Standing Waves in Stretched Elastic String

I.   THEORETICAL BACKGROUND

In the measurement we will investigate the propagation of waves in a string with both end fixed. We assume that the propagation is one dimensional (along the string), the wave is transverse (the displacement vectors of the points of the string is perpendicular to the string), in-plane polarized (the displacement vectors lie in the same plane). It follows that displacement of a point from its equilibrium can be described with only one scalar value. Assuming the string is parallel to the x axis, the wave can be described with

$$c^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{\partial^2 \Psi(x,t)}{\partial t^2}, \qquad (1)$$

the one-dimensional wave-equation, where x is the coordinate, t is the time, $\Psi(x,t)$ is the wave-function, which describes the displacement as a function of place and time, i.e. the propagation of the wave, and c is the speed of wave propagation. Deriving the wave-equation for the case of a simple, one-dimensional string, the speed of the propagation, c only depends only on the stretching force (tension), T and the linear mass density of the string, μ:

$$c = \sqrt{\frac{T}{\mu}}. \qquad (2)$$

The waves formed by an external excitation in the string are generally very complicated. Although it is known that exciting at given frequencies stationary, so called *standing waves* are formed due to the reflection from the ends of the string. A property of the standing waves is that the points of the string are moving in-phase, and amplitude of the oscillation is only the function of the place. Mathematically it means that eq. (1) has a solution, which is the product of a time-dependent and a place-dependent function (the eq. (1) is separable partial differential equation). If the excitation is harmonic, the solution can be written as

$$\Psi(x,t) = \varphi(x)\sin(\omega t + \alpha), \qquad (3)$$

where $\omega = 2\pi\nu$ is the angular frequency of the oscillation (ν is the frequency, Hz), and α is the initial phase.

Substituting eq. (3) into eq. (1) the time-dependent part cancels, and place-dependent part, which describes the amplitude along the string, gives the following ordinary differential equation:

$$\frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2} + k^2\varphi(x) = 0, \qquad (4)$$

where $k = \omega/c$, the wavenumber is introduced.

The general solution of eq. (4) is

$$\varphi(x) = A\sin(kx) + B\cos(kx), \qquad (5)$$

where A and B are arbitrary constants, determined by the boundary conditions. In this case both ends of the string are fixed, so the displacement at these points is always zero, i.e.

$$\varphi(0) = 0$$
$$\varphi(L) = 0, \qquad (6)$$

where one end of the string lies in origin of the coordinate system, and length of the string is L.

The first condition of eq. (6) implies that $B = 0$, the solution can have the form of

$$\varphi(x) = A\sin(kx). \qquad (7)$$

The second condition of eq. (6) restricts the allowed values of the wavenumber to

$$k_n = n\frac{\pi}{L}, \ (n = 1, 2, 3, ...). \qquad (8)$$

Since the wavenumber, k is in direct connection with the wavelength λ ($k = 2\pi/\lambda$), eq. (8) implies that standing waves are only formed in case of given discrete value of wavelenghts:

$$\lambda_n = \frac{2L}{n}, \ (n = 1, 2, 3, ...). \qquad (9)$$

Combining this result with the $\nu = c/\lambda$ formula, in case of fixed speed of propagation (it means fixed tension and linear density according to eq. (2)) the frequency values cannot be arbitrary, either, only given discrete frequency values of

$$\nu_n = n\frac{c}{2L} \ (n = 1, 2, 3, ...) \qquad (10)$$

are allowed. These frequencies are the resonance frequencies of the string.

As it follows from the discussion above, the solutions which suit the boundary conditions can be written as

$$\varphi_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right) \ (n = 1, 2, 3, ...). \qquad (11)$$

The maximal values of the amplitudes, A_n are determined by the details of the excitation (the initial conditions), but for the given measurement it is not relevant. Assuming that the actual standing wave contains only one given n component, the solution of eq. (1) can be written as

$$\Psi_n(x,t) = A_n \sin\left(\frac{2\pi}{\lambda_n}x\right)\sin(\omega_n t + \alpha_n), \qquad (12)$$

where n is an arbitrary positive integer. The amplitude-distribution (place-dependence) of the standing wave is given by eq. (11), some examples are shown in FIG. 1, for $n = 1, 2$ and 3. As it can be seen from FIG. 1 and

eq. (11), the solution for a given n contains n antinodes and $n + 1$ nodes (taking into account the fixed ends of the string). From eq. (11) the distance of two consecutive nodes,

