Chapter 7. Inside the Black Hole

- *Why would anyone volunteer to dive to the center of a black hole?*

- *Why does everything inside the event horizon inevitably move to smaller r?*

- *How massive must a black hole be so that 20 years pass on my wristwatch between crossing the event horizon and arrival at the crunch point?*

- *How can I construct a local inertial frame that is valid inside the event horizon?*

- *What do I see ahead of me and behind me as I approach the crunch point?*

- *Is my death quick and painless?*

C H A P T E R

7

₂₄

Inside the Black Hole

Edmund Bertschinger & Edwin F. Taylor [*]

₂₅ *Alice had not a moment to think about stopping herself before*
₂₆ *she found herself falling down what seemed to be a very deep*
₂₇ *well. Either the well was very deep, or she fell very slowly, for*
₂₈ *she had plenty of time as she went down to look about her,*
₂₉ *and to wonder what was going to happen next. First she tried*
₃₀ *to look down and make out what she was coming to, but it was*
₃₁ *too dark to see anything . . . So many out-of-the-way things*
₃₂ *had happened lately that Alice had begun to think that very few*
₃₃ *things indeed were really impossible.*

₃₄ —Lewis Carroll, *Alice in Wonderland*

7.1 ■ INTERVIEW OF A DIVING CANDIDATE

₃₆ *Few things are really impossible.*

₃₇ So you are applying to be a member of the black hole diving research group.

₃₈ *Yes.*

₃₉ Have you personally had experience diving into black holes?

₄₀ *This question is a joke, right?*

₄₁ Why do you want to be part of this diving group, since your research results
₄₂ cannot be reported back to us outside the event horizon?

₄₃ *We want to see for ourselves whether or not our carefully-studied*
₄₄ *predictions are correct. You know very well that 27 percent of*
₄₅ *qualified Galaxy Fleet personnel volunteered for this mission.*

₄₆ Tell me, why doesn't the black-hole diving group use local shell coordinates to
₄₇ make measurements inside the event horizon?

[*]Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*
Copyright © 2015 Edmund Bertschinger, Edwin F. Taylor, & John Archibald Wheeler. All
rights reserved. Latest drafts at dropsite exploringblackholes.com.

48 *Inside the event horizon no one can build a spherical shell that*
49 *stays at constant r in Schwarzschild coordinates. Instead we make*
50 *measurements in a series of local inertial frames that can exist*
51 *anywhere except on the singularity.*

52 Then how will you measure your r-coordinate without a spherical shell?

53 *We track the decreasing value of r by measuring the decreasing*
54 *separation between us and a test particle beside us that is also*
55 *diving radially inward.*

56 What clocks will you use in your experiments?

57 *Our wristwatches.*

58 When does your diving group cross the event horizon?

59 *As measured on whose clock?*

60 You *are* savvy. When does your diving group cross the event horizon as read
61 on your wristwatches?

62 *Zeroing different clocks in different locations is arbitrary. The*
63 *Astronautics Commission has a fancy scheme for coordinating the*
64 *various clock readings, mostly for convenience in scheduling. Want*
65 *more details?*

66 Not now. Is there any service that we on the outside can provide for your
67 diving group once you are inside the event horizon?

68 *Sure. We will welcome radio and video bulletins of the latest news*
69 *plus reports of scientific developments outside the event horizon.*

70 And will your outgoing radio transmissions from inside the event horizon
71 change frequency during their upward transit to us?

72 *Another joke, I see.*

73 Yes. Does your personal—ah—end seem mercifully quick to you?

74 *The terminal "spaghettification" will take place in a fraction of a*
75 *second as recorded on my wristwatch. Many of you outside the*
76 *event horizon would welcome assurance of such a quick end.*

77 What will you personally do for relaxation during the trip?

78 *I am a zero-g football champion and grandmaster chess player.*
79 *Also, my fiancé has already been selected as part of the team.*
80 *We will be married before launch.*

81 [We suppress the transcript of further discussion about the ethical and moral
82 status of bringing children into the diving world.]

Box 1. Eggbeater Spacetime?

Being "spaghettified" as you approach the center of a black hole is bad enough. But according to some calculations, your atoms will be scrambled by violent, chaotic tidal forces before you reach the center—especially if you fall into a young black hole.

The first theory of the creation of a black hole by J. Robert Oppenheimer and Hartland Snyder (1939) assumed that the collapsing structure is spherically symmetric. Their result is a black hole that settles quickly into a placid final state. A diver who approaches the singularity at the center of the Oppenheimer-Snyder black hole is stretched with steadily increasing force along the r-direction and compressed steadily and increasingly from all sides perpendicular to the r-direction.

In Nature an astronomical collapse is rarely spherically symmetric. Theory shows that when a black hole forms, the asymmetries exterior to the event horizon are quickly radiated away in the form of gravitational waves—in a few seconds measured on a distant clock! Gravitational radiation captured inside the event horizon, however, evolves and influences spacetime inside the black hole.

So what happens? There is no way to verify any predictions about events inside the event horizon (Objection 1), but that does not stop us from making them! Vladimir Belinsky, Isaac Markovich Khalatnikov, Evgeny Mikhailovich Lifshitz, and independently Charles Misner discovered that Einstein's equations predict more than one kind of singularity.

Their theory says that as a diving observer approaches the center point, spacetime can oscillate chaotically, squeezing and stretching the poor traveler in random directions like an electric mixer (eggbeater). These oscillations increase in both amplitude and frequency as the astronaut approaches the singularity of the black hole. Any physical object, no matter what stresses it can endure, is necessarily utterly destroyed at an eggbeater singularity.

However, there is some theoretical evidence that eggbeater oscillations will die away, so an astronaut who waits a while to dive after the black hole has formed may not encounter them. Before these eggbeater oscillations die away—if they do—spacetime in the chaotic regions is definitely NOT described by the Schwarzschild metric!

In the present chapter we assume the non-spinning black hole under exploration is an ancient one and that we can ignore possible eggbeater oscillations of spacetime. We predict (and hope!) that as our astronaut colony approaches the center, the "spacetime weather" is clear and calm.

?

83   **Objection 1.** *What kind of science are these people talking about?*
84   *Obviously nothing more than science fiction! No one who crosses the*
85   *event horizon of a black hole can report observations to the scientific*
86   *community outside the event horizon. Therefore all observations carried*
87   *out inside the event horizon—and conclusions drawn from them—remain*
88   *private communications. Private communication is not science!*

!

89   Yours is one sensible view of science, but if the "spacetime weather is
90   clear and calm" inside the event horizon (Box 1), then the diving research
91   group may have decades of life ahead of them, as recorded on their
92   wristwatches. They can receive news and science updates from friends
93   outside, view the ever-changing pattern of stars in the heavens (Chapter
94   12), carry out investigations, discuss observations among themselves, and
95   publish their own exciting research journal.

96   We recognize that the event horizon separates two communities of
97   investigators with a one-way surface or "membrane." Outsiders cannot
98   receive reports of experiments that test their predictions about life inside
99   the event horizon. They must leave it to insiders to verify or disprove these
100 predictions with all the rigor of a lively in-falling research community. Later
101 chapters on the *spinning* black hole raise the possibility that an explorer
102 might navigate in such a way as to reemerge from the event horizon,
103 possibly into a different spacetime region.
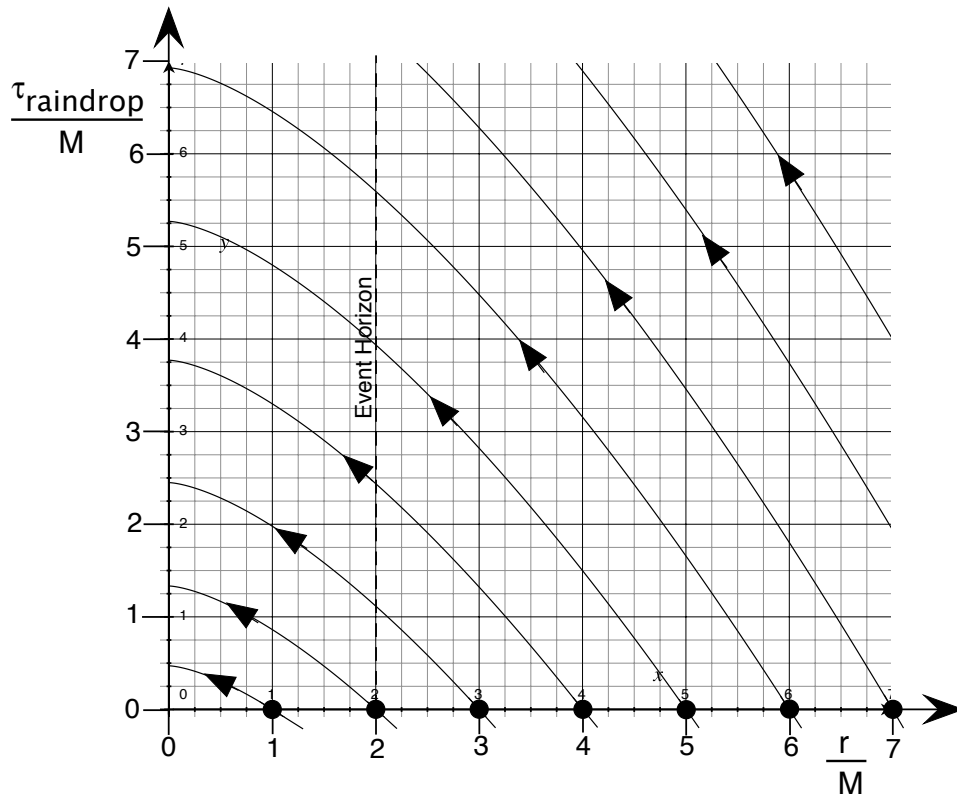
FIGURE 1 Raindrop wristwatch time vs. global r-coordinate from (2) for a series of raindrops that pass $r = M, 2M, ..., 9M$ at $\tau_{\text{raindrop}} = 0$ (little filled circles along the horizontal axis). The shapes of these curves are identical, just displaced vertically with respect to one another. Every raindrop moves smoothly across the event horizon when clocked on its wristwatch, but not when tracked with global Schwarzschild t-coordinate (Figure 2).

John A. Wheeler:
radical conservatism

104   **Comment 1. Wheeler's "radical conservatism"**
105   John Archibald Wheeler (1911-2008), who co-author the first edition of
106   *Exploring Black Holes*, rescued general relativity from obscurity in the 1950s and
107   helped to jump-start the present golden age of gravitational physics. He was
108   immensely inventive in research and teaching; for example he adopted and
109   publicized the name $black\ hole$ (initial quote, Chapter 3). Wheeler's professional
110   philosophy was **radical conservatism**, which we express as: *Follow*
111   *well-established physical principles while pushing each to its extreme limits.*
112   *Then develop a new intuition!* The black hole—both outside and inside its event
113   horizon—is a perfect structure on which to apply Wheeler's radical conservatism,
114   as we do throughout this book.

115   **Comment 2. Non-spinning vs. spinning black hole**
116   Chapters 2 through 13 describe spacetime around a black hole: a black hole
117   that does not rotate. We call this a **non-spinning black hole**. The Universe is
118   full of black holes that spin; many of them spin very fast, with deep

119   consequences for their structure and for spacetime around them. We call each of
120   these a **spinning black hole**, the subject of Chapters 17 through 21.

7.2 ■ RAINDROP WORLDLINE

122   *"Raindrops keep fallin' on my head . . ."* song by Hal David and Burt
123   Bacharach

124   We start with the raindrop of Chapter 6. The *raindrop* is a stone (wearing a
125   wristwatch) that drops from initial rest very far from the black hole
126   (Definition 2, Section 6.4). From equation (23) of that section:

$$d\tau_{\text{raindrop}} = -\left(\frac{r}{2M}\right)^{1/2} dr \qquad (1)$$

127   In the following Queries you integrate (1) and apply the result to a raindrop
128   that first falls past an $\underline{\text{A}}$bove r-coordinates r_{A} then falls past a sequence of
129   lower r-coordinates (Figure 1).

130

QUERY 1. Raindrop wristwatch time lapse between the above r_{A} and lower r

A. Integrate (1) to determine the elapsed raindrop wristwatch time from the instant the raindrop
   falls past the $\underline{\text{A}}$bove coordinate r_{A} until it passes a sequence of smaller r-coordinates. Express
   this elapsed raindrop wristwatch time with the notation $[r_{\text{A}} \to r]$ for r-limits.

$$\tau_{\text{raindrop}} [r_{\text{A}} \to r] = \frac{4M}{3} \left[\left(\frac{r_{\text{A}}}{2M}\right)^{3/2} - \left(\frac{r}{2M}\right)^{3/2} \right] \qquad (2)$$

   Figure 1 plots this equation for a series of raindrops after each passes through a different given
   r_{A} at $\tau_{\text{raindrop}} = 0$.
B. What happens to the value of the raindrop wristwatch time lapse in (2) when the initial r_{A}
   becomes very large? Explain why you are not disturbed by this result.

139

140

QUERY 2. Raindrop wristwatch time lapse from event horizon to crunch.
Suppose you ride the raindrop, and assume (incorrectly) that you survive to reach the center. This
Query examines how long (on your wristwatch) it takes you to drop from the event horizon to the
singularity—the crunch point.

