the year, say, 2000. The two are shifted with respect to one another, a rotation called *the advance (or precession) of Mercury's perihelion.* The unaccounted-for precession in one Earth-century is about 43 arcseconds, less than the thickness of a line in this figure.

Observation: perihelion advances.

with each orbit (Figure 1). The long ("major") axis of the ellipse rotates. We call this rotation of the axis the **advance (or precession) of the perihelion**.

The **aphelion** is the point of the orbit farthest from the Sun; it advances at the same angular rate as the perihelion (Figure 1).

Newton: Influence of other planets, predicts most of the perihelion advance . . .

. . . but leaves a *residual.*

Observation shows that the perihelion of Mercury precesses at the rate of 574 arcseconds (0.159 degree) *per Earth-century.* (One degree equals 3600 arcseconds.) Newton's mechanics accounts for 531 seconds of arc of this advance by computing the perturbing influence of the other planets. But a stubborn 43 arcseconds (0.0119 degree) per Earth-century, called a **residual**, remains after all these effects are accounted for. This residual (though not its modern value) was computed from observations by Urbain Le Verrier as early as 1859 and more accurately later by Simon Newcomb (Box 1). Le Verrier attributed the residual in Mercury's orbit to the presence of an unknown inner planet, tentatively named Vulcan. We know now that there is no planet Vulcan. (Sorry, Mr. Spock!)

Box 1. Simon Newcomb

FIGURE 2 Simon Newcomb
Born 12 March 1835, Wallace, Nova Scotia.
Died 11 July 1909, Washington, D.C.
(Photo courtesy of Yerkes Observatory)

From 1901 until 1959 and even later, the tables of locations of the planets (so-called **ephemerides**) used by most astronomers were those compiled by Simon Newcomb and his collaborator George W. Hill.

By the age of five Newcomb was spending several hours a day making calculations, and before the age of seven was extracting cube roots by hand. He had little formal education but avidly explored many technical fields in the libraries of Washington, D. C. He discovered the *American Ephemeris and Nautical Almanac,* of which he said, "Its preparation seemed to me to embody the highest intellectual power to which man had ever attained."

Newcomb became a "computer" (a person who computes) in the American Nautical Almanac office and by stages rose to become its head. He spent the greater part of the rest of his life calculating the motions of bodies in the solar system from the best existing data. Newcomb collaborated with Q. M. W. Downing to inaugurate a worldwide system of astronomical constants, which was adopted by many countries in 1896 and officially by all countries in 1950.

The advance of the perihelion of Mercury computed by Einstein in 1914 would have been compared to entries in the tables of Simon Newcomb and his collaborator.

Einstein correctly predicts residual precession.

Newton's mechanics says that there should be *no residual* advance of the perihelion of Mercury's orbit and so cannot account for the 43 seconds of arc per Earth-century which, though tiny, is nevertheless too large to be ignored or blamed on observational error. But Einstein's general relativity accounted for the extra 43 arcseconds on the button. Result: joyous excitement!

Method: Compare in-and-out time with round-and-round time for Mercury.

Preview, Newton: This chapter begins with Newton's approximations that lead to his no-precession conclusion (in the absence of other planets). Mercury moves in a near-circular orbit; Newton calculates the time for one orbit. The approximation also describes the small radial in-and-out motion of Mercury as if it were a harmonic oscillator moving back and forth about a potential energy minimum (Figure 3). Newton calculates the time for one in-and-out radial oscillation and compares it with the time for one orbit. The orbital and radial oscillation T-values are exactly equal (according to Newton), provided one considers only the Mercury-Sun interaction. He concludes that Mercury circulates around once in the same time that it oscillates radially inward and back out again. The result is an elliptical orbit that closes on itself. In the absence of other planets, Mercury repeats this exact elliptical path forever—according to Newton.

