Chapter 1. Speeding

- *What is the key idea of relativity?*
- *Everything is relative, right?*
- *"Space and time form a unity called spacetime." Huh?*
- *Do people in relative motion* age *differently? Do they* feel *the slowing down/speeding up of their aging?*
- *What is the farthest galaxy I can possibly visit in person?*
- *How do relativistic expressions for energy and momentum differ from those of Newton?*
- *When and why does special relativity break down, and what warns us that this is about to happen?*

CHAPTER 1

Speeding

Edmund Bertschinger & Edwin F. Taylor *

> *I've completely solved the problem. My solution was to analyze the concept of time. Time cannot be absolutely defined, and there is an inseparable relation between time and signal velocity.*
>
> —Albert Einstein, May 1905, to his friend Michele Besso

1.1 ■ SPECIAL RELATIVITY

Special relativity and general relativity

Special relativity distinguished from **General relativity**

Special relativity describes the very fast and reveals the unities of both space-time and mass-energy. **General relativity**, a **Theory of Gravitation**, describes spacetime and motion near a massive object, for example a star, a galaxy, or a black hole. The present chapter reviews a few key concepts of special relativity as an introduction to general relativity.

Begin relativity with a stone wearing a wristwatch.

What is at the root of relativity? Is there a single, simple idea that launches us along the road to understanding? At the beginning of *Alice in Wonderland* a rabbit rushes past carrying a pocket watch. At the beginning of our relativity adventure a small stone wearing a wristwatch flies past us.

Observe two events in laboratory frame.

The wristwatch ticks once at Event 1, then ticks again at Event 2. At each event the stone emits a flash of light. The top panel of Figure 1 shows these events as observed in the laboratory frame. We assume that the laboratory is an **inertial reference frame**.

Definition: **inertial frame**

DEFINITION 1. Inertial frame

An **inertial reference frame**, which we usually call an **inertial frame**, is a region of spacetime in which Newton's first law of motion holds: *A free stone at rest remains at rest; a free stone in motion continues that motion at constant speed in a straight line.*

We are interested in the records of these two events made by someone in the laboratory. We call this someone, the **observer**:

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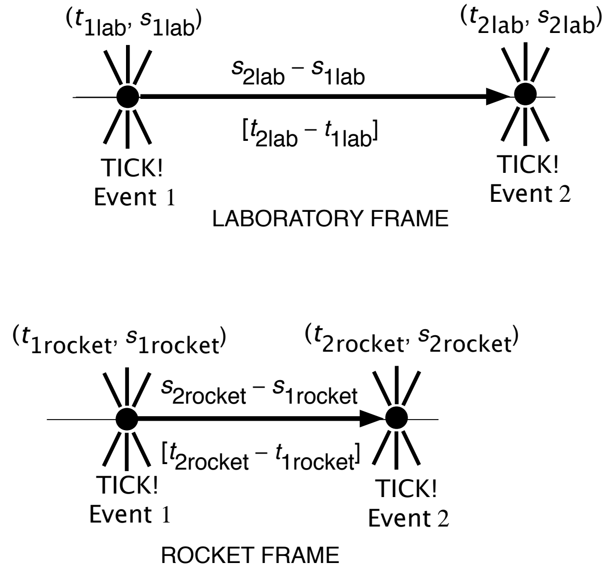

FIGURE 1 A free stone moves through a laboratory at constant speed. The stone wears a wristwatch that ticks as it emits a first flash at Event 1 and a second flash at Event 2.*Top panel:* The laboratory observer records Event 1 at coordinates $(t_{1\text{lab}}, s_{1\text{lab}})$ and Event 2 at coordinates $(t_{2\text{lab}}, s_{2\text{lab}})$. *Bottom panel:* An unpowered rocket ship streaks through the laboratory; the observer riding in the rocket ship records Event 1 at rocket coordinates $(t_{1\text{rocket}}, s_{1\text{rocket}})$ and Event 2 at $(t_{2\text{rocket}}, s_{2\text{rocket}})$. Each observer calculates the distance and time lapse between the two events, displayed on the line between them.

Definition:
inertial observer

DEFINITION 2. Observer ≡ inertial observer

An **inertial observer** is an observer who makes measurements using the space and time coordinates of any given inertial frame. In this book we *choose* to report *every* measurement and observation using an inertial frame. Therefore in this book **observer ≡ inertial observer**.

The top panel of Figure 1 summarizes the records of the laboratory observer, who uses the standard notation $(t_{1\text{lab}}, s_{1\text{lab}})$ for the lab-measured time and space coordinates of Event 1 and $(t_{2\text{lab}}, s_{2\text{lab}})$ for the coordinates of Event 2.

The laboratory observer calculates the *difference* between the time coordinates of the two events and the *difference* between the space coordinates of the two events that she measures in her frame. The top panel of Figure 1 labels these results.

Next an unpowered rocket moves through the laboratory along the line connecting Event 1 and Event 2. An observer who rides in the rocket measures the coordinates of the two events and constructs the bottom panel in Figure 1.

Now the key result of special relativity: There is a surprising relation between the coordinate differences measured in laboratory and rocket frames, both of which are inertial frames. Here is that expression:

Surprise:
Both observers calculate the same wristwatch time between two events.

$$\tau^2 = (t_{2\text{lab}} - t_{1\text{lab}})^2 - (s_{2\text{lab}} - s_{1\text{lab}})^2 = (t_{2\text{rocket}} - t_{1\text{rocket}})^2 - (s_{2\text{rocket}} - s_{1\text{rocket}})^2 \tag{1}$$

The expression on the left side of (1) is the square of the so-called **wristwatch time** τ, which we define explicitly in the following section. Special relativity says that the wristwatch time lapse of the stone that moves directly between events can be predicted (calculated) by both laboratory and rocket observers, each using his or her own time and space coordinates. The middle expression in (1) contains only laboratory coordinates of the two events. The right-hand expression contains only rocket coordinates of the same two events. Each observer predicts (calculates) the same value of the stone's wristwatch time lapse as it travels between these two events.

Fuller Explanation: *Spacetime Physics,* Chapter 1. Chapter 2, Section 2.6, shows how to synchronize the clocks in each frame with one another. Or look up **Einstein-Poincaré synchronization**.

1.2 ■ WRISTWATCH TIME

Every observer agrees on the advance of wristwatch time.

Einstein said to Besso (initial quote): "Time cannot be absolutely defined . . ." Equation (1) exhibits this ambiguity: the laboratory time lapse, rocket time lapse, and wristwatch time lapse between two ticks of the stone's wristwatch *can all be different from one another.* But equation (1) tells us much more: It shows how any inertial observer whatsoever can use the space and time coordinate separations between ticks measured in her frame to calculate the unique **wristwatch time** τ, the time lapse between ticks recorded on the stone's wristwatch as it moves from Event 1 to Event 2.

Example of **wristwatch time** or **aging**

DEFINITION 3. Wristwatch time = aging

Equation (1) and Figure 1 show an example of the **wristwatch time** τ between two events, in this case the time lapse recorded on a wristwatch that is present at both events and travels uniformly between them. Wristwatch time is sometimes called **aging**, because it is the amount by which the wearer of the wristwatch gets older as she travels directly between this pair of events. Another common name for wristwatch time is **proper time**, which we do not use in this book.

We, the authors of this book, rate (1) as one of the greatest equations in physics, perhaps in all of science. Even the famous equation $E = mc^2$ is a child of equation (1), as Section 1.7 shows.

Truth be told, equation (1) is not limited to events along the path of a stone; it also applies to any pair of events in flat spacetime, no matter how large their coordinate separations in any one frame. In the general case, equation (1) is called the spacetime **interval** between these two events.

Definition: **interval**

DEFINITION 4. Interval

The spacetime **interval** is an expression whose inputs are the distance separation and time separation between a pair of events measured in an inertial frame. The term “interval” refers to the whole equation (1). There are three different possible outputs, three types of interval:

Case 1: Timelike interval, $\tau^2 > 0$ this section
Case 2: Spacelike interval, $\tau^2 < 0$ Section 1.3
Case 3: Lightlike interval, $\tau^2 = 0$ Section 1.4

These three categories span all possible relations between a pair of events in special relativity. When $(t_{2\text{lab}} - t_{1\text{lab}})^2$ is greater than $(s_{2\text{lab}} - s_{1\text{lab}})^2$, then we have the case we analyzed for two events that may lie along the path of a stone. We call this a **timelike interval** because the magnitude of the time part of the interval is greater than that of its space part.

What happens when $(s_{2\text{lab}} - s_{1\text{lab}})^2$ is greater than $(t_{2\text{lab}} - t_{1\text{lab}})^2$ in (1), so the interval is negative? We call this a **spacelike interval** because the magnitude of the space part of the interval is greater than that of its time part. In this case we interchange $(t_{2\text{lab}} - t_{1\text{lab}})^2$ and $(s_{2\text{lab}} - s_{1\text{lab}})^2$ to yield a positive quantity we call σ^2, whose different physical interpretation we explore in Section 1.3.

What happens when $(s_{2\text{lab}} - s_{1\text{lab}})^2$ is equal to $(t_{2\text{lab}} - t_{1\text{lab}})^2$ in (1), so the interval has the value zero? We call this a **null interval** or **lightlike interval**, as explained in Section 1.4.