A. Adapt your result from Query 1 to show that:

$$\tau_{\text{raindrop}}[2M \to 0] = \frac{4M}{3} \qquad \text{(event horizon to crunch, in meters)} \qquad (3)$$

B. *Predict:* Does every curve in Figure 1 satisfy (3)?

C. Use constants inside the front cover to find the event horizon-to-crunch raindrop wristwatch time *in seconds* for a black hole of mass M/M_{Sun} times the mass of our Sun:

$$\tau_{\text{raindrop}}[2M \rightarrow 0] = 6.57 \times 10^{-6} \frac{M}{M_{\text{Sun}}} \qquad \text{(event horizon to crunch, in seconds)} \qquad (4)$$

D The monster black hole at the center of our galaxy has mass $M \approx 4 \times 10^6 M_{\text{Sun}}$: its mass is about four million times the mass of our Sun. Assume (incorrectly) that this black hole does not spin. How long, in seconds on your wristwatch, will it take you—riding on the raindrop—to fall from the event horizon of this monster black hole to its singularity?

E. **Discussion question:** How can the r-value of the event horizon $r = 2M$ possibly be greater than the wristwatch time $4M/3$ that it takes the raindrop to fall from the event horizon to the singularity? Does the raindrop move faster than light inside the event horizon? (*Hint:* Do global coordinate separations dependably predict results of our measurements?)

QUERY 3. Mass of the "20-year black hole."
The black hole we feature in this chapter has a mass such that it takes 20 years—recorded on the wristwatch of the raindrop—to fall from event horizon to singularity.

A. Find the mass of the "20-year black hole" (a) in meters, (b) as a multiple of the mass of our Sun, and (c) in light-years.

B. An average galaxy holds something like 10^{11} stars of mass approximately equal to that of our Sun. The "20-year black hole" has the mass of approximately how many average galaxies?

C. What is the value of the r-coordinate at the event horizon of the "20-year black hole" in light-years?

Now turn attention to the motion of the raindrop in global Schwarzschild coordinates. Equation (22) in Section 6.4 gives the Schwarzschild map velocity of the raindrop:

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop map velocity)} \qquad (5)$$

$$= \left(\frac{r}{2M}\right)^{-3/2}\left(1 - \frac{r}{2M}\right)$$

We want to find $r(t)$, the r-coordinate of the raindrop as a function of the t-coordinate. This function defines a worldline (Section 3.10).
To find the worldline of the raindrop, manipulate (5) to read:

$$dt = \frac{\left(\frac{r}{2M}\right)^{3/2} dr}{1 - \frac{r}{2M}} = \frac{4Mu^4 du}{1 - u^2} \qquad \text{(raindrop)} \qquad (6)$$
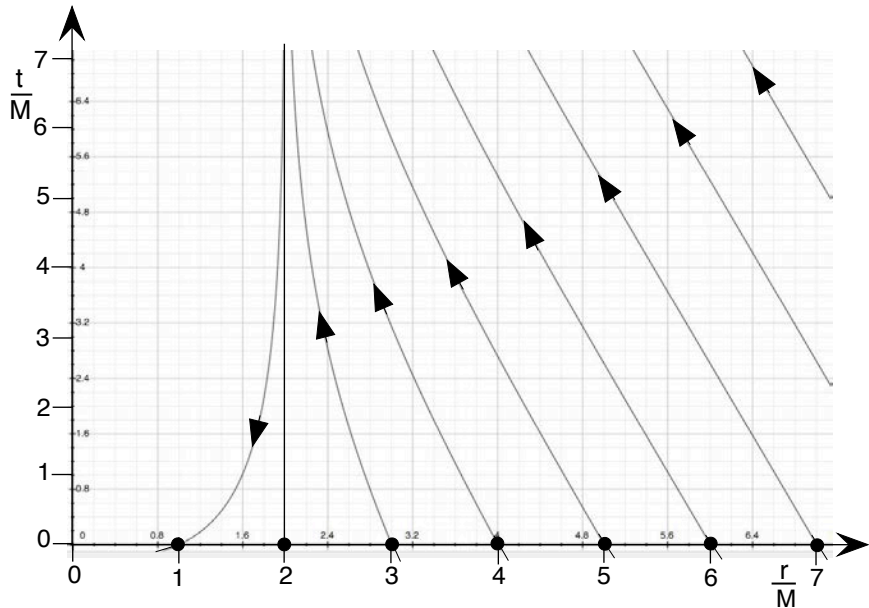
FIGURE 2 Schwarzschild worldlines of raindrops from (10), plotted on the $[r, t]$ slice. These particular raindrops pass $t_A/M = 0$ at different values of r_A/M (filled dots along the horizontal axis). The curves for $r_A/M > 2$ are identical in shape, simply displaced vertically with respect to one another. These worldlines are not continuous across the event horizon (compare Figure 1).

175 where the expression on the right side of (6) results from substitutions:

$$u \equiv \left(\frac{r}{2M}\right)^{1/2} \quad \text{so} \quad du = \frac{1}{4M}\left(\frac{r}{2M}\right)^{-1/2} dr \quad \text{and} \quad dr = 4M u \, du \quad (7)$$

176 From a table of integrals:

$$\int \frac{u^4 du}{1 - u^2} = -\frac{u^3}{3} - u + \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| \quad (8)$$

177 Integrate (6) from u_A to u, where A stands for Above. The integral of (6)
178 becomes:

$$t - t_A = 4M\left[\frac{u_A^3}{3} - \frac{u^3}{3} + u_A - u + \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{2}\ln\left|\frac{1+u_A}{1-u_A}\right|\right] \quad \text{(raindrop)} (9)$$

179 Substitute the expression for u from (7) into (9):

$$t - t_A = \frac{4M}{3}\left[\left(\frac{r_A}{2M}\right)^{3/2} - \left(\frac{r}{2M}\right)^{3/2} + 3\left(\frac{r_A}{2M}\right)^{1/2} - 3\left(\frac{r}{2M}\right)^{1/2}\right. \quad (10)$$

$$\left. + \frac{3}{2}\ln\left|\frac{1+\left(\frac{r}{2M}\right)^{1/2}}{1-\left(\frac{r}{2M}\right)^{1/2}}\right| - \frac{3}{2}\ln\left|\frac{1+\left(\frac{r_A}{2M}\right)^{1/2}}{1-\left(\frac{r_A}{2M}\right)^{1/2}}\right|\right] \quad \text{(raindrop)}$$

180   Equation (10) is messy, but the computer doesn't care and plots the curves in
181   Figure 2 for $t_A = 0$ and $r_A/M = 1$ through 9.

?

182   **Objection 2.** *Wait! The t/M versus r/M worldline in Figure 2 tells us that*
183   *the raindrop takes an unlimited t to reach the event horizon. Do you mean*
184   *to tell me that our raindrop does not cross the event horizon?*

!

185   Recall our "strong advice" in Section 5.6: "To be safe, it is best to assume
186   that global coordinate separations do not have any measured meaning."
187   The worldlines in Figure 2 that rise without limit in t-coordinate as
188   $r/M \to 2^+$ (from above) do not tell us directly what any observer
189   measures. In contrast, the observer riding on the raindrop reads and
190   records her wristwatch time τ as she passes each shell. At each such
191   instant on her wristwatch, she also her direct reading of the r-coordinate
192   stamped on the shell she is passing. When timed on her wristwatch, the
193   raindrop passes smoothly inward across the event horizon (Figure 1).

?

194   **Objection 3.** *There is still a terrible problem with Figure 2. Why does the*
195   *worldline of the raindrop that somehow makes it inward across $r/M = 2$*
196   *run **backward** in the Schwarzschild t-coordinate?*

!

197   We saw this earlier in the light-cone diagram of Figure 8 in Section 3.7, in
198   which the t-coordinate can run backward along a worldline. However, that
199   has no measurable consequence. You cannot grow younger by falling into
200   a black hole. Sorry! The global t-coordinate is *not* time. Does the idea of
201   backward-running global t-coordinate along a worldline make you
202   uncomfortable? Get used to it! In contrast, the time you read on your
203   wristwatch *always* runs forward along your worldline, in particular along the
204   raindrop worldline (Figure 1). In Section 7.4 we develop a set of **global**
205   **rain coordinates** that not only labels each event—which we require of
206   every set of global coordinates—but also yields predictions more
207   comfortable to our intuition about the "respectable sequence" of global
208   coordinates along a worldline. This (arbitrary) choice of global coordinates
209   is purely for our own convenience: Nature doesn't care!

7.3 ■ THE LOCAL RAIN FRAME IN SCHWARZSCHILD COORDINATES

211   *Carry out experiments as you pass smoothly through the event horizon*

212   Does passing through the event horizon disturb local experiments that we may
213   be conducting during this passage? To answer this questions go step by step
214   from the raindrop to the local inertial rain frame and (in Section 7.4) from the
215   local frame to a new global description. Start with local shell coordinates
216   outside the horizon and use special relativity to derive local rain frame
217   coordinates.
218   Equations (9) through (11) in Section 5.7 give us local shell coordinates
219   expressed in Schwarzschild global coordinates:

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \qquad \text{(from Schwarzschild)} \qquad (11)$$

$$\Delta y_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \qquad\qquad (12)$$

$$\Delta x_{\text{shell}} \equiv \bar{r}\Delta\phi \qquad\qquad (13)$$

220

221   Now use the Lorentz transformation equations of Section 1.10 to find to
222   the local time lapse Δt_{rain} in an inertial frame in which the raindrop is at rest.
223   With respect to the local shell frame, the raindrop moves with velocity v_{rel} in
224   the $-\Delta y_{\text{shell}}$ direction. From equation (18) in Section 6.4:

$$v_{\text{rel}} = -\left(\frac{2M}{r}\right)^{1/2} \quad \text{so} \quad \gamma_{\text{rel}} \equiv \frac{1}{\left(1 - v_{\text{rel}}^2\right)^{1/2}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \qquad (14)$$

225   Then from the first of equations (41) in Section 1.10:

$$\Delta t_{\text{rain}} = \gamma_{\text{rel}} \left(\Delta t_{\text{shell}} - v_{\text{rel}}\Delta y_{\text{shell}}\right) \qquad\qquad (15)$$

$$= \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \left[\left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r\right]$$

226   so that

$$\Delta t_{\text{rain}} = \Delta t + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r \qquad (16)$$

227

228   And from the second of equations (41) in Section 1.10:

$$\Delta y_{\text{rain}} = \gamma_{\text{rel}} \left(\Delta y_{\text{shell}} - v_{\text{rel}}\Delta t_{\text{shell}}\right) \qquad\qquad (17)$$

$$= \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \left[\left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t\right]$$

229   so that

$$\Delta y_{\text{rain}} = \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2} \Delta t \qquad (18)$$

230

231   Finally, from the third of equations (41) in Section 1.10, the shell and rain
232   coordinates transverse to the direction of relative motion have equal values:

$$\Delta x_{\text{rain}} = \Delta x_{\text{shell}} = \bar{r}\Delta\phi \qquad (19)$$

233

234   The right sides of local rain frame equations (16) and (18) suffer from the
235 same disease as their parent global Schwarzschild coordinates: They blow up at
236 the event horizon. Nevertheless, we can use these *local* coordinate equations to
237 derive a new set of *global* rain coordinates that, at long last, cures this disease
238 and allows us to predict observations made in the local rain frame as we ride
239 smoothly inward across the event horizon and all the way to the singularity.

7.4 ■ GLOBAL RAIN COORDINATES
240

241   *Convert from Schwarzschild-t to global rain T.*

242   Schwarzschild coordinates are completely legal, but they make us
243 uncomfortable because they do not describe the motion of a stone or light
244 flash inward across the event horizon in a finite lapse of the t-coordinate. So in
245 this section we find a new global coordinate—that we label the
246 T-coordinate—which advances smoothly along the global worldline of a
247 descending stone, even when the stone crosses the event horizon.
248   The result is a new set of global coordinates with the old global
249 coordinates r and ϕ but a new T-coordinate. We call this new set of

Global rain
coordinates or
"Painlevé-Gullstrand
coordinates."

250 coordinates **global rain coordinates**. They are often called
251 **Painlevé-Gullstrand coordinates** after Paul Painlevé and Alvar Gullstrand
252 who independently developed them in 1921 and 1922, respectively. The present
253 section develops global rain (Painlevé-Gullstrand) coordinates. Section 7.5 uses
254 these new global coordinates to derive the **global rain metric**

?

255 **Objection 4.** *Hold on! How can we have two different global coordinate*
256 *systems for the same spacetime?*

!

257 For the same reason that a flat Euclidean plane can be described by either
258 Cartesian coordinates or polar coordinates. More than one global
259 coordinate system can describe the same spacetime. Indeed, an unlimited
260 number of global coordinate systems exist for any configuration of
261 mass-energy-pressure (Box 3).

?

262 **Objection 5.** *I am awash in arbitrary global coordinates here. What makes*
263 *practical sense of all this formalism? What can I hang onto and depend*
264 *upon?*

!

Primary goal:
Predict result of
local measurements

265 The answer comes from invariant wristwatch time and invariant ruler
266 distance. These direct observables are outputs of the global metric. In
267 John Wheeler's words, *"No phenomenon is a real phenomenon until it is*
268 *an observed phenomenon."* Spacetime is effectively flat on a local patch;
269 on that flat patch we use the approximate global metric to derive local
270 coordinates that we choose to be inertial (Section 5.7). The observer
271 makes a measurement and expresses the result in those local

272   coordinates. **Every set of global coordinates must lead to the same**
273   **predicted result of a given measurement.** That is what makes sense of
274   the formalism; that is what you can hang onto and depend on.