Preview, Einstein: In contrast, our general relativity approximation shows that these two times—the orbital round-and-round and the radial in-and-out T-values—are *not quite equal.* The radial oscillation takes place more slowly, so that by the time Mercury returns to its inner limit, the

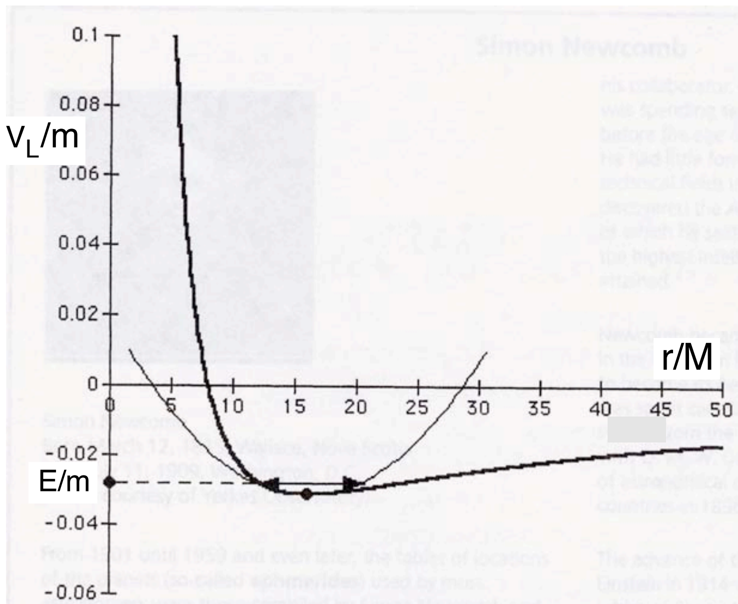

FIGURE 3 Newton's effective potential, equation (5) (heavy curve), on which we superimpose the parabolic potential of the simple harmonic oscillator (thin curve) with the shape given by equation (3). Near the minimum of the effective potential, the two curves closely conform to one another.

circular motion has carried it farther around the Sun than it was at the preceding minimum r-coordinate. From this difference Einstein reckons the residual angular rate of advance of Mercury's perihelion around the Sun and shows that this predicted difference is close to the observed residual advance. Now for the details.

Comment 1. Relaxed about Newton's time and coordinate T
In this chapter we speak freely about Newton's time or Einstein's change in global T-value, without worrying about which we are talking about. We get away with this sloppiness for two reasons: (1) All observations are made from Earth's surface. Every statement about time should in principle be followed by the phrase, "as observed on Earth." (2) For this system, the effects of spacetime curvature on the rates of local clocks are so small that all time or T-measures give essentially the same rate of precession, as summarized in Section 10.11.

10.2 ■ NEWTON'S SIMPLE HARMONIC OSCILLATOR

Assume radial oscillation is sinusoidal.

Why does the planet oscillate in and out radially? Look at the effective potential in Newton's analysis of motion, the heavy line in Figure 3. This heavy line has a minimum, the location at which the planet can ride around at constant r-value, tracing out a circular orbit. But with a slightly higher energy, it not only moves tangentially, it also oscillates radially in and out, as shown by the two-headed arrow in Figure 3.

How long does it take for one in-and-out oscillation? That depends on the shape of the effective potential curve near the minimum shown in Figure 3. But if the amplitude of the oscillation is small, then the effective part of the curve is very close to this minimum, and we can use a well-known mathematical theorem: If a continuous, smooth curve has a local minimum, then near that minimum a parabola approximates this curve. Figure 3 shows such a parabola (thin curve) superimposed on the (heavy) effective potential curve. From the diagram it is apparent that the parabola is a good approximation of the potential, at least near that local minimum.

In-and-out motion in parabolic potential . . .

. . . predicts simple harmonic motion.