Measure space and time separations in the same unit, which *you* choose.

Note: All separations in (1) must be measured in the same unit; otherwise they cannot appear as separate terms in the same equation. But we are free to choose the common unit: it can be **years** (of time) and **light-years** (of distance). A light-year is the distance light travels in a vacuum in one year. Or we can use **meters** (of distance) along with **light-meters** (of time). A light-meter of time is the time it takes light to travel one meter in a vacuum—about 3.34×10^{-9} second. Alternative expressions for light-meter are **meter of light-travel time** or simply **meter of time**.

Speed of light equals unity.

Distance and time expressed in the same unit? Then the *speed of light* has the value unity, with *no* units:

$$c = \frac{1 \text{ light-year of distance}}{1 \text{ year of time}} = \frac{1 \text{ meter of distance}}{1 \text{ light-meter of time}} = 1 \qquad (2)$$

Why the letter c? The Latin word *c̲eleritas* means “swiftness” or “speed.”

Stone’s speed: a fraction of light speed

So much for the speed of *light*. How do we measure the speed of a *stone* using space and time separations between ticks of its wristwatch? Typically the value of the stone’s speed depends on the reference frame with respect to which we measure these separations. In the top panel of Figure 1, its speed in

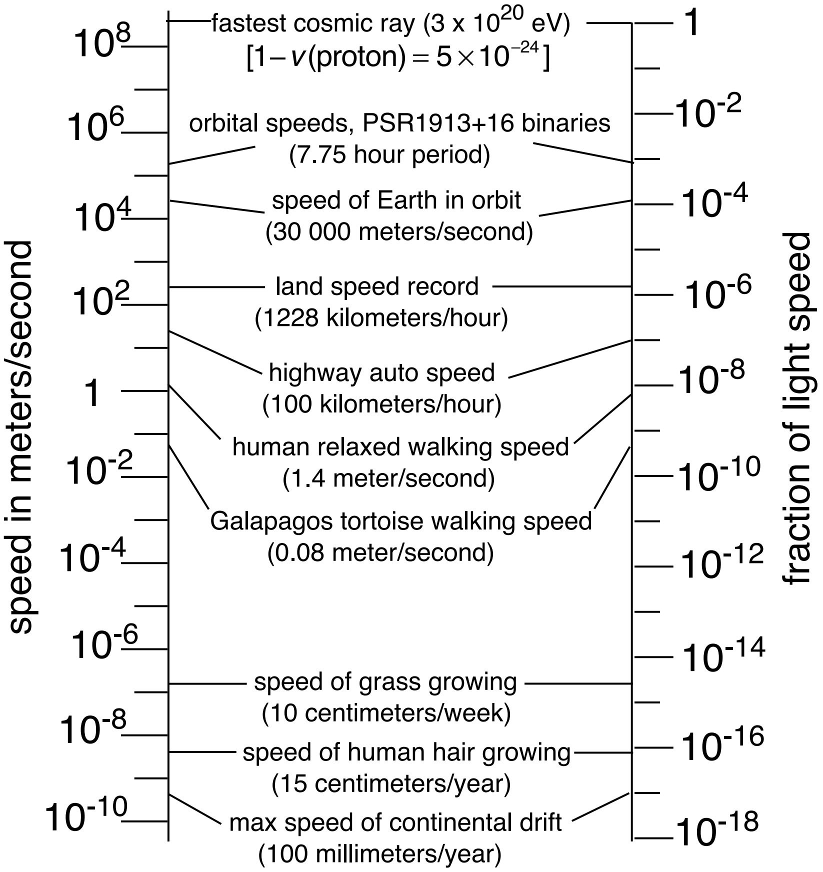

FIGURE 2 The speed ladder. Some typical speeds encountered in Nature.

the laboratory frame is $v_{\text{lab}} = (s_{2\text{lab}} - s_{1\text{lab}})/(t_{2\text{lab}} - t_{1\text{lab}})$. In the bottom panel, its speed in the rocket frame is $v_{\text{rocket}} = (s_{2\text{rocket}} - s_{1\text{rocket}})/(t_{2\text{rocket}} - t_{1\text{rocket}})$. Typically the values of these two speeds differ from one another. However, both values are less than one. Figure 2 samples the range of speeds encountered in Nature.

Equation (1) is so important that we use it to define **flat spacetime.**

DEFINITION 5. Flat spacetime

Definition:
flat spacetime

Flat spacetime is a spacetime region in which equation (1) is valid for every pair of events.

The interval in equation (1) has an important property that will follow us through special and general relativity: it has the same value when calculated using either laboratory or rocket coordinates. We say that wristwatch time is an **invariant quantity**.

Sample Problems 1. Wristwatch Times

PROBLEM 1A
An unpowered rocket ship moves at constant speed to travel 3 light-years in 5 years, this time and distance measured in the rest frame of our Sun. What is the time lapse for this trip recorded on a clock carried with the spaceship?

SOLUTION 1A
The two events that start and end the spaceship's journey are separated in the Sun frame by $s_{2\mathrm{Sun}} - s_{1\mathrm{Sun}} = 3$ light-years and $t_{2\mathrm{Sun}} - t_{1\mathrm{Sun}} = 5$ years. Equation (1) gives the resulting wristwatch time:

$$\tau^2 = 5^2 - 3^2 = 25 - 9 = 16 \text{ years}^2 \qquad (3)$$
$$\tau = 4 \text{ years}$$

which is *less* than the time lapse measured in the Sun frame.

PROBLEM 1B
An elementary particle created in the target of a particle accelerator arrives 5 meters of time later at a detector 4 meters from the target, as measured in the laboratory. The wristwatch of the elementary particle records what time between creation and detection?

SOLUTION 1B
The events of creation and detection are separated in the laboratory frame by $s_{2\mathrm{lab}} - s_{1\mathrm{lab}} = 4$ meters and $t_{2\mathrm{lab}} - t_{1\mathrm{lab}} = 5$ meters of time. Equation (1) tells us that

$$\tau^2 = 5^2 - 4^2 = 25 - 16 = 9 \text{ meters}^2 \qquad (4)$$
$$\tau = 3 \text{ meters}$$

Again, the wristwatch time for the particle is less than the time recorded in the laboratory frame.

PROBLEM 1C
In Problem 1B the two events are separated by a distance of 4 meters, which means that it takes light 4 meters of light-travel time to move between them. But Solution 1B says that the particle's wristwatch records only 3 meters of time as the particle moves from the first to the second event. Does this mean that the particle travels faster than light?

SOLUTION 1C
This difficulty is common in relativity. The phrase "time between two events" has no unique value (initial quote of this chapter). The *time* depends on *which clock* measures the time, in this case either the laboratory clocks, which measure laboratory time separation $t_{2\mathrm{lab}} - t_{1\mathrm{lab}}$, or the particle's wristwatch, which measures lapsed wristwatch time τ. Equation (1) already warns us that these two measures of time may not have the same value. Indeed a particle that moves faster and faster, covering a greater and greater distance $s_{2\mathrm{lab}} - s_{1\mathrm{lab}}$ in the same laboratory time lapse $t_{2\mathrm{lab}} - t_{1\mathrm{lab}}$, records a wristwatch time τ that gets smaller and smaller (Sample Problems 2), finally approaching—as a limit—the value zero, in which case a light flash has replaced the particle (Section 1.4). But for a particle with mass, the distance $s_{2\mathrm{lab}} - s_{1\mathrm{lab}}$ it travels in the laboratory frame is always less than the laboratory time $t_{2\mathrm{lab}} - t_{1\mathrm{lab}}$ that it takes the particle to move that distance. In other words, its laboratory speed will always be less than one, the speed of light. No particle can move faster than light moves in a vacuum. (Convince the scientific community that this statement is false, and your name will go down in history!)

Definition:
invariant

DEFINITION 6. Invariant
Formally, a quantity is an **invariant** when it keeps the same value under some transformation. Equation (1) shows the interval between any pair of events along the path of a free stone to have the same value when calculated using coordinate separations in any inertial frame. Transformations of coordinate separations between inertial frames are called **Lorentz transformations** (Section 1.10), so we say that the interval is a **Lorentz invariant**. However, the interval must also be an invariant under even more general transformations, not just Lorentz transformations, because all observers—not just those in inertial frames—will agree on the stone's wristwatch time lapse between any two given events. As a consequence, we most often drop the adjective *Lorentz* and use just the term **invariant**.

Fuller Explanation: *Spacetime Physics,* Chapter 1, Spacetime: Overview

Sample Problems 2. Speeding to Andromeda

At approximately what constant speed $v_{\rm Sun}$ with respect to our Sun must a spaceship travel so that its occupants age only 1 year during a trip from Earth to the Andromeda galaxy? Andromeda lies 2 million light-years distant from Earth in the Sun's rest frame.