275   The troublemaker in Schwarzschild coordinates is the t-coordinate, which
276 Figure 2 shows to be diseased at the event horizon. The cure is a new global
277 rain coordinate which we call capital T. The other two global rain coordinates,
278 r and ϕ remain the same as the corresponding Schwarzschild coordinates.

279   **Comment 3. Why conversion from Schwarzschild to global rain?**
280   Most often in this book we simply display a global metric with its global
281   coordinate system without derivation. In what follows, we make a "conversion" of
282   Schwarzschild coordinates to global rain coordinates that leads to the global rain
283   metric. Why this conversion? Why don't we simply display the global rain metric
284   and its coordinate system? We do this to show that there are two ways to derive
285   a valid global metric. The first way is to submit the (almost) arbitrary global
286   coordinates to Einstein's equations, which return the correct global metric. The
287   second way is simply to transform the already-validated global metric directly.
288   That conversion does not require Einstein's equations.

289   To find the new global T coordinate we do something apparently illegal:
290 We derive it from *local* rain coordinate $\Delta t_{\rm rain}$ in equation (16). From the
291 beginning of this book we have emphatically declared that you cannot derive
292 global coordinates from local coordinates. Why not? Because in considering
293 any flat local inertial frame, we lose details of the global curvature of the
294 spacetime region. The local $\Delta t_{\rm rainA}$ from equation (16) is unique to Frame A
295 which depends on the average value $\bar{r}_{\rm A}$; adjacent Frame B has a different

You cannot derive
global coordinates from
local coordinates. . . .

296 $\Delta t_{\rm rainB}$ which depends on a different average value $\bar{r}_{\rm B}$. This leads to a
297 discontinuity at the boundary between these two local frames. The local frame
298 with its local coordinates is useful for us because it allows us to apply special
299 relativity to an experiment or observation made in a limited spacetime region
300 in globally curved spacetime. But the local frame does have this major
301 drawback: We cannot connect adjacent flat frames smoothly to one another in
302 curved spacetime. That keeps us from generalizing from local frame
303 coordinates to global coordinates.

304   But there is an exception. To understand this exception, pause for a quick
305 tutorial in the mathematical theory of calculus (invented in the late 1600s by
306 both Isaac Newton and Gottfried Wilhelm Leibniz). In calculus we use
307 differentials, for example dt and dr. The technical name for the kind of

. . . except when
local coordinates
lead to an exact
differential.

308 differential we use in this book is **exact differential**, sometimes called a
309 **perfect differential**. Formally, an exact differential (as contrasted with an
310 inexact differential or a partial differential) has the form dQ or dT, where Q
311 and T are **differentiable functions**. What is a differentiable function? It is
312 simply a function whose derivative exists at every point in its domain.

313   *Question:* Are global coordinates differentiable? *Answer:* We choose global

Global coordinates
are differentiable.

314 coordinates ourselves, then submit them to Einstein's equations which are
315 differential equations that return to us the global metric. So if one of our
316 chosen global coordinates is not differentiable, it is our own fault. *Conclusion:*

317 For a global coordinate to be useful in general relativity, it must be
318 differentiable and thus have an exact differential (except at a physical
319 singularity).
320   So we purposely choose a set of global coordinates that are differentiable.
321 If we then make a transformation between sets of global coordinates, the
322 transformed global coordinate is also a differentiable function and therefore
323 has an exact differential.
324   A close look at equation (16) shows that we can turn it into an exact
325 differential, as follows:

$$\lim_{\Delta t \to 0} \Delta t_{\text{rain}} = dt + \frac{df(r)}{dr} dr = d[t + f(r)] \equiv dT \tag{20}$$

326 where, from (16), $T = t + f(r)$ is a differentiable function with

$$\frac{df(r)}{dr} = \left(\frac{2M}{r} \right)^{1/2} \left(1 - \frac{2M}{r} \right)^{-1} \tag{21}$$

327   Equation (20) is immensely significant. It tells us that there is an exact
328 differential of what we call a new global T-coordinate. This means that T is a
329 differentiable function of global rain coordinates. This coordinate $T = t + f(r)$
330 is global because (a) Schwarzschild's t and r (along with ϕ) already label every
331 event for $r > 0$, and (b) Schwarzschild coordinates and metric satisfy
332 Einstein's equations. From (20) and (21):

$$dT = dt + d[f(r)] = dt + \left(\frac{2M}{r} \right)^{1/2} \left(1 - \frac{2M}{r} \right)^{-1} dr \tag{22}$$

333   To validate the new global rain coordinate T, we need to express it in the
334 already-validated Schwarzschild coordinates. To start this process integrate
335 (21):

$$f(r) \equiv \int_0^r \left(\frac{2M}{r} \right)^{1/2} \left(1 - \frac{2M}{r} \right)^{-1} dr = -(2M)^{1/2} \int_0^r \frac{r^{1/2} dr}{2M - r} \tag{23}$$

336 A table of integrals helps us to integrate the right side of (23). The result is:

$$f(r) = 4M \left(\frac{r}{2M} \right)^{1/2} - 2M \ln \left| \frac{1 + (2M/r)^{1/2}}{1 - (2M/r)^{1/2}} \right| \tag{24}$$

337 Finally, from (22) and (24):

$$T = t + f(r) = t + 4M \left(\frac{r}{2M} \right)^{1/2} - 2M \ln \left| \frac{1 + (2M/r)^{1/2}}{1 - (2M/r)^{1/2}} \right| \tag{25}$$

338
339 where we have arbitrarily set some constants of integration equal to zero.
340 Equation (25) is the final proof that the new global rain T is valid, since it
341 transforms directly from valid Schwarzschild t and r global coordinates.

?

342   **Objection 6.** *I still think that your definition of T derived from a local*
343   *coordinate increment Δt_{rain} in equations (22) through (25) violates your*
344   *own prohibition in Section 5.7 of a local-to-global transformation.*

!

345   No, we do *not* transform from local rain coordinates to global rain
346   coordinates, which would be illegal. Instead, we start with the global
347   Schwarzschild expression for Δt_{rain} in (16), take its differential limit in
348   (20), which converts it to the global rain differential dT. The key step in
349   that conversion is to recognize that equation (20) contains an exact
350   differential in global Schwarzschild coordinates. Note that this does not
351   happen for shell time in Schwarzschild coordinates:

$$\lim_{\Delta t \to 0} \Delta t_{\text{shell}} \to \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad (26)$$

352   The right side of (26) cannot be expressed as $d[T(r,t)]$, the exact
353   differential of a coordinate $T(r,t)$.

354   The exact differential in (20) allows us to complete the derivation and
355   validation of the global rain T-coordinate in equation (25). Is this magic?
356   No, but it does require sophisticated use of calculus.

357

QUERY 4. Differentiate T (*Optional*)
Take the differential of (25) to show that the result is (22).

360

361   We worked in Schwarzschild coordinates to find a function
362     $T(t,r) = t + f(r)$ whose differential matches local rain frame time. We can free
363   this T of its origin and regard it as a new label for each spacetime event. But

Coordinate   364   after that addition there is no need to use both T and t to label events. We
transformation   365   can now eliminate t from the list, because we can always find it if we know the
to global rain   366   value of T (along with r) through $t = T - f(r)$. So we perform the **coordinate**
coordinates   367   **transformation** (25), in which we replace one set of global coordinates
                    368   (t,r,ϕ) with a new set of global coordinates (T,r,ϕ). We call the result **global**
                    369   **rain coordinates**—or historically, **Painlevé-Gullstrand coordinates**.

                    370   Why go to all this bother? In order to derive (in the next section) a global
Payoff: "Intuitive" 371   rain metric that describes the motion of a stone or light flash across the event
global rain metric 372   horizon in a manner more comfortable to our intuition. The global rain metric
                    373   encourages our unlimited, free exploration of *all* spacetime outside, at, and
                    374   inside the event horizon.

                    375   An immediate payoff of global rain coordinates is local shell coordinates
                    376   expressed in global rain coordinates (Box 2).

377   **Comment 4. Global T is defined everywhere.**
378   Although we started from local shell coordinates which exist only for $r > 2M$,
379   our new T coordinate is defined everywhere, even at and inside the event

---

Box 2. Local shell coordinates expressed in global rain coordinates

We now derive local shell coordinates as functions of global rain coordinates. Equation (11) gives Δt_{shell} as a function of the Schwarzschild t-coordinate increment:

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \qquad (27)$$

The Schwarzschild t-coordinate does increase without limit along the worldline of a stone that approaches $r = 2M$, but shells exist only outside this event horizon, where (27) is well-behaved. Write equation (22) in approximate form:

$$\Delta t \approx \Delta T - \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r \qquad (28)$$

Substitute (28) into (27) to yield:

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta T \qquad (29)$$
$$- \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r$$

Equations for Δy_{shell} and Δx_{shell} do not depend on Δt, so we copy them directly from (12) and (13).

$$\Delta y_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \qquad (30)$$

$$\Delta x_{\text{shell}} \equiv \bar{r}\Delta\phi \qquad (31)$$

Note: Expressions for Δt_{shell} and Δy_{shell} are real only for $r > 2M$, consistent with the conclusion that no shell can exist inside the event horizon.

---

380   horizon, because its defining equation (25) contains only global Schwarzschild
381   coordinates, which span all of spacetime.

382   Should we worry that the last term on the right side of (25) blows up as
383    $r \to 2M$? No, because Schwarzschild t in (25) blows up in the opposite
384   direction in such a way that T is continuous across $r = 2M$. Our only legitimate
385   worry is continuity of the resulting metric, which (32). Section 7.5 shows this
386   metric to be continuous across $r = 2M$.

7.5■ THE GLOBAL RAIN METRIC

388   *Move inward across the event horizon.*

389   Section 7.4 created a global T-coordinate, validated it by direct transformation
390   from already-approved global Schwarzschild coordinates, and installed it in a
391   new set of global rain coordinates. To derive the **global rain metric**, solve
392   the dT-transformation (22) for dt, substitute the result into the Schwarzschild
393   metric, and collect terms to yield:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dT^2 - 2\left(\frac{2M}{r}\right)^{1/2} dT\,dr - dr^2 - r^2 d\phi^2 \qquad (32)$$
$$-\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \le \phi < 2\pi \quad \text{(global rain metric)}$$

394

395   Metric (32), with its connectedness (topology), provides a complete
396   description of spacetime around a non-spinning black hole, just as the

397 Schwarzschild metric does. In addition, all Schwarzschild-based difficulties
398 with worldlines that cross the event horizon disappear.

399

QUERY 5. Global rain metric

400 Substitute dt from (22) into the Schwarzschild metric to verify the global rain metric (32).

402

403

QUERY 6. Flat spacetime far from the black hole

404 Show that as $r \to \infty$ metric (32) becomes the metric for flat spacetime in global coordinates T, r, ϕ.

406

407 **Comment 5. USE THE GLOBAL RAIN METRIC FROM NOW ON.**

Use global
408 **From now on in this book we use the global rain metric—and expressions**

rain metric
409 **derived from it—to analyze events in the vicinity of the non-spinning black**

from now on.
410 **hole.**

411 *Question:* When all is said and done, which set of coordinates is the
412 "correct" one for the non-spinning black hole: Schwarzschild coordinates or
413 rain coordinates or some other set of global coordinates? *Answer: Every global*
414 *coordinate system is valid provided it is either (a) submitted to Einstein's*
415 *equations, which return a global metric or (b) transformed from an*

Principle of
416 *already-validated global coordinate system.* This reflects a fundamental

General Covariance
417 principle of general relativity with the awkard technical name **Principle of**
418 **General Covariance**. In this book we repeat over and over again that no
419 single observer measures map coordinates directly (back cover). To overlook
420 the Principle of General Covariance by attaching physical meaning to global
421 coordinates is wrong and sets oneself up to make fundamental errors in the
422 predictions of general relativity.

423 **EVERY GENERATION MUST LEARN ANEW**
424 *One of the fundamental principles of general relativity (the principle of*
425 *general covariance) states that all [global] spacetime coordinate systems*
426 *are equally valid for the description of nature and that metrics that are*
427 *related by a coordinate transformation are physically equivalent. This*
428 *principle sounds simple enough, but one repeatedly finds in the literature*
429 *arguments that amount to advocacy for the interpretation based on one*
430 *set of [global] coordinates to the exclusion of the interpretation that is*
431 *natural when using another set of [global] coordinates for the same set of*
432 *events. It has been said that each generation of physicists must learn*
433 *anew (usually the hard way) the meaning of Einstein's postulate of*
434 *general covariance.*

435 —Richard C. Cook and M. Shane Burns

7-16 Chapter 7 Inside the Black Hole

Box 3. An Unlimited Number of Global Coordinate Systems

General relativity uses two methods to derive a global metric: *Method 1:* Choose an arbitrary set of global coordinates and submit them to Einstein's equations, which return the global metric expressed in those coordinates. Karl Schwarzschild did this to derive the Schwarzschild metric. *Method 2:* Transform an already-validated set of global coordinates to another set, then substitute the new coordinates into the already-verified metric of Method 1 to yield a metric in the new global coordinates. In the present chapter we use Method 2 to go from the Schwarzschild metric to the global rain metric.

How many different global coordinate systems are there for the non-spinning black hole? An unlimited number! Any one-to-one transformation from one valid set of global coordinates to another set of global coordinates makes the second set valid as well, provided it meets the usual criteria of uniqueness and smoothness (Section 5.8).