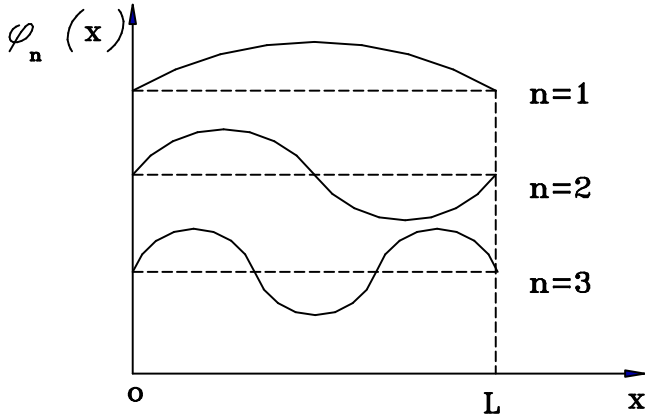$$d_n = \frac{L}{n} = \frac{\lambda_n}{2}. \tag{13}$$

FIG. 1. Possible first three modes for a stretched string fixed at the ends

According to eq. (10), different values of n result in different frequencies (pitches). The $n = 1$ is called fundamental frequency, and higher n values are called higher harmonics (overtones).

It is useful to note that exciting the string with an ordinary method (e.g. plucking a string) generally more harmonics appear at the same time. (Mathematically it means that the full solution of eq. (1) is the sum of eq. (12)-like solutions, with different values of n). But a string has a well defined pitch, since usually the amplitude of the fundamental frequency is much larger than the amplitude of the higher harmonics. The higher harmonics are always present, and the ratio for the different higher harmonics gives the tone of the string. (I.e. the difference of the sound of different musical instruments.)

In the measurement sinusoidal excitation will be used, which enables us to form standing waves with different n by setting the proper frequencies.

II. MEASUREMENT SETUP

The measurement setup (see FIG. 2) is a stretched steel string (2) fixed on a board (1). The ends of the string is fixed to an aluminum block (4), which is movable with a screw (3), and to a two-armed lever (5). The length of the oscillating section of the string can be set with two stands (6). The oscillation is induced by an exciter (driver) coil (8) via magnetic coupling. The wave in the string is transverse, almost in-plane polarized. The oscillation in the string is detected by a detector coil (9), whose signal (together with the excitation signal) is processed by

NI myDAQ data collector (FIG. 3). The driver coil is driven with this data collector, as well. The stretching force can be changed by putting weight (11) on the longer (horizontal) arm of the lever (5).

Fixing the string: The copper-pieced end of the string has to be hooked to the shorter (vertical) arm of the lever. The loop on the other end has to be hooked to the screw on the aluminum block. After fixing the string, it has to be tightened by moving the screw (3), so the longer arm of the lever is horizontal.

Relevant tunable parameters of the setup:

- The length of the string can be changed by moving the stands (6).

- The stretching force can be changed by changing the position of the weight on the longer arm of the lever. This arm is designed so putting the weight in the first slot, the stretching force is equal to the weight of the weight, putting in the second slot it is twice as large, and so on. Each time changing the stretching force, the longer arm should be fine tuned with the screw (3) to be horizontal.

- The linear mass density of the string can be changed by using another string with different diameter. The linear mass density has to be calculated from the diameter, which has to be measured by a screw-micrometer (the density of steel is 7800 kg/m^3).

- The shape of the standing wave can be changed by changing the excitation frequency on the function generator. The function generator is controlled by computer (PC).

The function generator has to be set to sine-wave output and the exciting coil has to be positioned close to one of the stands (about 5 cm distance), since it is most efficient close to a node. The function generator itself cannot drive the coil therefore an OHMEG power amplifier is connected to the coil. With the rotary knob set up the amplification in a way that the red LED above the input just does not light up. The detector coil should be placed to an antinode. If we look for the fundamental frequency, then put it in the middle; if for the overtone, put it to the appropriate location according to n. The signal of the detector is large when it is near to an antinode.

Equipments: String, fixed board, driver and detector coils, computer, NI myDAQ data acquistion device, cables, weights.