From introductory Newtonian mechanics, we know how a particle moves in a parabolic potential. The motion is called **simple harmonic oscillation**, described by the following expression:

$$x = \mathrm{A}\sin\omega t \qquad (1)$$

Here A is the amplitude of the oscillation and ω (Greek lower case omega) tells us how rapidly the oscillation occurs in radians per unit time. The potential energy per unit mass, V/m, of a particle oscillating in a parabolic potential follows the formula

$$\frac{V}{m} = \frac{1}{2}\omega^2 x^2 \qquad (2)$$

To find the rate of oscillation ω of the harmonic oscillator, take the second derivative with respect to x of both sides of (2).

$$\frac{d^2\left(V/m\right)}{dx^2} = \omega^2 \qquad (3)$$

10.3 ■ NEWTON'S ORBIT ANALYSIS

Round and round vs. in and out

The in-and-out radial oscillation of Mercury does not take place around $r = 0$ but around the r-value of the effective potential minimum. What is the r-coordinate of this minimum (call it r_0)? Start with Newton's equation (25) in Section 8.4:

Newton's equilibrium r_0

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \left(-\frac{M}{r} + \frac{L^2}{2m^2r^2}\right) = \frac{E}{m} - \frac{V_{\mathrm{L}}(r)}{m} \qquad \text{(Newton)} \qquad (4)$$

This equation defines the effective potential,

$$\frac{V_{\mathrm{L}}(r)}{m} \equiv -\frac{M}{r} + \frac{L^2}{2m^2r^2} \qquad \text{(Newton)} \qquad (5)$$

To locate the minimum of this effective potential, set its derivative equal to zero:

$$\frac{d(V_{\mathrm{L}}/m)}{dr} = \frac{M}{r^2} - \frac{L^2}{m^2r^3} = 0 \qquad \text{(Newton)} \qquad (6)$$

Solve the right-hand equation to find r_0, the r-value of the minimum:

$$r_0 = \frac{L^2}{Mm^2} \qquad \text{(Newton, equlibrium radius)} \qquad (7)$$

Newton: In-and-out time equals round-and-round time.

We want to compare the rate ω_{r} of in-and-out radial motion of Mercury with its rate ω_ϕ of round-and-round tangential motion. Use Newton's definition of angular momentum, with increment dt of Newton's universal time, similar to equation (10) of Section 8.2:

$$\frac{L}{m} \equiv r^2\frac{d\phi}{dt} = r^2\omega_\phi \qquad \text{(Newton)} \qquad (8)$$

where $\omega_\phi \equiv d\phi/dt$. Equation (8) gives us the angular velocity of Mercury along its almost-circular orbit.

Queries 1 and 2 show that for Newton the radial in-and-out angular velocity ω_{r} is equal to the orbital angular velocity ω_ϕ.

QUERY 1. Newton's angular velocity ω_ϕ of Mercury in orbit.
Set $r = r_0$ in (8) and substitute the result into (7). Show that at the equilibrium radius, $\omega_\phi^2 = M/r_0^3$ for Newton.

QUERY 2. Newton's radial oscillation rate ω_r for Mercury's orbit
We want to use (3) to find the angular rate of radial oscillation. Accordingly, take the second derivative of V_{L} in (5) with respect to r. Set $r = r_0$ in the resulting expression and substitute your value for L^2 in (7). Use (3) to show that at Mercury's orbital radius, $\omega_r^2 = M/r_0^3$, according to Newton.

Important result: *For Newton, Mercury's perihelion does not advance when one considers only the gravitational interaction between Mercury and the Sun.*

10.4 ■ EFFECTIVE POTENTIAL: EINSTEIN

Extra effective potential term advances perihelion.

Now we repeat the analysis of radial and tangential orbital motion for the general relativistic case. Chapter 9 predicts the radial motion of an orbiting satellite. Multiply equations (4) and (5) of Section 9.1 through by 1/2 to obtain an equation similar to (4) above for the Newton's case:

$$\begin{aligned}\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 &= \frac{1}{2}\left(\frac{E}{m}\right)^2 - \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right) \qquad (9)\\ &= \frac{1}{2}\left(\frac{E}{m}\right)^2 - \frac{1}{2}\left(\frac{V_\mathrm{L}(r)}{m}\right)^2 \qquad \text{(Einstein)}\end{aligned}$$

Set up general relativity effective potential.