As a simple case, make the transformation:

$$T^{\dagger} \equiv KT \tag{33}$$

where K is any real number—or even a simple minus sign! With this substitution, global rain metric (32) becomes

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)\left(\frac{dT^{\dagger}}{K}\right)^2 \tag{34}$$

$$- 2\left(\frac{2M}{r}\right)^{1/2}\frac{dT^{\dagger}}{K}dr - dr^2 - r^2 d\phi^2$$

If K is negative, then global $T^{\dagger}$ runs backward along the worldline of every stone. This does not bother us; Schwarzschild t does the same along some worldlines (Figure 2). Equations for local shell and local rain coordinates are similarly modified, which does not change any prediction or the result of any measurement in these local frames.

In a similar manner we can let $r^{\dagger} \equiv Qr$ and/or $\phi^{\dagger} \equiv -\phi$. You can write down new metrics with any one or any pair of these new coordinates. These changes may seem trivial, but they are not. For example, choose $K = Q = M^{-1}$. The result is the variables T/M and r/M, so-called **unitless coordinates**, in which curves are plotted in many figures of this book. Plots with unitless coordinates are correct for *every* non-spinning black hole, independent of its mass M.

Result: Global rain metric (32) is only one of an unlimited set of alternative, equally-valid global metrics for the non-spinning black hole, each one expressed in a different global coordinate system. The examples in this box may be no more useful than the original global rain metric, but there are other global metrics that highlight some special property of the non-spinning black hole. Look up **Eddington-Finkelstein coordinates** and **Kruskal-Szekeres coordinates** and their metrics.

---

⁴³⁶   *In particular:* Fixation on an interpretation based on one set of arbitrary
⁴³⁷ coordinates can lead to the mistaken belief that global coordinate differences

Reminder of
Einstein's error

⁴³⁸ correspond to measurable quantities (Section 2.7).

⁴³⁹   **Comment 6. You know some relativity that Einstein missed!**
⁴⁴⁰   Chapter 5 says that Einstein took seven years to appreciate that global
⁴⁴¹ coordinate separations have no measurable meaning. But even then he did not
⁴⁴² fully understandd this fundamental idea. At a Paris conference in 1922—seven
⁴⁴³ years after he completed general relativity—Einstein worried about what would
⁴⁴⁴ happen at a location where the denominator of the dr^2 term in the
⁴⁴⁵ Schwarzschild metric, namely $(1 - 2M/r)$, goes to zero. He said it would be "an
⁴⁴⁶ unimaginable disaster for the theory; and it is very difficult to say *a priori* what
⁴⁴⁷ would occur physically, because the theory would cease to apply." (In 1933 the
⁴⁴⁸ Belgian priest Georges Lemaître recognized that the apparent singularity in
⁴⁴⁹ Schwarzschild coordinates at $r = 2M$ is "fictional.") Einstein was also baffled by
⁴⁵⁰ the $dTdr$ cross term in the global rain metric (32) presented to him by Paul
⁴⁵¹ Painlevé and rejected Painlevé's solution out of hand. This led to the eclipse of
⁴⁵² this metric for decades.

⁴⁵³   *Congratulations:* You know some relativity that Einstein did not understand!

QUERY 7. Map energy in global rain coordinates

A. Use the global rain metric and the Principle of Maximal Aging to derive the map energy of a stone in global rain coordinates. You can model the procedure on the derivation of the Schwarzschild-coordinate expression for E/m in Section 6.2, but will need to alter some of the notation. Demonstrate this result:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau} \qquad \text{(global rain coordinates)} \qquad (35)$$

B. Substitute for dT from (22) and show that the result yields the Schwarzschild expression for map energy, equation (8) in Section 6.2.

C. Show that the map energy equation (35) applies to a stone in orbit around the black hole, not just to one that moves along the r-coordinate line.

D. Find an expression for dr/dT of the raindrop in rain map coordinates? Start with the first line of (5) and multiply both sides by dt/dT from (22). Show that the result is:

$$\frac{dr}{dT} = \frac{dr}{dt}\frac{dt}{dT} = -\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop)} \qquad (36)$$

E. Show that the map worldline of a raindrop is given by the equation:

$$T - T_{\rm A} = \frac{4M}{3}\left[\left(\frac{r_{\rm A}}{2M}\right)^{3/2} - \left(\frac{r}{2M}\right)^{3/2}\right] \qquad \text{(raindrop)} \qquad (37)$$

where $(r_{\rm A}, T_{\rm A})$ locates the initial event on the plotted curve (Figure 3).

?

Objection 7. *Wait! The right side of (37) is identical to the right side of (2). The two equations are both correct if and only if the left sides are equal:*

$$T - T_{\rm A} = \tau_{\rm raindrop}[r_{\rm A} \to r] \qquad (38)$$

But these two quantities are completely different—apples and oranges! The left side is the difference in a global coordinate, while the right side is the lapse of wristwatch time of a particular falling stone.

!

For our own convenience, we *chose* global rain coordinates so that equation (38) is valid. This is playing with fire, of course, because it is dangerous to assume that any given global coordinate corresponds to a measurable quantity (Section 5.8). But we are adults now, able to see the pitfalls of mature life. The goal, as always, is correct prediction of measurements and observations.

481   Raindrop equation (36) for dr/dT looks quite different from raindrop
482 equation (5) for dr/dt, but the two must predict the same raindrop wristwatch
483 time from $r_{\rm A}$ to a smaller r-coordinate given by (2). Let's check this: For a
484 raindrop, set $E/m = 1$ in (35) and multiply through by $d\tau_{\rm raindrop}$:

$$d\tau_{\rm raindrop} = \left(1 - \frac{2M}{r}\right) dT - \left(\frac{2M}{r}\right)^{1/2} dr \qquad (39)$$

485 Solve (36) for dT and substitute into (39).

$$d\tau_{\rm raindrop} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{-1/2} dr - \left(\frac{2M}{r}\right)^{1/2} dr \qquad (40)$$

$$= -\left(\frac{r}{2M}\right)^{1/2} dr$$

486 which is the same as equation (1), so its integral from $r_{\rm A}$ to r must be (2), as
487 required. This is an example of an important property of different global
488 metrics: *Every global metric must predict the same result of a given*
489 *measurement or observation* (Objection 5).
490   Box 4 carries out a Lorentz transformation from local shell coordinates in
491 Box 2 to local rain coordinates, then verifies that local rain coordinates are
492 valid everywhere, not just outside the event horizon.

493   **Comment 7. An observer passes through a sequence of local frames.**
494   The rain observer rides on a raindrop (Definition 4, Section 7.7). In curved
495   spacetime, local inertial frames are limited in both space and time. During her
496   fall, the rain observer passes through a series of local rain frames as shown in
497   Figure 3. Equations (42) through (44) contain an $\bar{r}$, assumed to have the same
498   value everywhere in that local frame. Although absent from the equations, similar
499   average $\bar{T}$ and $\bar{\phi}$ are implied by all three local rain coordinate equations. *Result:*
500   Each rain observer passes through a sequence of local inertial frames. Similar
501   statements also apply to, and may seem more natural for, local shell frames with
502   local coordinates (29) through (31).

503   Box 4 derives *local* rain coordinates expressed in *global* rain coordinates.
504 This simplifies local rain coordinates compared with those expressed in
505 Schwarzschild coordinates in equations (16) through (19).
506   Figure 3 displays several rain observer worldlines on the $[r, T]$ slice. We
507 surround one worldline with a worldtube—shown in cross section—that
508 contains local rain frames through which this rain observer passes in sequence.
509 With $\Delta t_{\rm rain} = \Delta T = 0$, equation (43) tells us that $\Delta y_{\rm rain}$ coordinate lines are
510 horizontal in this figure. Finally, $\Delta x_{\rm rain}$ coordinate lines, which are
511 perpendicular to both $\Delta t_{\rm rain}$ and $\Delta y_{\rm rain}$ coordinate lines, project outward,
512 perpendicular to the page in Figure 3.
513   We could cover the worldtube in Figure 3 with adjacent or overlapping
514 local rain frames. The resulting figure would be analogous to Figure 5 in
515 Section 2.2, which places overlapping local flat maps along the spatial path
516 from Amsterdam to Vladivostok along Earth's curved surface.

Box 4. Local rain coordinates expressed in global rain coordinates

Apply the Lorentz transformation to local shell coordinates in Box 2 to derive local rain coordinates. Relative velocity in the Lorentz transformation lies along the common Δy_{shell} and Δy_{rain} line. With this change, Lorentz transformation equations of Section 1.10 become:

$$\Delta t_{\text{rain}} = \gamma_{\text{rel}} \left(\Delta t_{\text{shell}} - v_{\text{rel}} \Delta y_{\text{shell}} \right) \qquad (41)$$

$$\Delta y_{\text{rain}} = \gamma_{\text{rel}} \left(\Delta y_{\text{shell}} - v_{\text{rel}} \Delta t_{\text{shell}} \right)$$

$$\Delta x_{\text{rain}} = \Delta x_{\text{shell}}$$

Substitute v_{rel} and γ_{rel} from (14), along with local shell coordinates from Box 2, into equations (41) to obtain expressions for local rain coordinates as functions of global rain coordinates:

$$\Delta t_{\text{rain}} \equiv \Delta T \qquad (42)$$

$$\Delta y_{\text{rain}} \equiv \Delta r + \left(\frac{2M}{\bar{r}} \right)^{1/2} \Delta T \qquad (43)$$

$$\Delta x_{\text{rain}} \equiv \bar{r} \Delta \phi \qquad (44)$$

Coefficients on the right sides of these equations remain real inside the event horizon, so local rain coordinates are valid there (Figure 3).

Wait! How can we justify our derivation of rain coordinates from local shell coordinates, which are valid only outside the event horizon? To do so, we need to show that local rain coordinates lead back to the global rain metric, which is valid everywhere outside the singularity.

$$\Delta \tau^2 \approx \Delta t_{\text{rain}}^2 - \Delta y_{\text{rain}}^2 - \Delta x_{\text{rain}}^2 \qquad (45)$$

Substitute into (45) from (42) through (44):

$$\Delta \tau^2 \approx \Delta T^2 - \left[\Delta r + \left(\frac{2M}{\bar{r}} \right)^{1/2} \Delta T \right]^2 \qquad (46)$$

$$- \bar{r}^2 \Delta \phi^2$$

Multiply out:

$$\Delta \tau^2 \approx \left(1 - \frac{2M}{\bar{r}} \right) \Delta T^2 \qquad (47)$$

$$- 2 \left(\frac{2M}{\bar{r}} \right)^{1/2} \Delta T \Delta r - \Delta r^2 - \bar{r}^2 \Delta \phi^2$$

In the calculus limit, equation (47) becomes the global rain metric (32). The global rain metric is valid everywhere down to the singularity; therefore local rain coordinates can be constructed down to the singularity.

---

517   Box 5 derives the global rain embedding diagram, Figure 4, from the
518   global rain metric (32) and compares it with the embedding diagram for
519   Schwarzschild coordinates.

7.6 ■ TETRAD FORMS OF THE GLOBAL RAIN METRIC

521   *A difference of squares hides the cross term.*

522   Global rain metric (32) has a cross term. The metric for any local inertial
523   frame derived from this global metric does not have a cross term. For example:

$$\Delta \tau^2 \approx \Delta t_{\text{shell}}^2 - \Delta y_{\text{shell}}^2 - \Delta x_{\text{shell}}^2 \qquad (49)$$

$$\Delta \tau^2 \approx \Delta t_{\text{rain}}^2 - \Delta y_{\text{rain}}^2 - \Delta x_{\text{rain}}^2 \qquad (50)$$

524   Why this difference between global and local metrics? Can we use a set of
525   local inertial coordinates to create a form of the global metric that consists of
526   the sum and difference of squares? Try it! From expressions for local shell and
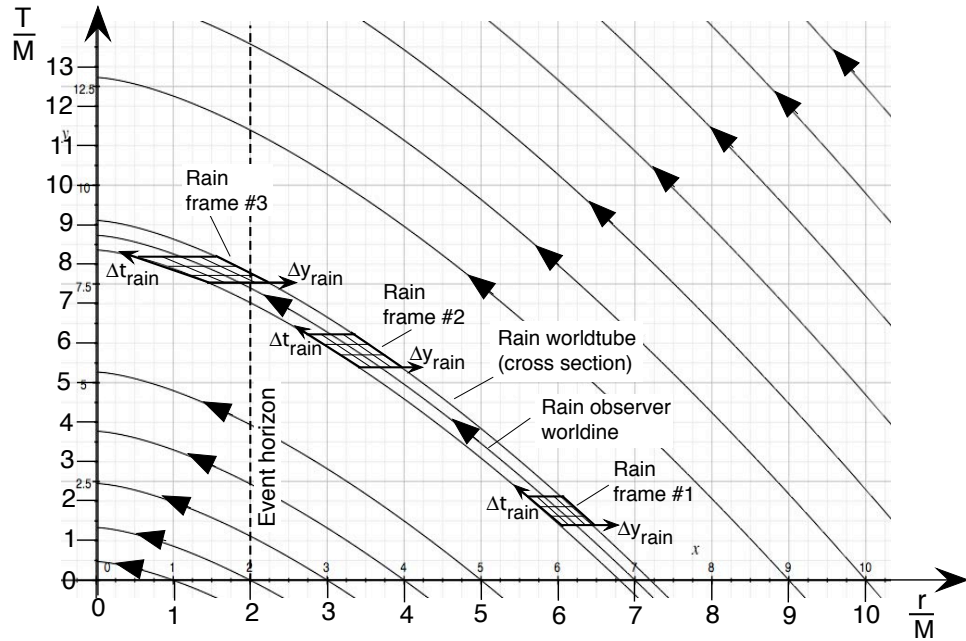
FIGURE 3 Raindrop worldlines plotted on an $[r, T]$ slice. Note that these worldlines are continuous through the event horizon (compare Figure 2). All these worldlines have the same shape and are simply displaced vertically with respect to one another. Around one of these worldlines we construct a worldtube (shown in cross section on this slice) that bounds local rain frames through which that rain observer passes.