SOLUTION The word *approximately* in the statement of the problem tells us that we may make some assumptions. We assume that a single inertial frame can stretch all the way from Sun to Andromeda, so special relativity applies. Equation (1) leads us to predict that the speed $v_{\rm Sun}$ of the spaceship measured in the Sun frame is very close to unity, the speed of light. That allows us to set $(1 + v_{\rm Sun}) \approx 2$ in the last of the following steps:

$$\begin{aligned}
\tau^2 &= (t_{\rm 2Sun} - t_{\rm 1Sun})^2 - (s_{\rm 2Sun} - s_{\rm 1Sun})^2 \qquad (5)\\
&= (t_{\rm 2Sun} - t_{\rm 1Sun})^2 \left[1 - \left(\frac{s_{\rm 2Sun} - s_{\rm 1Sun}}{t_{\rm 2Sun} - t_{\rm 1Sun}}\right)^2\right]\\
&= (t_{\rm 2Sun} - t_{\rm 1Sun})^2 \left(1 - v_{\rm Sun}^2\right)\\
&= (t_{\rm 2Sun} - t_{\rm 1Sun})^2 \left(1 - v_{\rm Sun}\right)\left(1 + v_{\rm Sun}\right)\\
&\approx 2\,(t_{\rm 2Sun} - t_{\rm 1Sun})^2 \left(1 - v_{\rm Sun}\right)
\end{aligned}$$

Equate the first and last expressions in (5) to obtain

$$1 - v_{\rm Sun} \approx \frac{\tau^2}{2\,(t_{\rm 2Sun} - t_{\rm 1Sun})^2} \qquad (6)$$

IF the spaceship speed $v_{\rm Sun}$ is very close to the speed of light, THEN the Sun-frame time for the trip to Andromena is very close to the time that light takes to make the trip: 2 million years. Substitute this value for $t_{\rm 2Sun} - t_{\rm 1Sun}$ and also demand that the wristwatch time on the spaceship (the aging of the occupants during their trip) be $\tau = 1$ year. The result is

$$\begin{aligned}
1 - v_{\rm Sun} &\approx \frac{1\,\text{year}^2}{2 \times 4 \times 10^{12}\,\text{year}^2} \qquad (7)\\
&= \frac{10^{-12}}{8} = 1.25 \times 10^{-13}
\end{aligned}$$

Equation (7) expresses the result in sensible scientific notation. However, your friends may be more impressed if you report the speed as a fraction of the speed of light: $v_{\rm Sun} = 0.999\,999\,999\,999\,875$. This result justifies our assumption that $v_{\rm Sun}$ is close to unity. *Additional question*: What is the *distance* $(s_{\rm 2rocket} - s_{\rm 1rocket})$ between Earth and Andromeda measured in the rocket frame?

1.3 ■ RULER DISTANCE

Everyone agrees on the ruler distance between two events.

Two firecrackers explode one meter apart and *at the same time,* as measured in a given inertial frame: in *this* frame the explosions are **simultaneous.** No stone—not even a light flash—can travel the distance between these two explosions in the zero time available in this frame. Therefore equation (1) cannot give us a value of the wristwatch time between these two events.

Use simultaneous explosions to measure length of a rod.

Simultaneous explosions are thus useless for measuring time. But they are perfect for measuring length. *Question:* How do you measure the length of a rod, whether it is moving or at rest in, say, the laboratory frame? *Answer:* Set off two firecrackers at opposite ends of the rod and *at the same time* $(t_{\rm 2lab} - t_{\rm 1lab} = 0)$ in that frame. Then *define* the rod's length in the laboratory frame as the *distance* $(s_{\rm 2lab} - s_{\rm 1lab})$ between this pair of explosions simultaneous in that frame.

Relativity of simultaneity

Special relativity warns us that another observer who flies through the laboratory typically does *not* agree that the two firecrackers exploded at the same time as recorded on her rocket clocks. This effect is called the **relativity of simultaneity**. The relativity of simultaneity is the bad news (and for many people the most difficult idea in special relativity). But here's the good news: All inertial observers, whatever their state of relative motion, can calculate the

Spacelike interval σ

distance σ between explosions as recorded in the frame in which they do occur simultaneously. This calculation uses Case 2 of the interval (Definition 4):

$$\begin{aligned}\sigma^2 \equiv -\tau^2 &= (s_{2\mathrm{lab}} - s_{1\mathrm{lab}})^2 - (t_{2\mathrm{lab}} - t_{1\mathrm{lab}})^2 \qquad \text{(spacelike interval)} \qquad (8)\\ &= (s_{2\mathrm{rocket}} - s_{1\mathrm{rocket}})^2 - (t_{2\mathrm{rocket}} - t_{1\mathrm{rocket}})^2\end{aligned}$$

The Greek letter *sigma*, σ, in (8)—equivalent to the Roman letter s—is the length of the rod defined as the distance between explosions at its two ends measured in a frame in which these explosions are simultaneous.

Equation (8) does not define a different kind of interval; it is merely shorthand for the equation for Case 2 in Definition 4 in which $\tau^2 < 0$.

Actually, we do not need a rod or ruler to make use of this equation (though we keep *ruler* as a label). Take any two events for which $\tau^2 < 0$. Then there exists an inertial frame in which these two events occur at the same time; we use this frame to define the **ruler distance** σ between these two events:

Definition: **ruler distance**

DEFINITION 7. Ruler distance

The **ruler distance** σ between two events is the distance between these events measured by an inertial observer in whose frame the two events occur at the same time. Another common name for ruler distance is **proper distance**, which we do not use in this book.

Equation (8) tells us that every inertial observer can calculate the ruler distance between two events using the space and time separations between these events measured in his or her own frame.

Fuller Explanation: *Spacetime Physics,* Chapter 6, Regions of Spacetime

1.4 ■ LIGHTLIKE (NULL) INTERVAL

Everyone agrees on the null value of the interval between two events connected by a direct light flash that moves in a vacuum.

Now think of the case in which the lab-frame space separation $(s_{2\mathrm{lab}} - s_{1\mathrm{lab}})$ between two events is equal to the time separation $(t_{2\mathrm{lab}} - t_{1\mathrm{lab}})$ between them. In this case anything that moves uniformly between them must travel at the speed of light $v_{\mathrm{lab}} = (s_{2\mathrm{lab}} - s_{1\mathrm{lab}})/(t_{2\mathrm{lab}} - t_{1\mathrm{lab}}) = 1$. Physically, only a direct light flash can move between this pair of events. We call the result a **lightlike interval**:

$$\begin{aligned}\tau^2 = -\sigma^2 = 0 &= (s_{2\mathrm{lab}} - s_{1\mathrm{lab}})^2 - (t_{2\mathrm{lab}} - t_{1\mathrm{lab}})^2 \quad \text{(lightlike interval)} \quad (9)\\ &= (s_{2\mathrm{rocket}} - s_{1\mathrm{rocket}})^2 - (t_{2\mathrm{rocket}} - t_{1\mathrm{rocket}})^2\end{aligned}$$

Because of its zero value, the lightlike interval is also called the **null interval**.

Definition: **lightlike interval** or **null interval**

DEFINITION 8. Lightlike (null) interval

A **lightlike interval** is the interval between two events whose space

Sample Problems 3. Causation

Three events have the following space and time coordinates as measured in the laboratory frame in meters of distance and meters of time. All three events lie along the x-axis in the laboratory frame. (Temporarily suppress the subscript "lab" in this Sample Problem.)

Event A: $(t_{\mathrm{A}}, x_{\mathrm{A}}) = (2, 1)$
Event B: $(t_{\mathrm{B}}, x_{\mathrm{B}}) = (7, 4)$
Event C: $(t_{\mathrm{C}}, x_{\mathrm{C}}) = (5, 6)$

Classify the intervals between each pair of these events as timelike, lightlike, or spacelike:

(a) between events A and B
(b) between events A and C
(c) between events B and C

In each case say whether or not it is possible for one of the events in the pair (which one?) to cause the other event of the pair, and if so, by what possible means.

SOLUTION
The interval between events A and B is:

$$\begin{aligned}\tau^2 &= (7-2)^2 - (4-1)^2 = 5^2 - 3^2 \qquad (10)\\ &= 25 - 9 = +16\end{aligned}$$

The time part is greater than the space part, so the interval between the events is *timelike.* Event A could have caused Event B, for example by sending a stone moving directly between them at a speed $v_{\mathrm{lab}} = 3/5$. (There are other possible ways for Event A to cause Event B, for example by sending a light flash that sets off an explosion between the two locations, with a fragment of the explosion reaching Event B at the scheduled time, and so forth. Our analysis says only that Event A *can* cause Event B, but it does not *force* Event A to cause Event B. Someone standing next to an object located at the x-coordinate of Event B could simply kick that object at the scheduled time of Event B.)

The interval between events A and C is:

$$\begin{aligned}\tau^2 &= (5-2)^2 - (6-1)^2 = 3^2 - 5^2 \qquad (11)\\ &= 9 - 25 = -16\end{aligned}$$

The space part is greater than the time part, so the interval between the events is *spacelike.* Neither event can cause the other, because to do so an object would have to travel between them at a speed greater than that of light.

The interval between events B and C is:

$$\begin{aligned}\tau^2 &= (7-5)^2 - (4-6)^2 = 2^2 - 2^2 \qquad (12)\\ &= 4 - 4 = 0\end{aligned}$$

The space part is equal to the time part, so the interval between the events is *lightlike.* Event C can cause Event B, but only by sending a direct light signal to it.