Equations (4) and (9) are of similar form, and we use this similarity to make a general relativistic analysis of the harmonic radial motion of Mercury in orbit. In this process we adopt the *algebraic manipulations* of Newton's analysis in Sections 10.2 and 10.3 but apply them to the general relativistic expression (9).

Before we proceed, note three characteristics of equation (9). First, $d\tau$ on the left side of (9) is the differential wristwatch time $d\tau$, not the differential dt of Newton's universal time t. This different reference time is not necessarily fatal, since we have not yet decided which relativistic measure of time should replace Newton's universal time t. You will show in Section 10.11 that for Mercury the choice of which time to use (wristwatch time, global map T-coordinate, or even shell time at the r-value of the orbit) makes a negligible difference in our predictions about the rate of advance of the perihelion.

Different time rates of different clocks do not matter.

Note, second, that in equation (9) the relativistic expression $(E/m)^2$ stands in the place of the Newtonian expression E/m in (4). However, both are constant quantities, which is all that matters in the analysis.

Evidence that we are on the right track results when we multiply out the second term of the first line of (9), which is the square of the effective potential, equation (20) of Section 8.4, with the factor one-half. Note that we have assigned the symbol $(1/2)(V_\mathrm{L}/m)^2$ to this second term.

$$\begin{aligned}\frac{1}{2}\left(\frac{V_\mathrm{L}(r)}{m}\right)^2 &= \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right) \qquad \text{(Einstein)} \qquad (10)\\ &= \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2m^2r^2} - \frac{ML^2}{m^2r^3}\end{aligned}$$

Details of relativistic effective potential

The heavy curve in Figure 4 plots this function. The second line in (10) contains the two effective potential terms that made up the Newtonian expression (5). The final term on the right of the second line of (10) describes an added attractive potential from general relativity. For the Sun-Mercury case at the r-value of Mercury's orbit, this term leads to the slight precession of the elliptical orbit. As r becomes small, the r^3 in the denominator causes this term to overwhelm all other terms in (10), which results in the downward plunge in the effective potential at the left side of Figure 4.

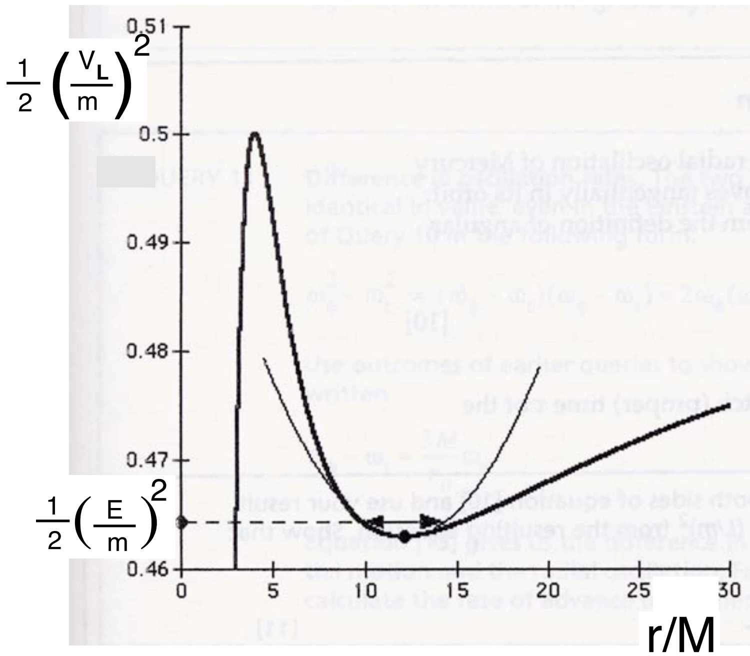

FIGURE 4 General-relativistic effective potential $(V_L/m)^2/2$ (heavy curve) and its approximation at the local minimum by a parabola (light curve) in order to analyse the radial excursion (double-headed arrow) of Mercury as simple harmonic motion. The effective potential curve is for a black hole, not for the Sun, whose effective potential near the potential minimum would be indistinguishable from the Newton's effective potential on the scale of this diagram. However, this minute difference accounts for the tiny residual precession of Mercury's orbit.