rain coordinates in Boxes 2 and 4, respectively, simply write down two differential forms of the global metric. From local shell coordinates (Box 2):

$$d\tau^2 = \left[\left(1 - \frac{2M}{r}\right)^{1/2} dT - \left(\frac{2M}{r}\right)^{1/2} \left(1 - \frac{2M}{r}\right)^{-1/2} dr \right]^2 \tag{51}$$

$$- \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \qquad \text{(global rain metric)}$$

$$-\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \le \phi \le 2\pi$$
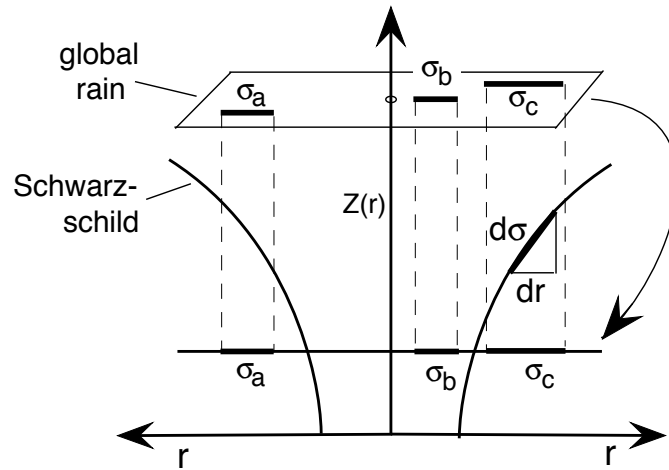
And from local rain coordinates (Box 4):

$$d\tau^2 = dT^2 - \left[dr + \left(\frac{2M}{r}\right)^{1/2} dT \right]^2 - r^2 d\phi^2 \qquad \text{(global rain metric)} \tag{52}$$

$$-\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \le \phi \le 2\pi$$

FIGURE 4   Figure for Box 5. Compare the embedding diagram outside the event horizon for Schwarzschild coordinates (the funnel) in Figures 11 through 13 in Section 3.9 with the flat embedding diagram of global rain coordinates (the flat surface shown in perspective across the top). The function $Z(r)$ is the fictional dimension we add in order to visualize these surfaces.

Box 5. Embedding diagrams for Schwarzschild and global rain coordinates.

Set $dT = 0$ in global rain metric (32), which then retains terms that include only r and ϕ:

$$d\sigma^2 = ds^2 = dr^2 + r^2 d\phi^2 \qquad (dT = 0) \qquad (48)$$

Surprise! The differential ruler distance $d\sigma$ obeys Euclidean flat-space geometry, which leads to the flat embedding diagram at the top of Figure 4 (point at $r = 0$ excluded). Because the global rain embedding diagram is flat, we can simply sum increments $d\sigma$ to draw arbitrary lines or curves with measured lengths σ_a, σ_b, and σ_c (for simplicity, drawn as parallel straight lines in Figure 4).

Figure 4 also repeats the embedding diagram outside the event horizon in Schwarzschild global coordinates from Figures 11 through 13 in Section 3.9.

Outside the event horizon both embedding diagrams are valid for what they describe. And the flat global rain embedding diagram is valid inside the event horizon as well.

What's going on here? Is space flat or funnel-shaped around this black hole? *That depends on our choice of global coordinates! Spacetime* as a unity is curved; but Nature does not care how we share the description of spacetime curvature among the terms of the global metric. In global rain coordinates the dT^2 and $dT dr$ terms describe spacetime curvature, which leaves the $[r, \phi]$ embedding diagram to show flat space. In contrast, for the Schwarzschild metric the dt^2 term and the dr^2 term share the description of spacetime curvature, which yields a funnel outside the event horizon on the $[r, \phi]$ embedding diagram. Each embedding diagram and global light cone diagram is a child of our (arbitrary!) choice of global coordinates in which they are expressed.

Recall Herman Minkowski's declaration (Section 2.7): "Henceforth space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." Each global metric displays that union in a different way.

₅₃₂   Are (51) and (52) valid global metrics? Yes! In Query 8 you multiply out
₅₃₃   these global metrics to show that they are algebraically equivalent to the
₅₃₄   original global rain metric (32).

₅₃₅

QUERY 8. The same global metric

Expand the right side of (51) and the right side of (52). Show that in both cases the result is equal to the right side of the original global rain metric (32).

Conclusion: All three forms of the global rain metric, (32), (51), and (52) are simply algebraic rearrangements of one another. So what? Why bother? Here's why: Suppose we are first given either metric (51) or metric (52). In that case we can immediately write down the expressions Δt_{frame}, Δy_{frame}, and Δx_{frame} for *some* local inertial frame. (We may not know right away *which* local inertial frame it is.)

A metric of the form (51) or (52) is called the **tetrad form** of the global metric. "Tetra" means "four," in this case the four dimensions of spacetime. A tetrad form rearranges the global metric in a form more useful to us.

DEFINITION 1. A tetrad form of a global metric

A **tetrad form of a global metric** consists of a sum and difference of squares, with no additional terms.

DEFINITION 2. Tetrad

A **tetrad** is a set of four differential expressions, each of which is squared in a "tetrad form" of a global metric.

Example: Equation (52) is the global rain metric in the tetrad form that leads to the local rain frame coordinates in Box 4.

?

Objection 8. *You said the tetrad consists of* four *differential expressions. I see only three.*

!

You're right. In most of this book we use only two global space dimensions, those on a slice through the center of the black hole. A third global space dimension would add a fourth Δz_{rain} component to the tetrad. We retain the professional terminology *tetrad* in spite of our simplification.

Comment 8. Tetrad as link

A tetrad is the link between a global metric and a local inertial frame. To specify a particular tetrad, give both the local inertial frame and the global metric. The local inertial frame stretches differentials d in the global metric to deltas Δ in the local inertial frame. This stretch creates elbow room to make measurements in the local inertial frame.

The global metric in tetrad form immediately translates into expressions for local inertial coordinates (Box 6). The remainder of this book will make primary use of such global tetrad metrics.

Box 6. A Brief History of Tetrads

The concept of a tetrad originates from the "repère mobile" (*moving frame*), introduced by Élie Cartan in the 1930s. Cartan showed that a sequence of local inertial coordinate systems, grouped along a set of curves such as worldlines, can provide a complete global description of a curved spacetime. He did so by introducing new calculus concepts on curved spaces, which extended the foundations of differential geometry laid by Bernhard Riemann in 1854.

Cartan's moving-frame theory was incomprehensible to Einstein, but later physicists found it useful and even necessary to study elementary particle physics in curved spacetime.

In this book we simplify the moving frame, or tetrad, to its most basic element: a set of local inertial frames in which motion is described using local inertial coordinates, with each frame and each local coordinate system related to the global coordinates provided by the metric. The tetrad is the bridge between the global metric and the local inertial metric in which we carry out all measurements and observations.

Because this metric is the sum and difference of the squares of the tetrad—such as in equations (51) and (52)—general relativists sometimes call the tetrad the **square root of the metric.**

7.7 ■ RAIN WORLDLINES OF LIGHT

573   *The light flash is ingoing or outgoing—or "outgoing."*

574   In the present chapter we consider only r-motions, motions that can be either
575   ingoing or outgoing along r-coordinate lines. (Chapter 11 analyzes the general
576   motion of light in global coordinates.) The r-motion of light is easily derived
577   from global rain metric (53), which you show in Query 8 to be equivalent to
578   global rain metric (32).

579

QUERY 9. Light r-motion in global rain coordinates

A. Multiply out the right side of (53) to show that it is equivalent to the global rain metric (32):

$$d\tau^2 = -\left[dr + \left\{1 + \left(\frac{2M}{r}\right)^{1/2}\right\}dT\right]\left[dr - \left\{1 - \left(\frac{2M}{r}\right)^{1/2}\right\}dT\right] - r^2 d\phi^2 \qquad (53)$$

$$\text{(global rain metric)}$$

582

B. For light ($d\tau = 0$) that moves along the r-coordinate line ($d\phi = 0$), show that (53) has two solutions for the rain map velocity of light, which are summarized by equation (54):

$$\frac{dr}{dT} = -\left(\frac{2M}{r}\right)^{1/2} \pm 1 \qquad \text{(light flash that moves along the } r\text{-coordinate line)} \qquad (54)$$

$$(\; - = \text{incoming light; } + = \text{outgoing or "outgoing" light})$$

C. Look separately at each element of the plus-or-minus sign in (54). Show that the solution with the lower sign ($-$) describes an *incoming* flash (light moving inward along the r-coordinate line). Then show that the solution with the upper sign ($+$) describes an *outgoing* flash (light moving outward along the r-coordinate line) outside the event horizon, but an "*outgoing*" flash inside the event horizon (Definition 3).

590

591   **DEFINITION 3. "Outgoing" light flash.**

592   The **"outgoing" light flash**—with quotes—is a flash inside the event
593   horizon whose r-motion is described by (54) with the plus sign. Inside
594   the event horizon dr is negative as T advances (positive dT), so that
595   even the "outgoing" light flash moves to smaller r-coordinate. The light
596   cone diagram Figure 5 shows this, and Figure 6 displays longer
597   worldlines of light flashes emitted sequentially by a plunging raindrop.
598   Compare the global rain worldline in Figure 6 with the Schwarzschild
599   worldlines in Figure 2.

600   We want to plot worldlines of incoming and outgoing (and "outgoing")
601   light flashes. Rewrite (54) to read

$$dT = \frac{dr}{-\left(\dfrac{2M}{r}\right)^{1/2} \pm 1} = \frac{r^{1/2}dr}{-(2M)^{1/2} \pm r^{1/2}} \qquad (r\text{-moving flash}) \qquad (55)$$

602   We carry the plus-or-minus sign along as we integrate to find expressions
603   for light that moves in either r-direction. Make the substitution:

$$u \equiv -(2M)^{1/2} \pm r^{1/2} \qquad (56)$$

604   From (56),

$$r^{1/2} = \pm[u + (2M)^{1/2}] \quad \text{and} \quad dr = +2\left[u + (2M)^{1/2}\right] du \qquad (57)$$

605   With these substitutions, equation (55) becomes

$$dT = \pm 2\frac{\left[u + (2M)^{1/2}\right]^2 du}{u} \qquad (r\text{-moving flash}) \qquad (58)$$

$$= \pm 2\left[u + 2\left(2M\right)^{1/2} + \frac{2M}{u}\right] du$$

606   Integrate the second line of (58) from an initial u_0 to a final u. The result
607   is:

$$\pm(T - T_0) = u^2 - u_0^2 + 4(2M)^{1/2}(u - u_0) + 4M \ln\left|\frac{u}{u_0}\right| \quad (r\text{-moving flash}) (59)$$

608   To restore global rain coordinates in (59), reverse the substitution in (56).
609   (*Hint:* To save time—and your sanity—replace $\pm$ in (59) with a symbol such
610   as Q.) There are two cases: *First case:* an incoming flash from a larger
611   r-coordinate $r_0 = r_A$ at coordinate $T_0 = T_A$ to a lower r at coordinate T. For

Incoming flash 612 this incoming flash, take the lower minus signs in (56) and (59). Multiply both
613 sides of the result by minus one to obtain:

$$T - T_{\rm A} = (r_{\rm A} - r) - 4M \left[\left(\frac{r_{\rm A}}{2M} \right)^{1/2} - \left(\frac{r}{2M} \right)^{1/2} \right] \tag{60}$$

$$+ 4M \ln \left[\frac{1 + (r_{\rm A}/2M)^{1/2}}{1 + (r/2M)^{1/2}} \right] \quad \text{(incoming flash)}$$

614

QUERY 10. Horizon-to-crunch global T-coordinate lapse for light

616
A. Verify that for $r = r_{\rm A}$ in (60), the elapsed global T-coordinate $T - T_{\rm A}$ is zero, as it must be for
light.   617

B. Show that from the event horizon $r_{\rm A} = 2M$ to the crunch point $r = 0$, the elapsed T-coordinate
$T - T_{\rm A} = 0.773M$.

C. Compare the result of Item B with the event-horizon-to-crunch wristwatch time
$\tau_{\rm raindrop} = (4/3)M$ in equation (3). Why is the result for a light flash in (60) less than the result
for the raindrop? Are the plots in Figure 6 consistent with this inequality?