Challenge: How can we rule out the possibility that event B causes event A, or that event B causes event C? Would your answers to these questions be different if the same events are observed in some other frame in rapid motion with respect to the laboratory? (Answer in Exercise 1.)

separation and time separation are equal in every inertial frame. Only a direct light flash can connect these two events. Because these space and time separations are equal, the interval has the value zero, so is also called the **null interval**.

Comment 1. Einstein's derivation of special relativity
Divide both sides of (9) by $(t_{2,\mathrm{frame}} - t_{1,\mathrm{frame}})^2$, where "frame" is either "lab" or "rocket." The result tells us that the speed in any inertial frame is one, $v_{\mathrm{lab}} = v_{\mathrm{rocket}} = 1$. Einstein derived (9) starting with the *assumption* that the speed of light is the same in all inertial frames.

Fuller Explanation: *Spacetime Physics,* Chapter 6, Regions of Spacetime.

1.5 ■ WORLDLINE OF A WANDERING STONE; THE LIGHT CONE

A single curve tells all about the motion of our stone.

Grasp a stone in your hand and move it alternately in one direction, then in the opposite direction along the straight edge of your desk. Choose the $x_{\rm lab}$ axis along this line. Then the stone's motion is completely described by the function $x_{\rm lab}(t_{\rm lab})$. No matter how complicated this back-and-forth motion is, we can view it at a glance when we plot $x_{\rm lab}$ along the horizontal axis of a graph whose vertical axis represents the time $t_{\rm lab}$. Figure 3 shows such a curve, which we call a **worldline**.

Definition: **worldline**

DEFINITION 9. Worldline

A **worldline** is the path through spacetime taken by a stone or light flash. By Definition 3, the total wristwatch time (aging) along the worldline is the sum of wristwatch times between sequential events along the worldline from a chosen initial event to a chosen final event. The wristwatch time is an invariant; it has the same value when calculated using either laboratory or rocket coordinates. Therefore specification of a worldline requires neither coordinates nor the metric.

Comment 2. Plotting the worldline

Figure 3 shows a worldline plotted in laboratory coordinates. Typically a given worldline will look different when plotted in rocket coordinates. We plot a worldline in whatever coordinates we are using. Worldlines can be plotted in spacetime diagrams for both flat and curved spacetime.

In the worldline of Figure 3 the stone starts at initial event O. As time passes—as time advances upward in the diagram—the stone moves first to the right. Then the stone slows down, that is it covers less distance to the right per unit time, and comes to rest momentarily at event Z. (The vertical tangent to the worldline at Z tells us that the stone covers zero laboratory distance there: it is instantaneously at rest at Z.) Thereafter the stone accelerates to the left in space until it arrives at event P.

Limits on worldline slope

What possible future worldlines are available to the stone that arrives at event P? Any material particle must move at less than the speed of light. In other words, it travels less than one meter of distance in one meter of light-travel time. Therefore its future worldline must make an "angle with the vertical" somewhere between minus 45 degrees and plus 45 degrees in Figure 3, in which space and time are measured in the same units and plotted to the same scale. These limits on the slope of the stone's worldline—which apply to every event on every worldline—emerge as dashed lines from event P in Figure 3. These dashed lines are worldlines of light rays that move in opposite $x_{\rm lab}$-directions and cross at the event P. We call these crossed light rays a **light cone.** Figure 4 displays the cone shape.

Definition: **light cone**

DEFINITION 10. Light cone

The **light cone** of an event is composed of the set of all possible worldlines of light that intersect at that event and define its past and

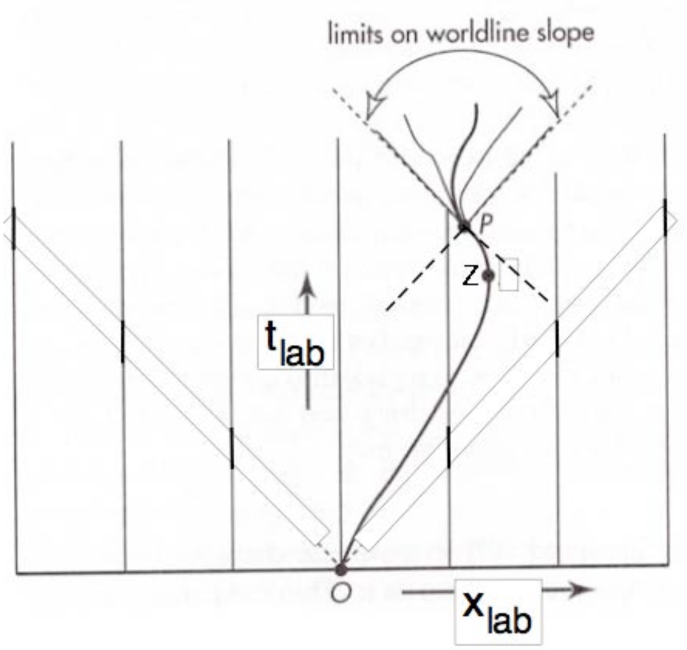

FIGURE 3 Curved **worldline** of a stone moving back and forth along a single straight spatial line in the laboratory. A point on this diagram, such as Z or P, combines x_{lab}-location (horizontal direction) with t_{lab}-location (vertical direction); in other words a point represents a spacetime *event*. The dashed lines through P are worldlines of light rays that pass through P. We call these crossed lines *the light cone of P*. For the cone shape, see Figure 4.

future (Figure 4). We also call it a light *cone* when it is plotted using one space dimension plus time, as in Figure 3, and when plotted using three space dimensions plus time—even though we cannot visualize the resulting four-dimensional spacetime plot.

THE LIGHT CONE AND CAUSALITY

. . the light cone provides a mathematical tool for the analysis of [general relativity] additional to the usual tools of metric geometry. We believe that this tool still remains to be put to full use, and that causality is the physical principle which will guide this future development.

—Robert W. Fuller and John Archibald Wheeler

More complete explanation: *Spacetime Physics,* Chapter 5, Trekking Through Spacetime

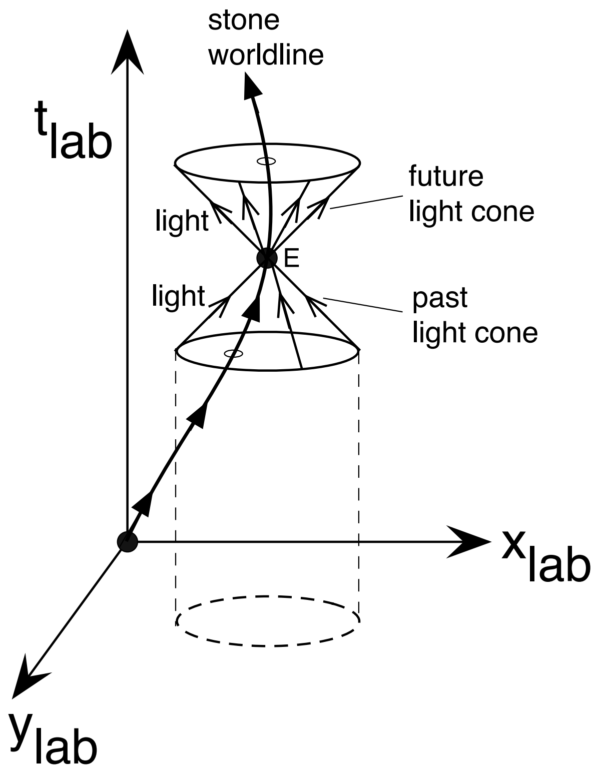

FIGURE 4 *Light cone* of Event E that lies on the worldline of a stone, plotted for two space dimensions plus time. The light cone consists of the upward-opening future light cone traced out by the expanding circular light flash that the stone emits at Event E, plus the downward-opening past light cone traced out by a contracting circular light flash that converges on Event E.

1.6 ■ THE TWIN “PARADOX” AND THE PRINCIPLE OF MAXIMAL AGING

The Twin Paradox leads to a definition of natural motion.

To get ready for curved spacetime (whatever that means), look more closely at the motion of a free stone in *flat spacetime* (Definition 5), where special relativity correctly describes motion.

Twin Paradox predicts motion of a stone.

A deep description of motion arises from the famous **Twin Paradox.** One twin—say a boy—relaxes on Earth while his fraternal twin sister frantically travels to a distant star and returns. When the two meet again, the stay-at-home brother has aged more than his traveling sister. (To predict this outcome, extend Sample Problem 1A to include return of the traveler to the point of origin.) Upon being reunited, the “twins” no longer look similar: the traveling sister is *younger*: she has aged less than her stay-at-home brother. Very strange! But (almost) no one who has studied relativity doubts the difference in age, and every minute of every day somewhere on Earth a measurement with a fast-moving particle verifies it.

Being at rest is one *natural* motion.