Finally, note third that the last term $(1/2)(V_L/m)^2$ in relativistic equation (9) takes the place of the Newton's effective potential V_L/m in equation (4).

In summary, we can manipulate general relativistic expressions (9) and (10) in nearly the same way that we manipulated Newton's expressions (4) and (5) in order to analyze the radial component of Mercury's motion and small perturbations of Mercury's elliptical orbit brought about by general relativity.

10.5 ■ EINSTEIN'S ORBIT ANALYSIS

Einstein tweaks Newton's solution.

Now analyze the radial oscillation of Mercury's orbit according to Einstein.

QUERY 3. Local minimum of Einstein's effective potential

Take the first derivative of the squared effective potential (10) with respect to r, that is find $d[(1/2)(V_L/m)^2]/dr$. Set this first derivative aside for use in Query 4. As a separate calculation, equate this derivative to zero, set $r = r_0$, and solve the resulting equation for the unknown quantity $(L/m)^2$ in terms of the known quantities M and r_0.

QUERY 4. Einstein's radial oscillation rate $\omega_{\rm r}$ for Mercury in orbit.
We want to use (3) to find the rate of oscillation $\omega_{\rm r}$ in the radial direction.

A. Take the second derivative of $(1/2)(V_{\rm L}/m)^2$ from (10) with respect to r. Set the resulting $r = r_0$ and substitute the expression for $(L/m)^2$ from Query 3 to obtain

$$\left[\frac{d^2}{dr^2}\left(\frac{1}{2}\frac{V_{\rm L}^2}{m^2}\right)\right]_{r=r_0} = \omega_r^2 = \frac{M}{r_0^3}\frac{\left(1-\dfrac{6M}{r_0}\right)}{\left(1-\dfrac{3M}{r_0}\right)} \qquad \text{(Einstein)} \qquad (11)$$

$$\approx \frac{M}{r_0^3}\left(1-\frac{6M}{r_0}\right)\left(1+\frac{3M}{r_0}\right) \qquad (12)$$

$$\approx \frac{M}{r_0^3}\left(1-\frac{3M}{r_0}\right) \qquad (13)$$

where we have made repeated use of the approximation inside the front cover in order to find a result to first order in the fraction M/r.

B. For our Sun, $M \approx 1.5 \times 10^3$ meters, while for Mercury's orbit $r_0 \approx 6 \times 10^{10}$ meters. Does the value of M/r_0 justify the approximations in equations (12) and (13)?

Note that the coefficient M/r_0^3 in these three equations equals Newton's expression for $\omega_{\rm r}^2$ derived in Query 1.

Now compare $\omega_{\rm r}$, the in-and-out oscillation of Mercury's orbital r-coordinate with the angular rate ω_ϕ with which Mercury moves tangentially in its orbit. The rate of change of azimuth ϕ springs from the definition of angular momentum in equation (10) in Section 8.2:

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} \qquad \text{(Einstein)} \qquad (14)$$

Note the differential wristwatch time $d\tau$ for the planet.

QUERY 5. Einstein's angular velocity
Square both sides of (14) and use your result from Query 3 to eliminate L^2 from the resulting equation. Show that at the equilibrium r_0 the result can be written

$$\omega_\phi^2 \equiv \left(\frac{d\phi}{d\tau}\right)^2 = \frac{M}{r_0^3}\left(1-\frac{3M}{r_0}\right)^{-1} \qquad \text{(Einstein)} \tag{15}$$

$$\approx \frac{M}{r_0^3}\left(1+\frac{3M}{r_0}\right) \tag{16}$$

where again we use our approximation inside the front cover. Compare this result with equation (13) and with Newton's result in Query 1.

10.6 ■ PREDICT MERCURY'S PERIHELION ADVANCE

Simple outcome, profound consequences

Einstein: in-out rate differs from circulation rate.