623

Outgoing flash 624 *Second case:* the outgoing and "outgoing" light flashes from an initial
625 r-coordinate $r_0 = r_{\rm L}$ at coordinate $T_{\rm L}$ to a final r at coordinate T. In this case
626 we must take the upper, plus signs in (56) and (59). The result is:

$$T - T_{\rm L} = (r - r_{\rm L}) + 4M \left[\left(\frac{r}{2M} \right)^{1/2} - \left(\frac{r_{\rm L}}{2M} \right)^{1/2} \right] \tag{61}$$

$$+ 4M \ln \left[\frac{1 - (r/2M)^{1/2}}{1 - (r_{\rm L}/2M)^{1/2}} \right] \quad \text{(outgoing flash)}$$

Inside event horizon, 627 What does equation (61) predict when $r_{\rm L} < 2M$? Worldlines of light that
"outward" means 628 sprout upward from the raindrop worldline in Figure 6 show that after the
worldline with $dr < 0$. 629 diver falls through the event horizon, even the "outward" flash moves to
630 smaller r in global rain coordinates.
631 Figure 5 uses equations (60) and (61) to plot light cones for a selection of
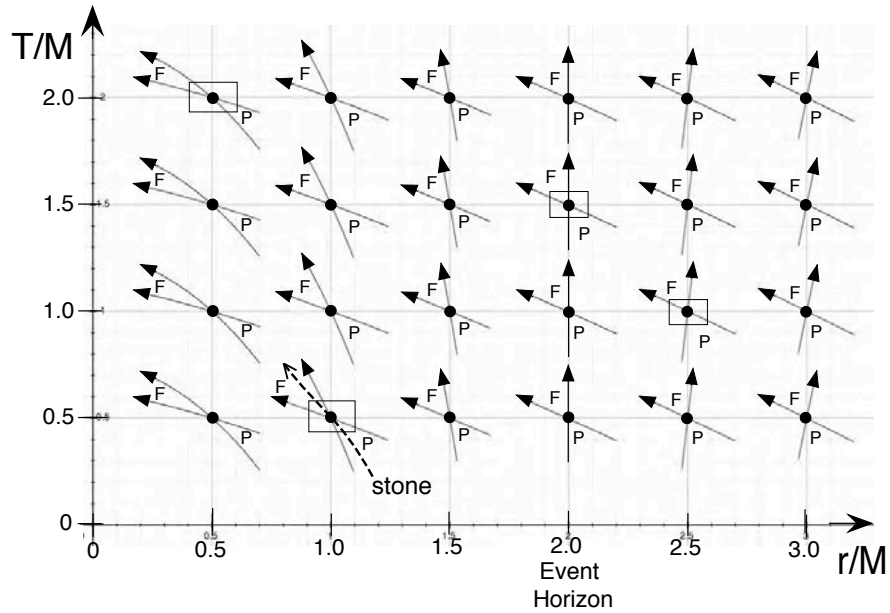632 events inside and outside of the event horizon.

633

QUERY 11. Motion to smaller r only
Use the dashed worldline of a stone in Figure 5 to explain, in one or two sentences, why "everything
moves to smaller r-coordinate" inside the event horizon. *Hint:* Think of the connection between
worldlines of stones and future light cones.

638

639

FIGURE 5 Light cone diagram on the $[r, T]$ slice, plotted from equations (60) and (61) for events both outside and inside the event horizon. Past and future events of each filled-dot-event are corralled inside the past (P) and future (F) light cone of that event. At each r-coordinate, the light cone can be moved up or down vertically without change of shape, as shown. Inside the event horizon, light and stones can move only to smaller r-coordinate. A few sample boxes show locally flat patches around a single event. The global rain T-coordinate conveniently runs forward along every worldline (in contrast to the Schwarzschild t-coordinate along some worldlines in the light cone diagram of Figure 8, Section 3.7).
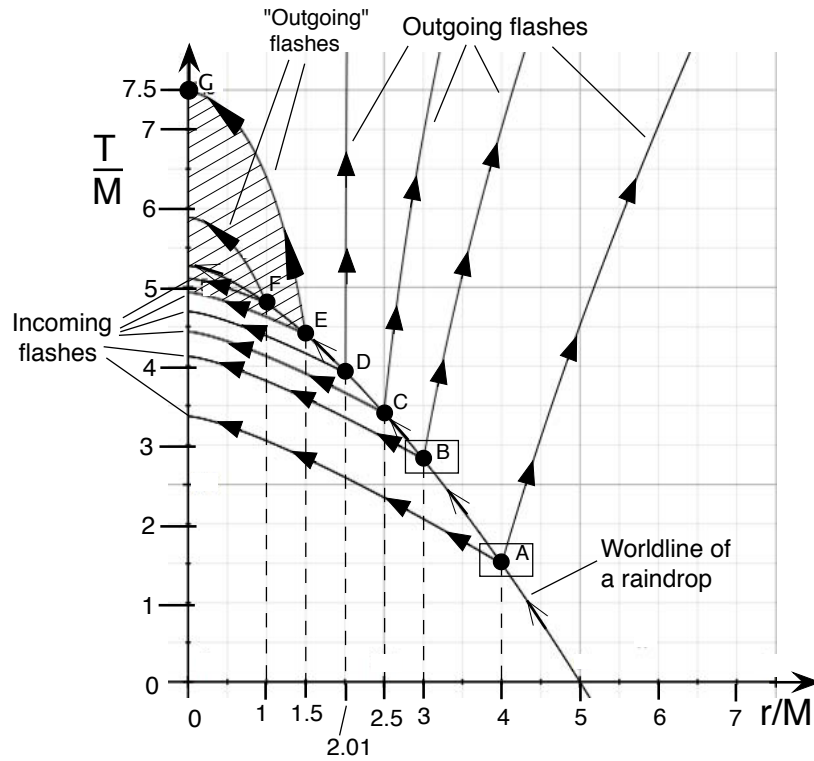
QUERY 12. Detailed derivation *Optional*
Show details of the derivation of equations (60) and (61) from equation (59). Recall the hint that follows equation (59).

Time to celebrate! The raindrop worldline in Figure 6 is continuous and smooth as it moves inward across the horizon. Global rain coordinates yield predictions that are natural and intuitive for us. With global rain coordinates, we no longer need to reconcile the awkward contrast between the discontinuous Schwarzschild worldlines of Figure 2 and the smooth advance of raindrop wristwatch time in Figure 1 (even though both of these plots are valid and technically correct). The simplicity of results for global rain coordinates leads us to use them from now on to describe the non-spinning black hole. Farewell, Schwarzschild metric!

FIGURE 6   A raindrop passes $r/M = 5$ at $T/M = 0$ and thereafter emits both incoming and outgoing flashes at events A through F. "Outgoing" flashes—with quotes—from events E and F move to smaller global r-coordinates, along with everything else inside the event horizon. Little boxes at A and B represent two of the many locally flat patches through which the rain observer passes as she descends. When the rain diver reaches event E, her "range of possible influence" consists of events in the shaded region, for example event F.

7.8 ■ THE RAIN OBSERVER LOOKS—AND ACTS

Which distant events can the rain observer see? Which can she influence?

You ride a raindrop; in other words, you fall from initial rest far from the black hole. What do you see radially ahead of you? behind you? Of all events that occur along this r-line, which ones can you influence from where you are? Which of these events can influence you? When can you no longer influence any events? To answer these questions we give the raindrop some elbow room, turn her into a **rain observer** who makes measurements and observations in a series of local rain frames through which she falls. This definition specializes the earlier general definition of an observer (Definition 4, Section 5.7).

Definition:
Rain observer

DEFINITION 4. Rain observer

A **rain observer** is a person or a data-collecting machine that rides a raindrop. As she descends, the rain observer makes a sequence of

666   measurements, each measurement limited to a local inertial rain frame
667   (Box 4).

?

668   **Objection 9.** *Wait: Go back! You have a fundamental problem that ruins*
669   *everything. The global rain metric (32) contains the r-coordinate, but you*
670   *have not defined the r-coordinate inside the event horizon. Section 3.3*
671   *defined the r-coordinate as "reduced circumference," that is, the*
672   *circumference of a shell divided by 2π. But you cannot build a shell inside*
673   *the event horizon, so you cannot define global coordinate r there.*
674   *Therefore you have no way even to describe the worldline of the rain*
675   *observer once she crosses the event horizon.*

!

676   Guilty as charged! Box 4 defined local rain coordinates and justified their
677   validity inside the event horizon, but we have not formally defined the
678   r-coordinate inside the event horizon, or how an observer might determine
679   its value there. Here is one way (Box 7): As the rain observer drops from
680   rest far from the black hole, she simultaneously releases a stone test
681   particle from rest beside her and perpendicular to her direction of motion.
682   Thereafter she uses radar or a meter stick to measure the distance to the
683   stone. In this way she monitors her r-coordinate as she descends inside
684   the event horizon.

685   Box 4 introduced local rain frames in which we can carry out and record
686   measurements using special relativity. Small boxes in Figures 5 and 6 represent
687   effectively flat patches on which we can construct local inertial frames. In this
688   chapter we allow the rain observer to look only at events that lie before and
689   behind her along her worldline. She can also send light flashes and stones to
690   influence (as much as possible) this limited set of events. (Chapter 11 allows
691   the raindrop observer to look all around her.)

692

QUERY 13. Observe ingoing and outgoing light flashes in a local rain frame.
How do light flashes that we describe as ingoing, outgoing, and "outgoing" in global rain coordinates
(Definition 3) move when observed entirely within a local rain frame? Answer this question with the
following procedure or some other method.

A. From (42) and (43) show that:

$$\Delta y_{\rm rain} = \left[\frac{\Delta r}{\Delta T} + \left(\frac{2M}{\bar{r}} \right)^{1/2} \right] \Delta t_{\rm rain} \quad \text{(light flash that moves along the r-coordinate line)} \tag{66}$$

B. Use an approximate version of (54) to replace the square bracket expression in (66):

$$\frac{\Delta y_{\rm rain}}{\Delta t_{\rm rain}} = \pm 1 \qquad \text{(light flash that moves along the r-coordinate line)} \tag{67}$$

C. Is equation (67) a surprise—or obvious? What does each sign mean for measurement of light
velocity inside a rain frame?

Box 7. Define the Value of r Inside the Event Horizon
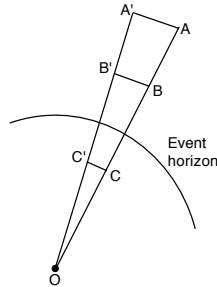
FIGURE 7 The rain observer measures her r-coordinate inside the event horizon.

Question: How can a rain observer inside the event horizon determine her current r-coordinate? *Answer:* To adapt the Chapter 3 definition of the r-coordinate—reduced circumference of the shell outside the event horizon—she measures only a tiny arc inside the event horizon.

The rain observer takes the path ABCO in Figure 7; the stone that accompanies her takes the converging path A'B'C'O. Draw a circular arc AA' and similar circular arcs BB' and CC'. The angle AOA' is the same for every arc, so the length of each arc represents the same fraction of the circumference of its corresponding circle. In equation form,

$$\frac{\left(\begin{array}{c}\text{length of}\\\text{arc AA'}\end{array}\right)}{\left(\begin{array}{c}\text{circumference}\\\text{of shell thru A}\end{array}\right)} = \frac{\left(\begin{array}{c}\text{length of}\\\text{arc BB'}\end{array}\right)}{\left(\begin{array}{c}\text{circumference}\\\text{of shell thru B}\end{array}\right)}\tag{62}$$

The values of r-coordinates r_A and r_B are stamped on the shells outside the event horizon, so the denominators of the two sides of the equation become $2\pi r_A$ and $2\pi r_B$, respectively, and we cancel the common factor 2π.

If the angle at the center is small enough, we can replace the length of each circular arc with the straight-line distance measured between, say, A and A' shown in Figure 7. Call this measured distance AA'. And call BB' the corresponding straight-line distance measured between B and B'. Then (62) becomes,

$$\frac{\text{AA}'}{r_A} \approx \frac{BB'}{r_B}\tag{63}$$

The rain observer monitors the distance to her accompanying stone as she descends, with radar or—if the stone lies near enough—directly with a meter stick. While she is outside the event horizon, the rain observer reads the value of the r-coordinate r_A stamped on that spherical shell as she passes it and the measured distance AA' between the two rain frames, and later BB' as she passes and reads off r_B. She verifies that this direct reading with the value of r_B is the same as that calculated with the equation:

$$r_B \approx \frac{BB'}{\text{AA}'}r_A\tag{64}$$

At any point C inside the event horizon, the observer measures distance CC' and *defines* her instantaneous r-coordinate r_C as:

$$r_C \equiv \frac{\text{CC}'}{\text{AA}'}r_A \qquad \text{(definition)}\tag{65}$$

This definition of the r-coordinate inside the event horizon is a direct extension of its definition outside the event horizon and is valid for any observer falling along an r-coordinate line.

D. From observations inside a rain frame, is there any difference between a light flash we describe
   as *outgoing* and one we describe as *"outgoing"*? More generally, can observations carried out
   entirely inside a rain frame tell us whether that rain frame is outside of, at, or inside the event
   horizon?

₇₀₄

₇₀₅

As our rain observer arrives at any of the emission points A through F in
Figure 6, she can try—by firing an ingoing or outgoing stone or light flash—to
influence a later event located within the region embraced by the worldlines of
the incoming and outgoing (or "outgoing") flashes from that event. The
farther toward the singularity the rain observer falls, the smaller is this "range

"Range of
possible influence"

711 of possible influence." When she arrives at event E, for example, she can
712 influence only events in the shaded region in Figure 6, including event F.

713

QUERY 14. Future events that the rain observer can still influence.
714
Make four photocopies of Figure 6. On each copy, choose one emission event A through D or F.
715

A. Shade the spacetime region in which a rain observer can influence future events once she has
716
   arrived at that emission event.
717

B. Which of these emission points is the last one from which the rain observer can influence events
718
   that occur at $r > 2M$?
719

720

721   As she crosses the event horizon, how long will it be on her wristwatch
722 before she reaches the singularity? Equation (3) tells us this wristwatch time is
723 $4M/3$ meters.