Which twin has the motion we can call *natural*? Isaac Newton has a definition of natural motion. He would say, “A twin at rest tends to remain at rest.” So it is the stay-at-home twin who moves in the natural way. In contrast, the out-and-back twin suffers the acceleration required to change her state of motion, from outgoing motion to incoming motion, so the twins can meet again in person. At least at her turnaround, the motion of the traveling twin is forced, *not natural.*

Viewed from the second, relatively moving, inertial frame of the twin sister, the stay-at-home boy initially moves away from her with constant speed in a straight line. Again, his motion is *natural.* Newton would say, “A twin in uniform motion tends to continue this motion at constant speed in a straight line.” So the motion of the stay-on-Earth twin is also natural from the viewpoint of his sister’s frame in uniform relative motion—or from the viewpoint of any frame moving uniformly with respect to the original frame. In *any* such frame, the time lapse on the wristwatch of the stay-at-home twin can be calculated from the interval (1).

Moving uniformly is another *natural* motion.

But there *is* a difference between the stay-at-home brother on Earth and the sister: She moves outward to a star, *then turns around* and returns to her Earthbound brother. So when her trip is over, everyone must agree: It is the brother who follows “natural” motion from parting event to reunion event. And it is the stay-at-home brother—whose wristwatch records the greater elapsed time—who **ages** the most.

Natural motion: Maximal wristwatch time.

The lesson we draw from the Twin Paradox in flat spacetime is that *natural* motion is the motion that maximizes the wristwatch time between *any* pair of events along its path. Now we can state the **Principle of Maximal Aging in flat spacetime**.

Definition: **Principle of Maximal Aging**

DEFINITION 11. The Principle of Maximal Aging (flat spacetime)

The **Principle of Maximal Aging** states that the worldline a free stone follows between a pair of events in flat spacetime is the worldline for which the wristwatch time is a maximum compared with every possible alternative worldline between these events. The free stone follows the worldline of *maximal aging* between these two events.

?

Objection 1. *Why should I believe the Principle of Maximal Aging? Newton never talks about this weird idea! What does this so-called “Principle” mean, anyway?*

!

Response: For now the Principle of Maximal Aging is simply a restatement of the observation that in flat spacetime a free stone follows a straight worldline. It repeats Newton’s First Law of Motion: A free stone at rest or in motion maintains that condition. Why bother? Because general relativity revises and extends the Principle of Maximal Aging to predict the motion of a free stone in curved spacetime.

?

Objection 2. *Wait! Have you really resolved the Twin Paradox? Both the twin sister and the twin brother sees his or her twin moving away, then moving back. Motion is relative, remember? The view of each twin is symmetrical, not only during the outward trip but also during the return trip. There is no difference between them. The experience of the two twins is identical; you cannot wriggle out of this essential symmetry! You have failed to explain why their wristwatches have different readings when they reunite.*

!

Nice point. But you forget that the experience of the two twins is *not* identical. Fill in details of the story: When the twin sister arrives at the distant star and reverses her starship's direction of motion, that reversal throws her against the forward bulkhead. Ouch! She starts home with a painful lump on the right side of her forehead. Then when her ship slows down so she can stand next to her stay-at-home brother, she forgets her seat belt again. *Result:* a second painful lump, this time on the left side of her forehead. In contrast, her brother remains relaxed and uninjured during their entire separation. When the twins stand side by side, can *each* of them tell *which twin* has gone to the distant star? Of course! *More: Every passing observer*—whatever his or her speed or direction of motion—sees and reports the difference between the twins: "injured sister; smiling brother." *Everyone* agrees on this difference. No contradiction and no confusion. "Paradox" resolved.

Comment 3. The Quintuplet "Paradox"

An infinite number of alternative worldlines: the free stone chooses one.

In the last sentence of Definition 11, The Principle of Maximal Aging, notice the word "every" in the phrase "is a maximum compared with *every* alternative path...between the given initial and final events." We are not just talking twins here, but triplets, quadruplets, quintuplets—indeed endless multiple births. *Example,* Figure 5: One quintuplet—**Quint** #1—follows the worldline of maximal aging between the two anchoring events by moving uniformly between them. Each of the other quints also starts from the same Initial Event A and ends at the same Final Event B, but follows a different alternative worldline—changes velocity—between initial and final events. When all the quints meet at the final event, all four traveling quints are younger than their uniformly-moving sibling, but typically by different amounts. *Every traveler, #2 through #5, who varies velocity between the two end-events is younger than its uniformly-moving sibling, Quint #1.* The Principle of Maximal Aging singles out one worldline among the limitless number of alternative worldlines between two end-events and demands that the free stone follow **this** worldline—and no other.

QUERY 1. Analyze the Quintuplet Paradox

Answer the following questions about the Quintuplet Paradox illustrated in Figure 5.

A. Which of the five quints ages the *most* between end-events A and B? (Trick question!)

B. Which of the five quints ages the *least* between end-events A and B?

C. List the numbered worldlines in order, starting with the worldline along which the aging is the *least* and ending with the worldline along which the aging is the *most.*

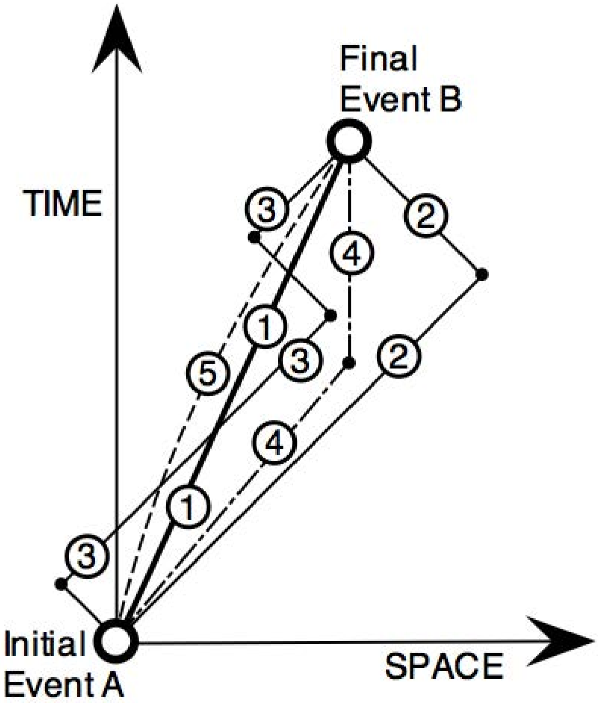

FIGURE 5 The Quintuplet Paradox: Five alternative worldlines track the motion of five different quintuplets (**quints**) between Initial Event A and Final Event B along a spatial straight line. Quint #1 follows the (thick) worldline of maximal aging between A and B. Quint #2 moves along the (thin) worldline at 0.999 of the speed of light outward and then back again. Quint #3 follows a worldline (also a thin line) at the same *speed* as #2, but with three reversals of direction. Quint #4 shuffles (dot-dash line) to the spatial position of Final Event B, then relaxes there until her siblings join her at Event B. The (dashed) worldline of Quint #5 hugs worldline #1—the worldline of Maximal Aging—but does not quite follow it.

D. True or false? If the dashed worldline of Quint #5 skims close enough to that of Quint #1—while still being separate from it—then Quint #5 will age the same as Quint #1 between end-events A and B.

E. *Optional:* Suppose we view the worldlines of Figure 5 with respect to a frame in which Event A and Event B occur at the same spatial location. Whose *inertial* rest frame does this correspond to? Will your answers to Items A through D be different in this case?

Fuller Explanation: Twin “paradox:” *Spacetime Physics,* Chapter 4, Section 4.6.

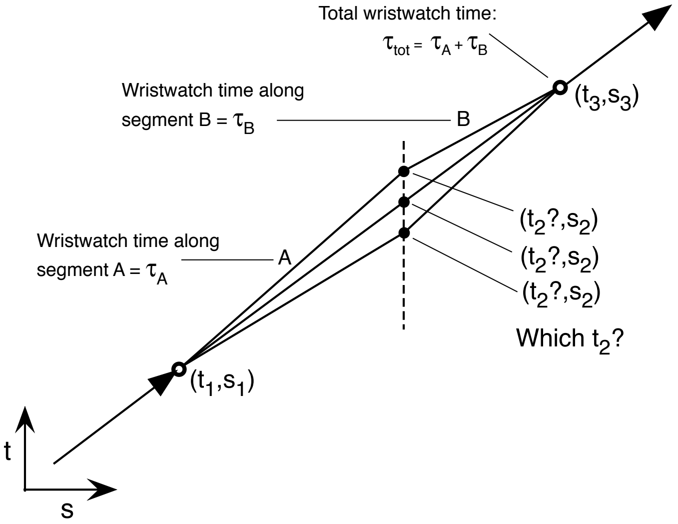

FIGURE 6 Figure for the derivation of the energy of a stone. Examine two adjacent segments, A and B, along an extended worldline plotted in, say, the laboratory frame. Choose three events at the endpoints of these two segments with coordinates (t_1, s_1), (t_2, s_2), and (t_3, s_3). All coordinates are fixed except t_2. Vary t_2 to find the maximum value of the total aging $\tau_{\rm tot}$ (Principle of Maximal Aging). *Result:* an expression for the stone's energy E.

1.7 ■ ENERGY IN SPECIAL RELATIVITY

The Principle of Maximal Aging tells us the energy of a stone.

Here is a modern translation (from Latin) of Isaac Newton's famous First Law of Motion:

Newton's First Law of motion

> **Newton's first law of motion:** Every body perseveres in its state of being at rest or of moving uniformly straight forward except insofar as it is compelled to change its state by forces impressed.