According to Einstein, the advance of Mercury's perihelion springs from the difference between the frequency with which the planet sweeps around in its orbit and the frequency with which it oscillates in and out in r. In Newton's analysis these two frequencies are equal (for the interaction between Mercury and the Sun). But Einstein's theory shows that these two frequencies are *slightly* different; Mercury reaches its minimum r (its perihelion) at an incrementally greater angular position in each successive orbit. *Result:* the advance of Mercury's perihelion. In this section we compare Einstein's prediction with observation. But first we need to define what we are calculating.

What do we mean by the phrase "the period of a planet's orbit"? The period with respect to what? Here we choose what is technically called the **synodic period** of a planet, defined as follows:

Definition: synodic period

DEFINITION 1. Synodic period of a planet

The **synodic period** of a planet is the lapse in time (Newton) or lapse in global T-value (Einstein) for the planet to revolve once around the Sun with respect to the fixed stars.

"Fixed" stars?

Comment 2. Fixed stars?

What are the "fixed stars"? Chapter 14 The Expanding Universe shows that stars are anything but fixed. With respect to our Sun, stars move! However, stars that we now know to be very distant do not change angle rapidly from our point of view. Over a few hundred years—the lifetime of the field of astronomy itself—these stars may be called *fixed*.

The value T_r to make a complete in-and-out radial oscillation is

$$T_\mathrm{r} \equiv \frac{2\pi}{\omega_\mathrm{r}} \qquad \text{(period of radial oscillation)} \tag{17}$$

In global coordinate lapse T_r, Mercury goes around the Sun, completing an angle

$$\omega_\phi T_\mathrm{r} = \frac{2\pi\omega_\phi}{\omega_\mathrm{r}} = (\text{Mercury revolution angle in } T_\mathrm{r}) \tag{18}$$

which exceeds one complete revolution in radians by:

$$\omega_\phi T_\mathrm{r} - 2\pi = T_\mathrm{r}\left(\omega_\phi - \omega_\mathrm{r}\right) = (\text{excess angle per revolution}) \tag{19}$$

QUERY 6. Difference in Einstein's oscillation rates

The two angular rates ω_ϕ and ω_r are *almost* identical in value, even in the Einstein analysis. Therefore we can write approximately:

$$\omega_\phi^2 - \omega_\mathrm{r}^2 = (\omega_\phi + \omega_\mathrm{r})(\omega_\phi - \omega_\mathrm{r}) \approx 2\omega_\phi(\omega_\phi - \omega_\mathrm{r}) \tag{20}$$

A. Substitute equations (13) and (16) into the left side of (20):

$$\omega_\phi^2 - \omega_\mathrm{r}^2 \approx \frac{M}{r_0^3}\left[\left(1 + \frac{3M}{r_0}\right) - \left(1 - \frac{3M}{r_0}\right)\right] = \frac{M}{r_0^3}\frac{6M}{r_0} \tag{21}$$

B. Equation (20) becomes:

$$\omega_\phi^2 - \omega_\mathrm{r}^2 \approx \frac{M}{r_0^3}\frac{6M}{r_0} \approx \omega_\phi^2\frac{6M}{r_0} \approx 2\omega_\phi(\omega_\phi - \omega_\mathrm{r}) \tag{22}$$

C. Simplify the right-hand equation in (22), write the result as:

$$\omega_\phi - \omega_\mathrm{r} \approx \frac{3M}{r_0}\omega_\phi \qquad \text{(angular rates, Einstein)} \tag{23}$$

Equation (23) shows the difference in angular velocity between the tangential motion and the radial oscillation. From this rate difference we will calculate the advance of the perihelion of Mercury in one Earth-century.

Comment 3. What is X?

Symbols ω in (23) express rotation rates in radians per unit of—what? *Question:* What is X in the denominator of $d\phi/dX \equiv \omega$? Does X equal global coordinate T? planet wristwatch time τ? shell time t_shell at the average r-value of the orbit? *Answer:* It does not matter which of these quantities X represents, as long as this measure is the *same* on both sides of any resulting equation. Comment 1 told us to be relaxed about time. In the following Queries you use (23) to calculate the precession rate of Mercury in radians/second, then to convert this result to arcseconds/Earth-century.