HOMEWORK

Before coming to the exercise, calculate the followings. The results must be shown to the instructor before starting the measurement.
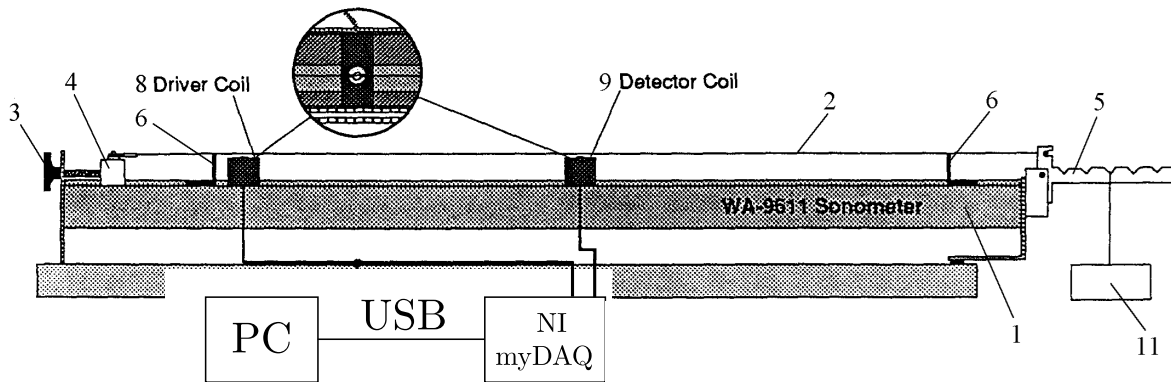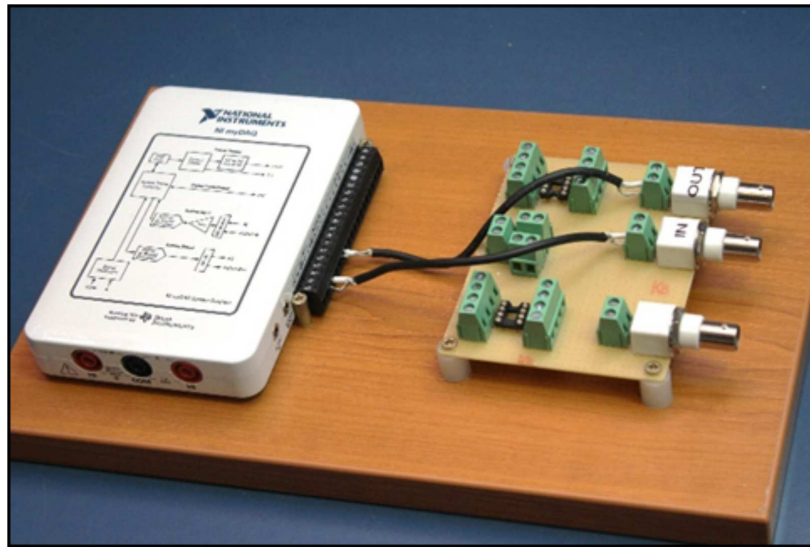
FIG. 2. The measurement setup

FIG. 3. MyDAQ acquistion device with connections

1. Determine the speed of wave propagation in a steel string of diameter 0.35 mm, 0.40 mm, and 0.50 mm, if the tension is 60 N (density of steel is 7800 kg/m^3);

2. Determine the fundamental frequencies ($n = 1$) for the above mentioned three strings. The length of a string is 60 cm.

III. MEASUREMENT TASKS

1. *Frequency and wavelength of first, second and third harmonics*

   Measure the diameter of the given string. Set the length of the string to 60 cm, and stretch it with 60 N (by putting 2 kg to the third slot of the lever, make sure that the lever arm is horizontal), and set the fundamental frequency of the given string on the function generator according to the previously calculated value. *The half of the calculated value should be set up on the function generator* since the magnet attracts the string two times within one sine period.

   Setup of the function generator (see FIG. 4 left):

   - Waveform Settings: Sine is highlighted,
   - Turn "Amplitude" to maximum (10 Vpp) by using the mouse,
   - Frequency: Click in the cell and type in the appropriate number;
   - Press RUN.

   Setup of the spectrum analyzer (Dynamic Signal Analyzer, see FIG. 4 right):
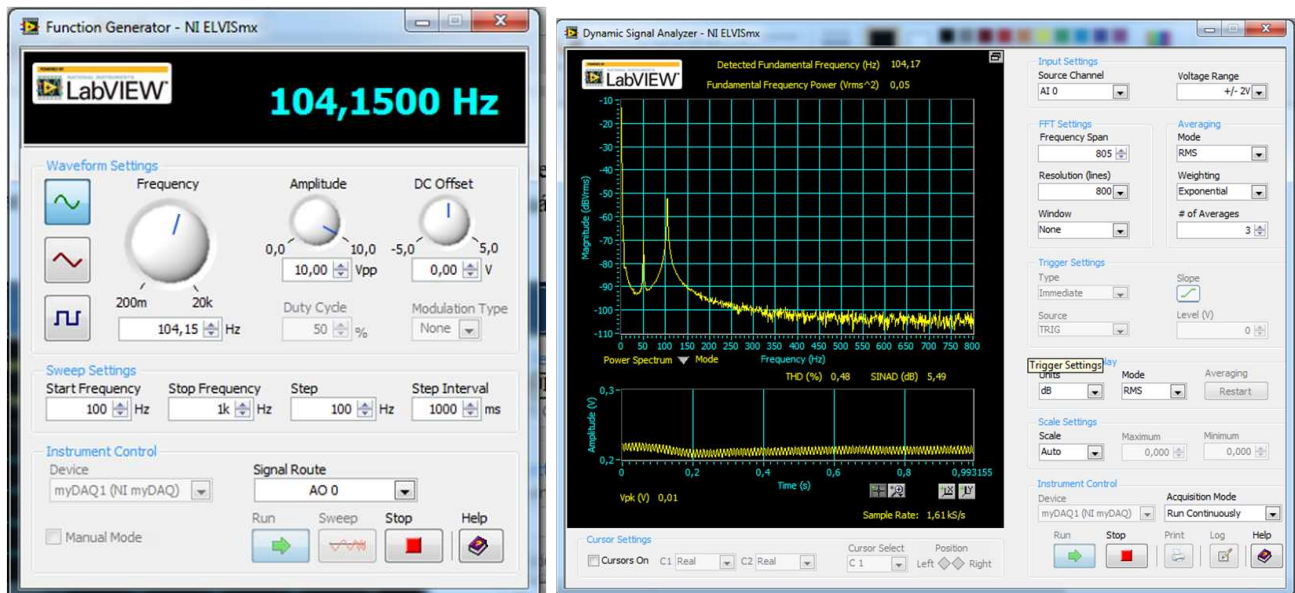
   - Input Settings; Voltage Range: ±2 V;