Validity of Newton's First Law in special relativity . . .

. . . leads to relativistic expression for energy.

In modern terminology, Newton's First Law says that, as measured in an inertial frame in flat spacetime, a free stone moves along a *straight worldline*, that is with constant speed along a straight path in space. We assumed the validity of Newton's First Law in defining the inertial frame (Definition 1, Section 1.1). In the present section the Principle of Maximal Aging again verifies this validity of the First Law. *Extra surprise!* This process will help us to derive the relativistic expression for the stone's energy E.

Figure 6 illustrates the method: Consider two adjacent segments, A and B, of the stone's worldline with fixed events at the endpoints. Vary t_2 of the middle event to find the value that gives a maximum for the total wristwatch time $\tau_{\rm tot}$ along the adjacent segments. Now the step-by-step derivation:

1. The wristwatch time between the first and second events along the worldline is the square root of the interval between them:

$$\tau_{\rm A} = \left[(t_2 - t_1)^2 - (s_2 - s_1)^2\right]^{1/2} \tag{13}$$

To prepare for the derivative that leads to maximal aging, differentiate this expression with respect to t_2. (All other coordinates of the three events are fixed.)

$$\frac{d\tau_{\rm A}}{dt_2} = \frac{t_2 - t_1}{\left[(t_2 - t_1)^2 - (s_2 - s_1)^2\right]^{1/2}} = \frac{t_2 - t_1}{\tau_{\rm A}} \tag{14}$$

2. The wristwatch time between the second and third events along the worldline is the square root of the interval between them:

$$\tau_{\rm B} = \left[(t_3 - t_2)^2 - (s_3 - s_2)^2\right]^{1/2} \tag{15}$$

Again, to prepare for the derivative that leads to extremal aging, differentiate this expression with respect to t_2:

$$\frac{d\tau_{\rm B}}{dt_2} = -\frac{t_3 - t_2}{\left[(t_3 - t_2)^2 - (s_3 - s_2)^2\right]^{1/2}} = -\frac{t_3 - t_2}{\tau_{\rm B}} \tag{16}$$

3. The total wristwatch time $\tau_{\rm tot}$ from event #1 to event #3—the total aging between these two events—is the sum of the wristwatch time $\tau_{\rm A}$ between the first two events plus the wristwatch time $\tau_{\rm B}$ between the last two events:

$$\tau_{\rm tot} = \tau_{\rm A} + \tau_{\rm B} \tag{17}$$

Principle of Maximal Aging finds time t_2 for middle event.

4. Now ask: At what intermediate t_2 will a free stone pass the intermediate point in space s_2 and emit the second flash #2? Answer by using the Principle of Maximal Aging: The time t_2 will be such that the total aging $\tau_{\rm tot}$ in (17) is a maximum. To find this maximum take the derivative of τ with respect to t_2 and set the result equal to zero. Add the final expressions (14) and (16) to obtain:

$$\frac{d\tau_{\rm tot}}{dt_2} = \frac{t_2 - t_1}{\tau_{\rm A}} - \frac{t_3 - t_2}{\tau_{\rm B}} = 0 \tag{18}$$

Quantity whose value is the same for adjoining segments

6. In equation (18) the time $(t_2 - t_1)$ is the lapse of laboratory time for the stone to traverse segment A. Call this time $t_{\rm A}$. The time $(t_3 - t_2)$ is the lapse of laboratory time for the stone to traverse segment B. Call this time $t_{\rm B}$. Then rewrite (18) in the simple form

$$\frac{t_{\rm A}}{\tau_{\rm A}} = \frac{t_{\rm B}}{\tau_{\rm B}} \tag{19}$$

This result yields a maximum $\tau_{\rm tot}$, *not* a minimum; see Exercise 4.

7. We did not say *which* pair of adjoining segments along the worline we were talking about, so equation (19) must apply to *every* pair of adjoining segments *anywhere* along the path. Suppose that there are three such adjacent segments. If the value of the expression is the same for, say, the first and second segments and also the same for the second and third segments, then it must be the same for the first and third segments. Continue in this way to envision a whole series of adjoining segments, labeled A, B, C, D,..., for each of which equation (19) applies, leading to the set of equations

$$\frac{t_\mathrm{A}}{\tau_\mathrm{A}} = \frac{t_\mathrm{B}}{\tau_\mathrm{B}} = \frac{t_\mathrm{C}}{\tau_\mathrm{C}} = \frac{t_\mathrm{D}}{\tau_\mathrm{D}} \rightarrow \frac{dt_\mathrm{lab}}{d\tau} \tag{20}$$

where all coordinate values are given in the laboratory frame.

Differences shrink to differentials

Comment 4. Differences to differentials

The last step, with the arrow, in (20) is a momentous one. We take the calculus limit by shrinking to differentials—infinitesimals—all the differences in physical quantities. In Figure 6, for example, segments A and B shrink to infinitesimals. Why is this step important? Because in general relativity, curvature of spacetime means that relations between adjacent events are described accurately only when *adjacent* events are differentially close to one another. If they are far apart, the two events may be in regions of different spacetime curvature.

What does the result (20) mean? We now show that $dt_\mathrm{lab}/d\tau$ in (20) is the expression for energy per unit mass of a free stone in the laboratory frame. The differential form of (1) yields:

$$d\tau^2 = dt_\mathrm{lab}^2 - ds_\mathrm{lab}^2 = dt_\mathrm{lab}^2\left(1 - ds_\mathrm{lab}^2/dt_\mathrm{lab}^2\right) = dt_\mathrm{lab}^2\left(1 - v_\mathrm{lab}^2\right) \tag{21}$$

Combine (20) with (21):

$$\frac{dt_\mathrm{lab}}{d\tau} = \frac{1}{\left(1 - v_\mathrm{lab}^2\right)^{1/2}} \tag{22}$$

Working in a single inertial frame, we have just found that $dt/d\tau$ is unchanging along the worldline of a free stone, which by Definition 11 is the worldline of maximal aging. It follows that v_lab is constant. Hence the Principle of Maximal Aging leads to the result that in flat spacetime the free stone moves at constant speed. (The derivation of relativistic momentum in Section 1.8 shows that the free stone's *velocity* is also constant, so that it moves along a straight worldline in every inertial frame.)

We show below that at low speeds (22) reduces to Newton's expression for kinetic energy plus rest energy, all divided by the stone's mass m. This supports our decision to call the expression in (22) the energy per unit mass of the stone:

$$\boxed{\frac{E_{\mathrm{lab}}}{m} = \frac{dt_{\mathrm{lab}}}{d\tau} = \frac{1}{(1 - v_{\mathrm{lab}}^2)^{1/2}} = \gamma_{\mathrm{lab}} \qquad (23)}$$

The last expression in (23) introduces a symbol—Greek lower case gamma—that we use to simplify later equations.

$$\boxed{\gamma_{\mathrm{lab}} \equiv \frac{1}{\left(1 - v_{\mathrm{lab}}^2\right)^{1/2}} \qquad (24)}$$

We call E_{lab}/m a **constant of motion** because the free stone's energy does not change as it moves in the laboratory frame. This may seem trivial for a stone that moves with constant speed in a straight line. In general relativity, however, we will find an "energy" that is a constant of motion for a free stone in orbit around a center of gravitational attraction.

We applied the Principle of Maximal Aging to motion in the laboratory frame. An almost identical derivation applies in the rocket frame. Coordinates of the initial and final events will differ from those in Figure 6, but the result will still be that $dt_{\mathrm{rocket}}/d\tau$ is constant along the free stone's worldline:

$$\boxed{\frac{E_{\mathrm{rocket}}}{m} = \frac{dt_{\mathrm{rocket}}}{d\tau} = \frac{1}{(1 - v_{\mathrm{rocket}}^2)^{1/2}} = \gamma_{\mathrm{rocket}} \qquad (25)}$$

Typically the value of the energy will be different in different inertial frames. We expect this, because the speed of a stone is not necessarily the same in different frames.

Equations (23) and (25) tell us that the energy of a stone in a given inertial frame increases without limit when the stone's speed approaches the value one, the speed of light, in that frame. Therefore the speed of light is the limit of the speed of a stone—or of any particle with mass—measured in any inertial frame. The other limit of (23) is a stone at rest in the laboratory. In this case, equation (23) reduces to

$$E_{\mathrm{lab}} = m \qquad \text{(when speed of stone } v_{\mathrm{lab}} = 0\text{)} \qquad (26)$$

We express m, the mass of the stone, in units of energy. If you insist on using conventional units, such as joules for energy and kilograms for mass, then a conversion factor c^2 intrudes into our simple expression. The result is the most famous equation in all of physics:

$$E_{\mathrm{lab,conv}} = m_{\mathrm{conv}}c^2 \qquad \text{(when speed of stone } v_{\mathrm{lab}} = 0\text{)} \qquad (27)$$

Here the intentionally-awkward subscript "conv" means "conventional units."

Rest energy

Equations (26) and (27) both quantify the *rest energy* of a stone; both tell us

Sample Problems 4. Energy Magnitudes

PROBLEM 4A

The “speed ladder” in Figure 2 shows that the fastest wheeled vehicle moves on land at a speed approximately $v \approx 10^{-6}$. The kinetic energy of this vehicle is what fraction of its rest energy?