10.7 ■ COMPARE PREDICTION WITH OBSERVATION

Check out Einstein!

Now compare our approximate relativistic prediction with observation.

QUERY 7. Mercury's angular velocity

The synodic period of Mercury's orbit is 7.602×10^6 seconds. To one significant digit, $\omega_\phi \approx 8 \times 10^{-7}$ radian/second. What is its value to three significant digits?

QUERY 8. Calculated coefficient

The mass M of the Sun is 1.477×10^3 meters and r_0 of Mercury's orbit is 5.80×10^{10} meters. To one significant digit, the coefficient $3M/r_0$ in (23) is 1×10^{-7}. Find this result to three significant digits.

QUERY 9. Advance of Mercury's perihelion in radians/second

From equation (23) and results of Queries 7 and 8, derive a numerical prediction of the advance of the perihelion of Mercury's orbit in radians/second. To one significant digit the result is 6×10^{-14} radians/second. Find the result to three significant digits.

QUERY 10. Advance of Mercury's perihelion in arcseconds per Earth-century.

Estimate the general relativity prediction of advance of Mercury's perihelion in arcseconds per century. Use results from preceding queries plus conversion factors inside the front cover plus the definition that 3600 arcseconds equals one degree. To one significant digit, the answer is 40 arcseconds/century. Find the result to three significant digits.

Observation and careful calculation agree.

A more accurate relativistic analysis predicts 42.980 arcseconds (0.011939 degrees) per Earth-century (Table 1). The observed rate of advance of the perihelion is in perfect agreement with this value: 42.98 ± 0.1 arcseconds per Earth-century. By what percentage did your prediction differ from observation?

10.8 ■ ADVANCE OF THE PERIHELIA OF THE INNER PLANETS

Help from a supercomputer.

All planet orbits precess.

Do the *perihelia* (plural of *perihelion*) of other planets in the solar system also advance as described by general relativity? Yes, but these planets are farther from the Sun, and their orbits are less eccentric, so the magnitude of the predicted advance is less than that for Mercury. In this section we compare our

TABLE 1 Advance of the perihelia of the inner planets

Planet	Advance of perihelion in seconds of arc per Earth-century (JPL calculation)	r-value of orbit in AU*	Period of orbit in years
Mercury	42.980 ± 0.001	0.38710	0.24085
Venus	8.618 ± 0.041	0.72333	0.61521
Earth	3.846 ± 0.012	1.00000	1.00000
Mars	1.351 ± 0.001	1.52368	1.88089

*Astronomical Unit (AU): average r-value of Earth's orbit; inside front cover.

estimated advance of the perihelia of the inner planets Mercury, Venus, Earth, and Mars with results of an accurate calculation.

Computer analysis of precessions.

The Jet Propulsion Laboratory (JPL) in Pasadena, California, supports an active effort to improve our knowledge of the positions and velocities of the major bodies in the solar system. For the major planets and the moon, JPL maintains a database and set of computer programs known as the Solar System Data Processing System. The input database contains the observational data measurements for current locations of the planets. Working together, more than 100 interrelated computer programs use these data and the relativistic laws of motion to compute locations of planets at in the past and the future. The equations of motion take into account not only the gravitational interaction between each planet and the Sun but also interactions among all planets, Earth's moon, and 300 of the most massive asteroids, as well as interactions between Earth and Moon due to nonsphericity and tidal effects.

JPL multi-program computation.

To help us with our project on perihelion advance, Myles Standish, Principal Member of the Technical Staff at JPL, kindly used the numerical integration program of the Solar System Data Processing System to calculate orbits of the four inner planets over four centuries, from A.D. 1800 to A.D. 2200. In an overnight run he carried out this calculation twice, first with the full program including relativistic effects and second "with relativity turned off." Standish "turned off relativity" by setting the speed of light to 10^{10} times its measured value, making light speed effectively infinite.