?
724   **Objection 10.** *Ha! I can live a lot longer inside the event horizon than your*
725   *measly $4M/3$ meters of time. All I have to do, once I get inside the event*
726   *horizon, is to turn on my rockets and boost myself radially outward. For*
727   *example, I can fire super-powerful rockets at event E and follow the*
728   *"outgoing" photon flash from E that reaches the singularity at Event G (top*
729   *left corner of that figure). That final T-value is much greater than the*
730   *T-value where the raindrop worldline reaches the singularity.*

!
731   Be careful! You want to maximize wristwatch time, not the span of global T
732   which, remember, is usually *not* measureable time. The wristwatch time is
733   zero along the worldline of a light flash, so the closer you come to that
734   worldline the smaller will be your wristwatch time during descent from
735   Event E to an event just below G in Figure 6.

?
736   **Objection 11.** *Okay, then! I'll give up the rocket blast, but I still want to*
737   *know what is the longest possible wristwatch time for me to live after I*
738   *cross the event horizon.*

!
739   Part B of Exercise 3 at the end of this chapter shows how to extend your
740   lifetime to πM meters after you cross the event horizon, which is a bit
741   longer than the raindrop $4M/3$ meters. You will show that the way to
742   achieve this is to drip from the shell just outside the event horizon; that is,
743   you release yourself from rest in global coordinates at $r = 2M^+$.

?
744   **Objection 12.** *Can I increase my lifetime inside the event horizon by*
745   *blasting rockets in either ϕ direction to add a tangential component to my*
746   *global velocity?*
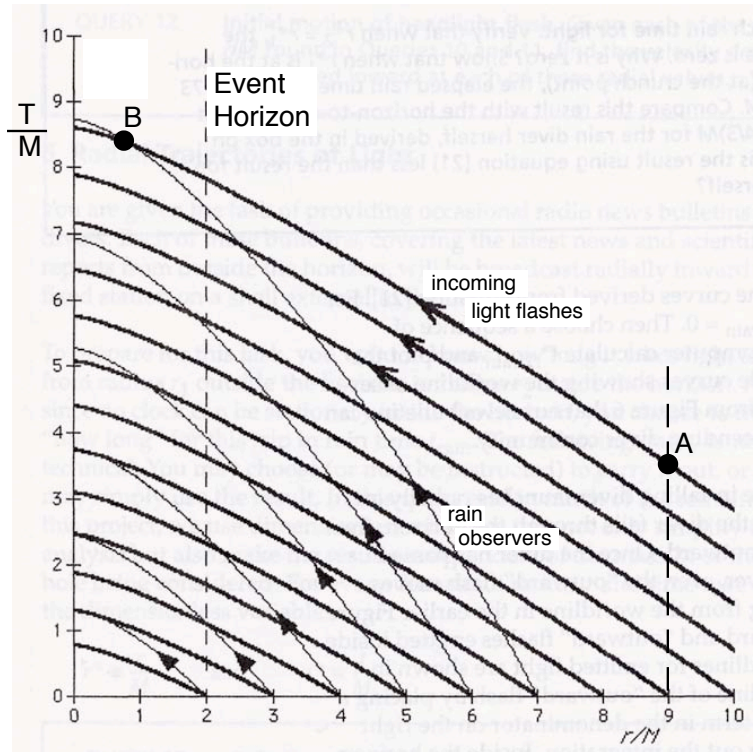
FIGURE 8   Worldlines of rain observers (thin curves) and incoming light flashes (thick curves) plotted on the $[r, T]$ slice. All rain observer worldlines have the same form and can be moved up and down without change in shape. Light-flash worldlines also have the same form. A shell observer at $r/M = 9$ emits a signal from Event A that a rain observer receives inside the event horizon at Event B.

!
  747   The present chapter analyzes only r-motion. In the exercises of Chapter 8
  748   you will show that the answer to your question is no; a tangential rocket
  749   blast *decreases* your lifetime inside the event horizon. A wristwatch time
  750   lapse of πM is the best you can do. Sorry.

  751   Figure 8 displays both global worldlines of light flashes—thick curves
  752   derived from (60)—and worldlines of raindrops—thin curves derived from (2).
  753   It is evident from Figure 8 that news bulletins—incoming or outgoing
  754   electromagnetic radio bursts fired outside the event horizon—can be scheduled
  755   to catch up with the diver community at any predetermined r-coordinate.

  756

QUERY 15. Can you see a rain diver ahead of you?
Compare Figures 6 and 8. Label as #1 the rain diver whose worldline is plotted in Figure 6. Label as #2 a second rain diver who falls along the *same r-coordinate line in space* as rain diver #1, but at a T-coordinate greater by ΔT. Use worldlines of Figure 8 to answer the following questions. (*Optional:*

Derive analytic solutions to these questions and compare the results with your answers derived from the figure.)   762

    A. Over what range of delays ΔT will rain diver #2 be able to see the flash from Event E emitted by diver #1 but not the flash from Event F?

    B. Over what range of delays ΔT will rain diver #2 be able to see the flash from Event D emitted by diver #1 but not the flash from Event E?

    C. Answer this question decisively: Can any later diver #2 see a flash emitted by diver #1 at say, $r/M = 0.1$, just before diver #1 reaches the singularity?

769

770

QUERY 16. Can you see a rain diver behind you?

Extend the results of Query 15 to analyze a third rain diver labeled #3 who falls along the same r-coordinate line in space as the earlier two rain divers, but at a T-coordinate that is smaller by $-\Delta T$. Diver #3 looks outward at light pulses emitted by diver #1. Use the worldlines of Figure 6 to answer the following questions. (*Optional:* Derive analytic solutions to these questions and compare the results with your answers derived from the figure.)

    A. Over what range of earlier launches $-\Delta T$ will rain diver #3 be able to see the flash from Event D emitted by diver #1 but not the flash from Event F?

    B. Over what range of earlier launches $-\Delta T$ will rain diver #3 be able to see the flash from Event A emitted by diver #1 but not the flash from Event D?

    C. Answer this question decisively: Can any earlier diver #3 see a flash emitted by diver #1 at say, $r/M = 0.1$, just before diver #1 reaches the singularity?

783

784

## QUERY 17. Can you see the crunch point ahead of you?

You are the rain diver whose worldline is plotted in Figure 6. By some miracle, you survive to reach the center of the black hole. Show that you *cannot* see the singularity ahead of you before you arrive there. (What a disappointment after all the training, preparation, and sacrifice!)

789

790

QUERY 18. Rain frame energy

The rain frame is inertial. Therefore the expression for energy of a stone in rain frame coordinates is that of special relativity, namely $E_{\rm rain}/m = dt_{\rm rain}/d\tau$ (Section 1.7). Recall also from the differential version of (42) that $dt_{\rm rain} = dT$.

    A. Use (35) together with the special relativity expression for the rain frame energy of the stone and the above identification of global rain T with rain frame time from (42) to show that:

Box 8. The River Model
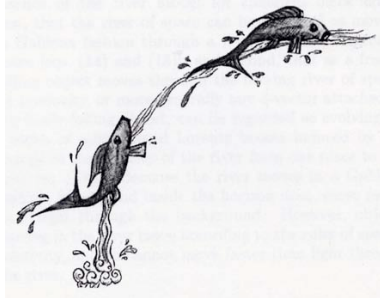
FIGURE 9   In the river model of a black hole, fish that swim at different rates encounter a waterfall. The fastest fish represents a photon. "The fish upstream can make way against the current, but the fish downstream is swept to the bottom of the waterfall." The event horizon corresponds to that point on the waterfall at which the upward-swimming photon-fish stands still. [From Hamilton and Lisle, see references]

Andrew Hamilton and Jason Lisle created a **river model** of the black hole. In their model, water "looks like ordinary flat space, with the distinctive feature that space itself is flowing inward at the Newtonian escape velocity. The place where the infall velocity hits the speed of light . . . marks the event horizon. . . . Inside the event horizon, the infall velocity exceeds the speed of light, carrying everything with it." At every r-coordinate near a black hole the river of space flows past at the speed of a raindrop, namely a stone that falls from initial rest far from the waterfall.

Envision flat spacetime distant from a black hole as still water in a large lake with clocks that read raindrop wristwatch time τ_{raindrop} floating at rest with respect to the water. At one side

of the lake the water drifts gently into a river and that carries the raindrop clocks with it. River water moves faster and faster as it approaches and flows over the brink of the waterfall. Each jet of falling water narrows as it accelerates downward. Fish represent objects that move in the river/space; the fastest fish represents a photon. At some point below the lip of the waterfall, not even the photon-fish can keep up with the downward flow and is swept to the bottom of the falls (Figure 9) The black hole event horizon corresponds to the point at which the upward-swimming "photon-fish" stands still.

The river model helps us to visualize many effects observed near the black hole. Hamilton and Lisle write, "It explains why light cannot escape from inside the event horizon, and why no star can come to rest within the event horizon. It explains how an extended object will be stretched radially by the inward acceleration of the river, and compressed transversely by the spherical convergence of the flow. It explains why an object that falls through the event horizon appears to an outsider redshifted and frozen at the event horizon: as the object approaches the event horizon, light [a photon-fish] emitted by it takes an ever-longer global time to forge against the onrushing current of space and eventually to reach the outside observer."

Hamilton and Lisle show that the river model is consistent with the results of general relativity. In that sense the river model is *correct* and *complete*.

The river model is a helpful visualization, but that visualization comes at a price. It carries two misleading messages: First, that space itself—represented by the river—is observable. We easily observe various flows of different rivers on Earth, but no one—and no instrument—registers or observes any "flow of space" into a black hole. Second, the river model embodies global rain coordinates, but we have seen that there are an unlimited number of global coordinates for the black hole, many of which cannot be envisioned by the river model.

$$\frac{E_{\text{rain}}}{m} \equiv \lim_{\Delta\tau \to 0} \frac{\Delta t_{\text{rain}}}{\Delta\tau} \equiv \frac{dT}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \left[\frac{E}{m} + \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau}\right] \quad (68)$$

B. Is it possible for E_{rain}/m to become negative inside the event horizon? Would any observer complain if it did?

C. Same questions as Item B for E/m, the global map energy per unit mass.

D. Perform a Lorentz transformation (Section 1.10) with v_{rel} and γ_{rel} from (14) to obtain E_{shell} in terms of E. Compare with E and E_{shell} in Schwarzschild coordinates from Sections 6.2 and 6.3.

Box 9. The Planck length

General relativity is a classical—non-quantum—theory (Box 7, Section 6.7). One of its beauties is that, when applied to the black hole, general relativity points to its own limits. The Schwarzschild metric plus the Principle of Maximal Aging predict that everything which moves inward across the horizon will end up on the singularity, a point. We know that this is does not satisfy quantum mechanics: The Heisenberg uncertainty principle of quantum mechanics tells us that a single electron confined to a point has unlimited momentum. So not even a single electron—much less an entire star gobbled up by the black hole—can be confined to the singularity. In this book we assume that classical general relativity is valid until very close to the singularity. How close? One estimate is the so-called **Planck length**, derived from three fundamental constants:

$$\text{Planck length} = \left(\frac{hG}{2\pi c^3}\right)^{1/2} = 1.616\,199 \times 10^{-35}$$

(69)

in meters, with an uncertainty of ± 97 in the last two digits. The presence in this equation of Planck's constant $h = 1.054\,571\,726 \times 10^{-34}$ Joule-second (± 15 in the last two digits) tells us that we have entered the realm of quantum mechanics, where classical general relativity is no longer valid. Cheer up! Before any part of you arrives at the Planck distance from the singularity, you will no longer feel any discomfort.

What happens when a single electron arrives at a Planck length away from the singularity? Nobody knows!

7.9 ■ A MERCIFUL ENDING?

How long does the "terminal spaghettification" process last?

To dive into a black hole is to commit suicide, which may go against religious, moral, or ethical principles—or against our survival instinct. Aside from such considerations, no one will volunteer for your black-hole diver research team if she predicts that as she approaches the crunch point her death will be painful. Your task is to estimate the **ouch time** τ_{ouch}, defined as the lapse of time on the wristwatch of the diver between her first discomfort and her arrival at the singularity, $r = 0$.

Is death near
the singularity
painful?

QUERY 19. Preliminary: Acceleration g in units of inverse meters.
Newton's expression for gravitational force in conventional units:

$$F_{\text{conv}} \equiv m_{\text{conv}} g_{\text{conv}} = -\frac{GM_{\text{conv}} m_{\text{conv}}}{r^2} \qquad \text{(Newton)} \qquad (70)$$

A. Verify the resulting gravitational acceleration in units of inverse meters:

$$g \equiv \frac{g_{\text{conv}}}{c^2} = -\frac{M}{r^2} \qquad \text{(Newton)} \qquad (71)$$

B. Show that at Earth's surface the Newtonian acceleration of gravity has the value given inside the front cover, namely

$$|g_{\text{Earth}}| \equiv |g_{\text{E}}| = \left| -\frac{M_{\text{Earth}}}{r_{\text{Earth}}^2} \right| = 1.09 \times 10^{-16} \text{ meter}^{-1} \qquad \text{(Newton)} \qquad (72)$$

Rain frame
observer
feels tides.