SOLUTION 4A

For such an “everyday” speed, the approximation on the right side of equation (28) should be sufficiently accurate. Then $v^2 \approx 10^{-12}$ and approximate equation (28) tells us that:

$$\frac{\text{kinetic energy}}{\text{rest energy}} = \frac{mv^2}{2m} = \frac{v^2}{2} \approx 5 \times 10^{-13} \qquad (29)$$

PROBLEM 4B

With what speed v must a stone move so that its kinetic energy equals its rest energy?

SOLUTION 4B

This problem requires relativistic analysis. Equation (23) gives total energy and (26) gives rest energy. Kinetic energy is the difference between the two:

$$\frac{E_{\text{lab}} - m}{m} = \frac{1}{\left(1 - v^2\right)^{1/2}} - 1 = 1 \qquad (30)$$

from which

$$1 - v^2 = \frac{1}{2^2} = \frac{1}{4} \qquad (31)$$

so that

$$v = \left(\frac{3}{4}\right)^{1/2} = 0.866 \qquad (32)$$

This speed is a fraction of the speed of light, which means that $v_{\text{conv}} = 0.866 \times 3.00 \times 10^8$ meters/second $= 2.60 \times 10^8$ meters/second.

PROBLEM 4C

Our Sun radiates 3.86×10^{26} watts of light. How much mass does it convert to radiation every second?

SOLUTION 4C

This problem provides exercise in converting units. One watt is one joule/second. The units of energy are the units of (force × distance) or (mass × acceleration × distance). Therefore the units of joule are kilogram-meter^2/second^2. From (27):

$$\begin{aligned} m &= \frac{E_{\text{conv}}}{c^2} && (33)\\ &= \frac{3.86 \times 10^{26}\ \text{kilogram-meters}^2/\text{second}^2}{\left(3.00 \times 10^8\ \text{meters/second}\right)^2} \\ &\approx 4.3 \times 10^9\ \text{kilograms} \\ &\approx 4.3 \times 10^6\ \text{metric tons} \end{aligned}$$

This is the mass—a few million metric tons—that our Sun, a typical star, converts into radiation every second.

that mass itself is a treasure trove of energy. On Earth, nuclear reactions release less than one percent of this available energy. In contrast, a particle-antiparticle annihilation can release *all* of the mass of the combining particles in the form of radiant energy (gamma rays).

At everyday speeds, the expression for E_{lab} in (23) reduces to an expression that contains Newton’s kinetic energy. How do we get to Newton’s case? Simply ask: How fast do things move around us in our everyday lives? At this writing, the fastest speed achieved by a wheeled vehicle on land is 1228 kilometers per hour (Figure 2), which is 763 miles per hour or 280 meters per second. As a fraction of light speed, this vehicle moves at $v = 9.3 \times 10^{-7}$ (no units). For such a small fraction, we can use a familiar approximation (inside the front cover):

$$\begin{aligned} E_{\text{lab}} &= \frac{m}{\left(1 - v_{\text{lab}}^2\right)^{1/2}} = m\left(1 - v_{\text{lab}}^2\right)^{-1/2} \approx m\left(1 + \frac{v_{\text{lab}}^2}{2}\right) && (28)\\ &\approx m + \frac{1}{2} m v_{\text{lab}}^2 = m + (KE)_{\text{Newton}} \qquad (v_{\text{lab}} \ll 1) \end{aligned}$$

You can verify that the approximation is highly accurate when v_{lab} has the value of the land speed record—and is an even better approximation for the everyday speeds of a bicycle or football. The final term in (28) is Newton's (low speed) expression for the kinetic energy of the stone. The first term is the rest energy of the stone, equation (26).

We can also separate the relativistic expression for energy into rest energy and kinetic energy. Define the relativistic kinetic energy of a stone in any frame with the equation

$$KE \equiv E - m = m(\gamma - 1) \qquad \text{(any frame, any speed)} \tag{34}$$

Comment 5. Deeper than Newton?
Newton's First Law of Motion, quoted at the beginning of this section, was his brilliant assumption. In the present section we have derived this result using the Principle of Maximal Aging. Is our result deeper than Newton's? We think so, because the Principle of Maximal Aging has wider application than special relativity. It informs our predictions for the motion of a stone around both the non-spinning and the spinning black hole. Deep indeed!

Fuller Explanation: Energy in flat spacetime: *Spacetime Physics,* Chapter 7, Momenergy.

1.8 ■ MOMENTUM IN SPECIAL RELATIVITY

The interval plus the Principle of Maximal Aging give us an expression for the linear momentum of a stone.

To derive the relativistic expression for the momentum of a stone, we use a method similar to that for the derivation of energy in Section 1.7. Figure 7 corresponds to Figure 6, which we used to derive the stone's energy. Momentum has components in all three space directions; first we derive its x_{lab} component, which we write as $p_{x,\text{lab}}$. In the momentum case the time t_2 for the intermediate flash emission is *fixed,* while we vary the space coordinate s_2 of this intermediate event to find the location that yields maximum wristwatch time between initial and final events. We ask you to carry out this derivation in the exercises. The result is a second expression whose value is constant for a free stone in either the laboratory frame or the rocket frame:

$$\frac{p_{x,\text{lab}}}{m} = \frac{dx_{\text{lab}}}{d\tau} = \frac{v_{x,\text{lab}}}{\left(1 - v_{\text{lab}}^2\right)^{1/2}} = \gamma_{\text{lab}}\, v_{x,\text{lab}} \tag{35}$$

$$\frac{p_{x,\text{rocket}}}{m} = \frac{dx_{\text{rocket}}}{d\tau} = \frac{v_{x,\text{rocket}}}{\left(1 - v_{\text{rocket}}^2\right)^{1/2}} = \gamma_{\text{rocket}}\, v_{x,\text{rocket}} \tag{36}$$

where v_{lab} and v_{rocket} are each constant in the respective frame, and γ was defined in (24). Expressions for the y_{lab} and z_{lab} components of momentum

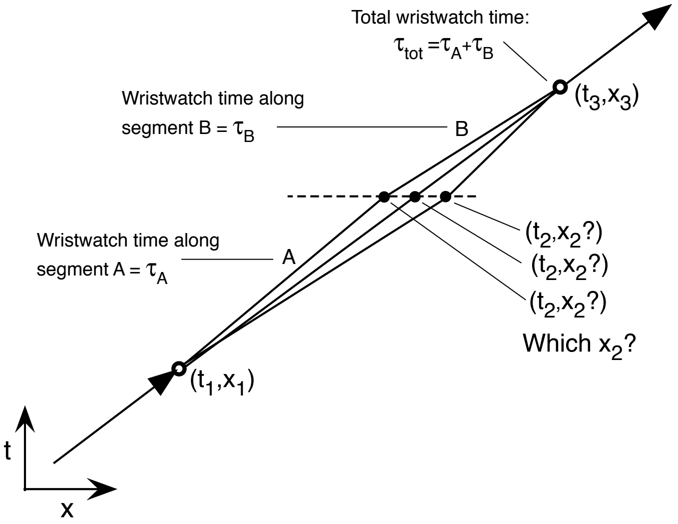

FIGURE 7 Figure for the derivation of the x-component of momentum of a stone. You will carry out this derivation in the exercises.

$p_{x,\text{lab}}/m = dx_{\text{lab}}/d\tau$ is a constant of motion.

are similar to (35) and (36). The result for each component of momentum reminds us that the free stone moves with constant speed in a straight line in every inertial frame.

Each component of the free stone's momentum in the laboratory frame is a *constant of motion*, like its energy E_{lab}/m in the laboratory frame, because each component of momentum does not change as the free stone moves in the laboratory frame. Momentum components of the stone in the rocket frame are also constants of motion, though equations (35) and (36) show that corresponding components in the two frames are not equal, because the stone's velocity is not the same in the two frames.

At slow speed, $v \ll 1$, we recover Newton's components of momentum in both frames. This justifies our calling components in (35) and (36) *momentum.*

Fuller Explanation: Momentum in flat spacetime: *Spacetime Physics,* Chapter 7, Momenergy.

1.9 ■ MASS IN RELATIVITY

The mass m of a stone is an invariant!

Find mass from energy and momentum.

An important relation among mass, energy, and momentum follows from the timelike interval and our relativistic expressions for energy and momentum. Suppose a moving stone emits two flashes differentially close together in distance ds_{lab} and in time dt_{lab}, with similar differentials in the rocket frame. Then (1) gives the lapse of wristwatch time $d\tau$:

$$d\tau^2 = dt_{\text{lab}}^2 - ds_{\text{lab}}^2 = dt_{\text{rocket}}^2 - ds_{\text{rocket}}^2 \qquad (37)$$

Box 1. No Mass Change with Speed!

The fact that no stone moves faster than the speed of light is sometimes "explained" by saying that "the mass of a stone increases with speed," leading to what is called "relativistic mass" whose increase prevents acceleration to a speed greater than that of light. This interpretation can be applied consistently, but what could it mean in practice? Someone riding along with the faster-moving stone detects no change in the number of atoms in the stone, nor any change whatever in the individual atoms, nor in the binding energy between atoms. Where's the "change" in what is claimed to be a "changing mass"? We observe no change in the stone that can possibly account for the varying value of its "relativistic mass."