For each of the two runs, the perihelia of the four inner planets were computed for the four centuries. The results from the nonrelativistic run were subtracted from those of the relativistic run, revealing advances of the perihelia per Earth-century accounted for only by general relativity. The second column of Table 1 shows the results, together with the estimated computational error.

QUERY 11. Approximate advances of the perihelia of the inner planets

Compare the JPL-computed advances of the perihelia of Venus, Earth, and Mars in Table 10.1 with approximate results calculated using equation (23).

10.9 ■ CHECK THE STANDARD OF TIME

Whose clock?

We have been casual about whose time tracks the advance of the perihelion of Mercury and other planets; we even treated the global T-coordinate as a time, which is against our usual rules. Does this invalidate our approximations?

QUERY 12. Difference between shell time and Mercury's wristwatch time.
Use special relativity to find the fractional difference between planet Mercury's wristwatch time increment $\Delta\tau$ and the time increment $\Delta t_{\mathrm{shell}}$ read on shell clocks at the same average r_0 at which Mercury moves in its orbit at the average velocity 4.8×10^4 meters/second. By what fraction does a change of time from $\Delta\tau$ to $\Delta t_{\mathrm{shell}}$ change the total angle covered in the orbital motion of Mercury in one century? Therefore by what fraction does it change the predicted angle of advance of the perihelion in that century?

QUERY 13. Difference between shell time and global rain map T.
Find the fractional difference between shell time increment $\Delta t_{\mathrm{shell}}$ at r_0 and global map increment ΔT for r_0 equal to the average r-value of the orbit of Mercury. By what fraction does a change from $\Delta t_{\mathrm{shell}}$ to a lapse in global T alter the predicted angle of advance of the perihelion in that century?

QUERY 14. Does the time standard matter?
From your results in Queries 12 and 13, say whether or not the choice of a time standard—wristwatch time of Mercury, shell time, or map t—makes a detectable difference in the numerical prediction of the advance of the perihelion of Mercury in one Earth-century. Would your answer differ if the time were measured with clocks on Earth's surface?

DEEP INSIGHTS FROM MORE THAN THREE CENTURIES AGO

> *Newton himself was better aware of the weaknesses inherent in his intellectual edifice than the generations that followed him. This fact has always roused my admiration.*
>
> —Albert Einstein

We agree with Einstein. In the following quote from the end of his great work *Principia*, Isaac Newton summarizes what he knows about gravity and what he does not know. We find breathtaking the scope of what Newton says—and the integrity with which he refuses to say what he does not know. In the following, "feign" means "invent," and since Newton's time "experimental philosophy" has come to mean "physics."

"I do not 'feign' hypotheses."

> *Thus far I have explained the phenomena of the heavens and of our sea by the force of gravity, but I have not yet assigned a cause to gravity. Indeed, this force arises from some cause that penetrates as far as the centers of the sun and planets without any diminution of its power to act, and that acts not in proportion to the quantity of the* surfaces *of the particles on which it acts (as mechanical causes are wont to do) but in proportion to the quantity of* solid *matter, and whose action is extended everywhere to immense distances, always decreasing as the squares of the distances. Gravity toward the sun is compounded of the gravities toward the individual particles of the sun, and at increasing distances from the sun decreases exactly as the squares of the distances as far as the orbit of Saturn, as is manifest from the fact that the aphelia of the planets are at rest, and even as far as the farthest aphelia of the comets, provided that those aphelia are at rest. I have not as yet been able to deduce from phenomena the reason for these properties of gravity, and I do not "feign" hypotheses. For whatever is not deduced from the phenomena must be called a hypothesis; and hypotheses, whether metaphysical or physical, or based on occult qualities, or mechanical, have no place in experimental philosophy. In this experimental philosophy, propositions are deduced from the phenomena and are made general by induction. The impenetrability, mobility, and impetus of bodies, and the laws of motion and the law of gravity have been found by this method. And it is enough that gravity really exists and acts according to the laws that we have set forth and is sufficient to explain all the motions of the heavenly bodies and of our sea.*

—Isaac Newton

10.10 ■ REFERENCES

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