FIG. 4. Left: Function generator; Right:Dynamic Signal Analyzer

- FFT settings; Frequency spam: 800, Resolution: 800, Window: Hanning;
- Averaging; # of Averages: 2 or 3 (for low average number the spectrum follows the changes faster but in this case due to the noise the fluctuation is larger);
- Please do not change the other not mentioned settings;
- Press RUN.

By changing the frequency, find the standing wave for $n = 1$ (fundamental frequency). Place the detector where one expects node. In the Dynamic Signal Anayzator the frequrency distribution of the oscillating string can be obtained. The magnitude of the frequency components is measured in dB (decibel):

$$X_{\mathrm{dB}} = 20 \log_{10} \frac{U_{\mathrm{rms}}}{1\ \mathrm{V}},$$

which is a logarithmic value of the real rms value of the measured voltage relative to 1V. It is useful if the ratio can change over more order of magnitude in a single measurement. For example 1 V is 0 dB, 0.5 V is –6 db, 0.1 V is –20 dB, 0.01 V is –40 dB.

When changing the frequency, also change the tenths and hundredths digits. Set the frequency in a way that the signal of the fundamental frequency is the largest and that's of the overtones are the lowest.

Then find the second and the third standing waves ($n = 2,\ 3$). For each of them read the frequency on the screen of the function generator (*remember that the frequency of the string is two times larger than that read by the function generator*), and measure the coordinates (from one of the stands) of the nodes and antinodes (by moving the detector coil, and checking the amplitude by using oscilloscope), and determine the wavelength of the standing wave from the measured coordinates. Make a table and check eq. (9).

2. *The $\nu_n - n$ function*

   Plot the frequency of the standing waves (ν_n) as a function of n, fit a linear function to the data. From the slope, determine the speed of wave propagation, c, and compare it with value calculated from eq. (2).

3. *First harmonic, different length*

   Find the $n = 1$ standing wave for 40, 50, 60, 70 cm length using tension of 60 N, and read the resonance frequency in each cases. For finding the frequencies, use that we already know the frequency for the string of length 60 cm. Plot the frequency as a function of the reciprocal of the length, fit a linear function, and determine the speed of wave from the slope. Compare this value with previous ones.

4. *First harmonic, different tensions*

   Set the length back to 60 cm, and put the 2 kg weight into the first slot of the lever, and set the first standing wave corresponding to $n = 1$. Again, use that we know the frequency for the 60 N tension. Measure the resonance frequency and wavelength and calculate the speed of wave in the string by using the $c = \lambda\nu$ formula. Make a table and write the values of the tension T, the linear mass

density μ, the fundamental frequency ν, and the speed c. Repeat the measurement using four different tensions by putting the weight farther. Write the measured values into the table. Using the collected data verify eq. (2). (According to the equation if μ is constant, $c - T^{1/2}$ function is linear.) The corresponding data fall along a straight line, the slope of the line is $(1/\mu)^{1/2}$

5. *First harmonic, different diameters*

   Set the standing wave corresponding to $n = 1$ with 60 cm length and 60 N tension for other strings given by the instructor. Do not forget to measure the diameter of each wire. Give the wavelength and the frequency of the standing wave, and calculate the speed of wave in the strings by using the $c = \lambda \nu$ formula. Make a table and write the values of the tension T, the linear mass density μ, the fundamental frequency ν, and the speed c. Using the collected data verify eq. (2). (According to the equation if T is constant, $c - \mu^{-1/2}$ function is linear.) The corresponding data fall along a straight line, the slope of the line is $T^{1/2}$.