820   The rain observer is in free fall and does not feel any net force as a result
821   of local acceleration. However, she does feel radially stretched due to a
822   *difference* in acceleration between her head and her feet, along with a
823   compression from side to side. We call these differences **tidal accelerations**.
824

QUERY 20. Tidal acceleration along the r-coordinate line
We want to know how much this acceleration *differs* between the head and the feet of an in-falling rain
observer. Take the differential of g in (71). Convert the result to increments over a patch of average $\bar{r}$.
Show that

$$\Delta g \approx \frac{2M}{\bar{r}^3}\Delta r \qquad \text{(Newton)} \qquad (73)$$

830   What does Einstein say about tidal acceleration? Section 9.7 displays the
831   correct general relativistic expressions for the variation of local gravity with
832   spatial separation. *Surprise:* The expression for tidal acceleration in *any*
833   inertial frame falling along the r-coordinate line has a form identical to the
834   Newtonian result (73), and thus for the local rain frame becomes:

$$\Delta g_{\rm rain} \approx \frac{2M}{\bar{r}^3}\Delta y_{\rm rain} \qquad (74)$$

Define "discomfort."

835   What are the criteria for discomfort? Individual rain observers will have
836   different tolerance to tidal forces. To get a rough idea, let the rain observer's
837   body be oriented along the r-coordinate line as she falls, and assume that her
838   stomach is in free fall, feeling no stress whatever. Assume that the rain
839   observer first becomes uncomfortable when the *difference* in local acceleration
840   between her free-fall stomach and her head (or her feet), that stretches her, is
841   equal to the acceleration at Earth's surface. By this definition, the rain
842   observer becomes uncomfortable when her feet are pulled downward with a
843   force equal to their weight on Earth and her head is pulled upward with a
844   force of similar magnitude. Let her height be h in her frame, and the ruler
845   distance between stomach and either her head or her feet be half of this, that
846   is, $\Delta y_{\rm rain} = h/2$.
847

QUERY 21. The r-value for the start of "ouch."
From our criteria above for discomfort, we have:

$$\Delta g_{\rm rain\,ouch} \equiv g_{\rm E} \qquad \text{(stomach-to-foot distance)} \qquad (75)$$

Show that the r-value for the start of "ouch," namely $r_{\rm ouch}$, is:

$$r_{\rm ouch} = \left(\frac{Mh}{g_{\rm E}}\right)^{1/3} \qquad (h = \text{head-to-foot height}) \qquad (76)$$

851

852

QUERY 22. Three cases for the start of "ouch."
Approximate h, the head-to-foot distance, as 2 meters. Find the value of the ratio $r_{\text{ouch}}/(2M)$ for these cases:

   A. A black hole with ten times the mass of our Sun.
   B. The "20-year black hole" in Query 3.
   C. Suppose that the ouch r-coordinate is at the event horizon. What is the mass of the black hole as a multiple of the Sun's mass?

QUERY 23. The wristwatch ouch time τ_{ouch}

   A. Use equation (2) to show that the raindrop ouch time τ_{ouch} (the wristwatch time between initial ouch and arrival at the singularity) is *independent of the mass of the black hole*:

$$\tau_{\text{ouch}} = \frac{1}{3}\left(\frac{2h}{g_{\text{E}}}\right)^{1/2} \qquad \text{(raindrop wristwatch ouch time in meters)} \qquad (77)$$

   Here, recall, h is the height of the astronaut, about 2 meters, and the value of g_{E} is given in (72). Show that the wristwatch ouch time in seconds, the same for *all* non-spinning black holes, is:

$$\tau_{\text{ouch}} = \frac{1}{3c}\left(\frac{2h}{g_{\text{E}}}\right)^{1/2} \qquad \text{(raindrop wristwatch ouch time in seconds)} \qquad (78)$$

   B. Substitute numbers into equation (78). Show that the duration of raindrop wristwatch ouch time is about 2/9 of a second for *every* non-spinning black hole, independent of its mass M. *Guess:* Will pain signals travel from your extremities to your brain during this brief wristwatch ouch time?

7.10 ■ EXERCISES

1. Crossing the Event Horizon

Pete Brown disagrees with the statement, "No special event occurs as we fall
through the event horizon." He says, "Suppose you go feet first through the
event horizon. Since your feet hit the event horizon before your eyes, then your
feet should disappear for a short time on your wristwatch. When your eyes
pass across the event horizon, you can see again what's inside, including your
feet. So tie your sneakers tightly or you will lose them in the dark!" Is Pete
correct? Analyze his argument without criticizing him.

2. Equations of Motion of the Raindrop

From equations in Chapter 6 we can derive the equations of motion for a
raindrop in Schwarzschild coordinates. From the definition of the raindrop,

$$\frac{E}{m} = 1 \quad \text{and} \quad \frac{d\phi}{d\tau} = 0 \qquad \text{(raindrop in Schwarzschild coordinates} \quad (79)$$
$$\text{and in global rain coordinates)}$$

In addition, equation (23) in Section 6.4 tells us that

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop in Schwarzschild coordinates} \qquad (80)$$
$$\text{and in global rain coordinates)}$$

From equation (13) in Section 6.4, you can easily show that

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad \text{(raindrop in Schwarzschild coordinates} \qquad (81)$$
$$\text{and in global rain coordinates)}$$

Now derive the raindrop equations of motion in global rain coordinates.
First, show that both $dr/d\tau$ and $d\phi/d\tau$ have the same form in global rain
coordinates as in Schwarzschild coordinates, as stated in the labels of
equations (80) and (81). Second, use equations (80) and (81) plus equation
(35) to show that

$$\frac{dT}{d\tau} = 1 \qquad \text{(raindrop in global rain coordinates)} \qquad (82)$$

Comment 9. Simple definition of global rain T
Equation (82) can be used as the definition of global rain coordinate differential
dT. In other words, we *choose* dT equal to the differential lapse of wristwatch
time on a falling raindrop.

910 **3. Different masses for the "20-year black hole."**

911 This chapter describes a "20-year black hole," defined as one for which the
912 wristwatch on a raindrop registers a 20-year lapse between its crossing of the
913 event horizon and its arrival at the singularity. But the wristwatch may be on
914 a hailstone—flung radially inward from far away from the black hole—or a
915 drip—dropped from rest from a shell outside the event horizon. What is the
916 required mass of the "20-year black hole" in these two cases?

917   A. We fling an incoming **hailstone** inward along the r-coordinate line
918   with initial shell speed $|v_{\text{far}}|$ from far away from the black hole. A
919   lengthy derivation of the wristwatch time from event horizon to the
920   singularity yields the result:

$$\tau_{\text{Aail}}[2M \to 0] = M \left[\frac{2}{v_{\text{far}}^2 \gamma_{\text{far}}} - \frac{1}{v_{\text{far}}^3 \gamma_{\text{far}}^3} \ln \left(\frac{1 + v_{\text{far}}}{1 - v_{\text{far}}} \right) \right] \qquad (83)$$

921   where, remember, $\gamma \equiv (1 - v^2)^{-1/2}$ and we treat v_{far} as a (positive)
922   speed. Answer questions in the following items:

923   a. *Guess:* In the case of the hailstone, will the mass of "20-year black
924   hole" be greater or less than that for the raindrop?

925   b. Consider $\gamma_{\text{far}} = 2$. What is the value of v_{far}?

926   c. Show that for this particular value $\gamma_{\text{far}} = 2$, the first term inside
927   the square bracket in (83) alone gives the same result as the
928   raindrop in equation (3).

929   d. Was your guess in Item **a** correct or incorrect?

930   e. What is the mass of the "20-year black hole" for that hailstone?
931   How does it compare to the mass of the "20-year black hole" for
932   the raindrop?

933   B. A **drip** drops from rest on a shell of global coordinate $r_0 > 2M$.
934   Another lengthy derivation of the wristwatch time from event horizon
935   to the singularity yields the result:

$$\tau_{\text{drip}}[2M \to 0] = \qquad\qquad\qquad\qquad\qquad\qquad\qquad (84)$$

$$2M \left(\frac{2M}{r_0} \right)^{-3/2} \left[-\left(\frac{2M}{r_0} \right)^{1/2} \left(1 - \frac{2M}{r_0} \right)^{1/2} + \arctan \left(\frac{2M/r_0}{1 - 2M/r_0} \right)^{1/2} \right]$$

936   a. *Guess:* In the case of a drip, will the mass of "20-year black hole"
937   be greater or less than that for the raindrop?

938   b. Next, take the limiting case $r_0 \to 2M$. Show that in this limit
939   arctan takes the value $\pi/2$.

940   c. Show that in this case $\tau_{\text{drip}}[2M \to 0] \to \pi M$.

941   d. Was your guess in Item **a** correct or incorrect?

e. What is the mass of the "20-year black hole" for that drip? How
does it compare to the mass of the "20-year black hole" for the
raindrop?

C. *Fascinating but optional:* The next to last paragraph in Box 8 states
that every stone that passes inward across the event horizon at $r = 2M$
moves at that r-value with shell velocity $v_{\text{shell}} = -1$, the speed of light
(as a limiting case). Since this holds for *all* r-diving stones, how can the
masses of "20-year black holes" possibly differ for raindrops, hailstones,
and drips?

4. Map energy of a drip released from r_0

A. Derive the following expression for E/m in global rain coordinates for a
drip released from rest with respect to the local shell frame at $r_0 > 2M$:

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} \quad \text{(drip released from rest at } r_0) \qquad (85)$$

Compare with equation (33) of Chapter 6. Are you surprised by what
you find? Should you be?

B. What are the maximum and minimum values of E/m in (85) as a
function of r_0? How can the minimum value possibly be less than the
rest energy m of the stone measured in an inertial frame?

C. Is expression (85) consistent with the value $E/m = 1$ for a raindrop?

D. Is expression (85) valid for $r_0 < 2M$? What is the physical reason for
your answer?

E. For $r_0 > 2M$, is expression (85) still valid when that stone arrives
inside the event horizon?

5. Map energy of a hailstone

A. Derive the following expression for E/m in global rain coordinates for a
hailstone hurled radially inward with speed v_{far} from a shell very far
from the black hole.

$$\frac{E}{m} = \gamma_{\text{far}} \equiv \left(1 - v_{\text{far}}^2\right)^{-1/2} \qquad \text{(hailstone)} \qquad (86)$$

Compare this expression with results of Exercise 7 in Chapter 6. Are
you surprised by what you find? Should you be?

B. Is expression (86) for the hailstone consistent with expression (85) for
the drip? consistent with $E/m = 1$ for a raindrop?

972 **6. Motion of outgoing light flash outside and at the event horizon**

973 Find the maximum value of r/M at which the "outgoing" flash moves to
974 larger r, that is $dr > 0$, at each of these global map velocities:

975   A. $dr/dT = 0.99$
976   B. $dr/dT = 0.9$
977   C. $dr/dT = 0.5$
978   D. $dr/dT = 0$

979 **7. Motion of the "outgoing" flash inside the event horizon.**

980 Find the value of r/M at which the "outgoing" flash moves to smaller r, that is
981 $dr < 0$, at each of these global map velocities:

982   A. $dr/dT = -0.1$
983   B. $dr/dT = -0.5$
984   C. $dr/dT = -1$
985   D. $dr/dT = -9$

986 **8. Motion of the incoming flash**

987 At each value of r/M found in Exercises 6 and 7, find the value of dr/dT for
988 the *incoming* flash.

7.11 989 ■ **REFERENCES**

990 Box 1, Eggbeater Spacetime comes in part, from Chapter 13 of *Black Holes*
991   *and Time Warps* by Kip Thorne. The original description of a spherically
992   symmetric collapse into an uncharged, non-spinning black hole was by J. R.
993   Oppenheimer and H. Snyder, *Physical Review,*Volume 56, pages 455–459
994   (1939). Vladimir Belinsky, Isaac Markovich Khalatnikov, Evgeny
995   Mikhailovich Lifshitz, *Advances in Physics*, Volume 31, page 639 (1982).
996   Charles W. Misner, *Physical Review Letters*, Volume 22, page 1071.

997 Wristwatch time from event horizon to singularity: Geraint F. Lewis and
998   Juliana Kwan, "No Way Back: Maximizing survival time below the
999   Schwarzschild event horizon," arXiv:0705.1029v1 [physics.ed-ph] 8 May 2007

1000 Box 8. The River Model: Andrew J. S. Hamilton and Jason P. Lisle, "The
1001   river model of black holes," *American Journal of Physics*, Volume 76,
1002   Number 6, June 2008, pages 519-532.

1003 Global rain coordinates and the global rain metric: Paul Painlevé, "La
1004   mécanique classique et la théorie de la relativité", C. R. Acad. Sci. (Paris)

1005   173, 677-680(1921). Allvar Gullstrand, "Allgemeine Lösung des statischen
1006   Einkörperproblems in der Einsteinschen Gravitationstheorie", Arkiv. Mat.
1007   Astron. Fys., Volume 16, Number 8, 1-15 (1922). For a more modern
1008   derivation and references, see "Regular coordinate systems for Schwarzschild
1009   and other spherical spacetimes," by Karl Martel and Eric Poisson, *American*
1010   *Journal of Physics,*Volume 69, Number 4, April 2001, pages 476-480.

1011   Comment 6: "Einstein in Paris: Einstein Presents and Discusses His Theory"
1012   by Charles Nordmann, translated by Shawna Halevy, *21st Century Science*
1013   *& Technology,* Summer 2011, pages 19-37.

1014   Realization that the apparent singularity at $r = 2M$ in the Schwarzschild
1015   metric is "fictional": G. Lemaître (1933). *Annales de la Société Scientifique*
1016   *de Bruxelles.* Volume **A53** pages $51-85$.

1017   Quote on mutable physical law: John Archibald Wheeler with Kenneth Ford,
1018   *Geons, Black Holes, and Quantum Foam*, New York, W. W. Norton
1019   Company, 1998, page 351.