Our viewpoint in this book is that mass is a *Lorentz invariant*, something whose value is the same for all inertial observers when they use (39) or (40) to reckon the mass. In relativity, every invariant is a diamond. Do not throw away a diamond!

To preserve the diamond of invariant mass, we will never—outside the confines of this box—use the phrase "rest mass." (Horrors!). Why not? Because "rest mass" (Ouch!) implies that there is such a thing as "non-rest mass"—mass that changes with speed. Oops, there goes your precious diamond down the drain.

In contrast, the phrase *rest energy* is fine; it *is true* that energy changes with speed; the energy of a stone *does* have different values as measured by inertial observers in uniform relative motion. In the special case of a stone at rest in any inertial frame, however, the value of its rest energy *in that frame* is equal to the value of its mass—equation (26)—provided you use the same units for mass as for energy.

"Rest mass"? NO!
Rest energy? YES!

For more on this subject see *Spacetime Physics,* **Dialog: Use and Abuse of the Concept of Mass,** pages 246–251.

Divide equation (37) through by the invariant $d\tau^2$ and multiply through by the invariant m^2 to obtain

$$m^2 = \left(m\frac{dt_{\text{lab}}}{d\tau}\right)^2 - \left(m\frac{ds_{\text{lab}}}{d\tau}\right)^2 = \left(m\frac{dt_{\text{rocket}}}{d\tau}\right)^2 - \left(m\frac{ds_{\text{rocket}}}{d\tau}\right)^2 \qquad (38)$$

Substitute expressions (23) and (35) for energy and momentum to obtain:

$$\boxed{m^2 = E_{\text{lab}}^2 - p_{\text{lab}}^2 = E_{\text{rocket}}^2 - p_{\text{rocket}}^2 \qquad (39)}$$

In (39) mass, energy, and momentum are all expressed in the same units, such as kilograms or electron-volts. In conventional units (subscript "conv"), the equation has a more complicated form. In either frame:

$$(m_{\text{conv}}c^2)^2 = E_{\text{conv}}^2 - p_{\text{conv}}^2c^2 \qquad (40)$$

Stone's energy (also momentum) may be different for different observers. . .

. . . but its mass has the same (invariant!) value in all frames.

Equations (39) and (40) are central to special relativity. There is nothing like them in Newton's mechanics. The stone's energy E typically has different values when measured in different inertial frames that are in uniform relative motion. Also the stone's momentum p typically has different values when measured in different frames. However, the values of these two quantities in *any* given inertial frame can be used to determine the value of the stone's mass m, which is independent of the inertial frame. *The stone's mass* m *is a Lorentz invariant* (Definition 6 and Box 1).

Fuller Explanation: Mass and momentum-energy in flat spacetime: *Spacetime Physics,* Chapter 7, Momenergy.

1.10 ■ THE LORENTZ TRANSFORMATION

Relative motion; relative observations

Lorentz transform from lab to rocket

To develop special relativity, Einstein assumed that the laws of physics are the same in every inertial frame, an assertion called **The Principle of Relativity**. Let two different inertial frames, such as those of a laboratory and an unpowered rocket ship, be in uniform relative motion with respect to one another. Special relativity is valid in each of these frames. *More:* Special relativity links the coordinates of an event in one frame with the coordinates of the same event in the other frame; it also relates the energy and momentum components of a stone measured in one frame to the corresponding quantities measured in the other frame. Let an inertial (unpowered) rocket frame pass with relative velocity v_{rel} in the x-direction through an overlapping laboratory frame. Call the laboratory coordinate separations between two events $(\Delta t_{\text{lab}}, \Delta x_{\text{lab}}, \Delta y_{\text{lab}}, \Delta z_{\text{lab}})$ and the rocket coordinate separations between the same events $(\Delta t_{\text{rocket}}, \Delta x_{\text{rocket}}, \Delta y_{\text{rocket}}, \Delta z_{\text{rocket}})$. From now on we use the Greek letter capital delta, Δ, as a shorthand for separation, to avoid lengthy expressions, for example $\Delta t_{\text{lab}} = t_{2,\text{lab}} - t_{1,\text{lab}}$. These separations are related by the **Lorentz transformation equations**:

$$\begin{aligned}
\Delta t_{\text{rocket}} &= \gamma_{\text{rel}}\left(\Delta t_{\text{lab}} - v_{\text{rel}}\Delta x_{\text{lab}}\right) \qquad (41)\\
\Delta x_{\text{rocket}} &= \gamma_{\text{rel}}\left(\Delta x_{\text{lab}} - v_{\text{rel}}\Delta t_{\text{lab}}\right)\\
\Delta y_{\text{rocket}} &= \Delta y_{\text{lab}} \quad \text{and} \quad \Delta z_{\text{rocket}} = \Delta z_{\text{lab}}
\end{aligned}$$

Lorentz transform from rocket to lab

where equation (24) defines γ_{rel}. We do not derive these equations here; see Fuller Explanation at the end of this section. The reverse transformation, from rocket to laboratory coordinates, follows from symmetry: replace v_{rel} by $-v_{\text{rel}}$ and interchange rocket and lab labels in (41) to obtain

$$\begin{aligned}
\Delta t_{\text{lab}} &= \gamma_{\text{rel}}\left(\Delta t_{\text{rocket}} + v_{\text{rel}}\Delta x_{\text{rocket}}\right) \qquad (42)\\
\Delta x_{\text{lab}} &= \gamma_{\text{rel}}\left(\Delta x_{\text{rocket}} + v_{\text{rel}}\Delta t_{\text{rocket}}\right)\\
\Delta y_{\text{lab}} &= \Delta y_{\text{rocket}} \quad \text{and} \quad \Delta z_{\text{lab}} = \Delta z_{\text{rocket}}
\end{aligned}$$

Transform energy and momentum from lab to rocket

For a pair of events infinitesimally close to one another, we can reduce differences in (42) and (41) to coordinate differentials. Further: It is also valid to divide the resulting equations through by the Lorentz invariant differential $d\tau$ and multiply through by the invariant mass m. Then substitute from equations (23) and (35). *Result:* Two sets of equations that transform the energy E and the components $(p_{\text{x}}, p_{\text{y}}, p_{\text{z}})$ of the momentum of a stone between these two frames:

$$\begin{aligned} E_{\text{rocket}} &= \gamma_{\text{rel}}\left(E_{\text{lab}} - v_{\text{rel}} p_{\text{x,lab}}\right) \\ p_{\text{x,rocket}} &= \gamma_{\text{rel}}\left(p_{\text{x,lab}} - v_{\text{rel}} E_{\text{lab}}\right) \\ p_{\text{y,rocket}} &= p_{\text{y,lab}} \quad \text{and} \quad p_{\text{z,rocket}} = p_{\text{z,lab}} \end{aligned} \tag{43}$$

Transform energy and momentum from rocket to lab

Here $p_{\text{x,rocket}}$ is the x-component of momentum in the rocket frame, and so forth. The reverse transformation, again by symmetry:

$$\begin{aligned} E_{\text{lab}} &= \gamma_{\text{rel}}\left(E_{\text{rocket}} + v_{\text{rel}} p_{\text{x,rocket}}\right) \\ p_{\text{x,lab}} &= \gamma_{\text{rel}}\left(p_{\text{x,rocket}} + v_{\text{rel}} E_{\text{rocket}}\right) \\ p_{\text{y,lab}} &= p_{\text{y,rocket}} \quad \text{and} \quad p_{\text{z,lab}} = p_{\text{z,rocket}} \end{aligned} \tag{44}$$

We can now predict and compare measurements in inertial frames in relative motion. And remember, special relativity assumes that every inertial frame extends without limit in every direction and for all time.

Comment 6. Nomenclature: Lorentz boost

Lorentz boost

Often a Lorentz transformation is called a **Lorentz boost**. The word *boost* does not mean sudden change, but rather a change in the frame from which we make measurements and observations.

Comment 7. Constant of motion *vs.* invariant

An *invariant* is not the same as a *constant of motion.* Here is the difference:

An invariant is a quantity that has the same value *in all inertial frames.* Two sample invariants: (a) the wristwatch time between any two events, (b) the mass of a stone. The term *invariant* must always tell or imply what the change is that leads to the same result. Carefully stated, we would say: “The wristwatch time between two events and the mass of a stone are each invariant with respect to a Lorentz transformation between the laboratory and the rocket frame.”

By contrast, a *constant of motion* is a quantity that stays unchanged along the worldline of a free stone *as calculated in a given inertial frame.* Two sample constants of motion: (a) the energy and (b) the momentum of a free stone as observed or measured in, say, the laboratory frame. In other inertial frames moving relatively to the lab frame, the energy and momentum of the stone are also constants of motion; however, these quantities typically have *different values in different inertial frames.*

Conclusion: Invariants (diamonds) and constants of motion (rubies) are both truly precious.

Fuller Explanation: *Spacetime Physics,* Special Topic: Lorentz Transformation.

1.11 ■ LIMITS ON LOCAL INERTIAL FRAMES

Limits on the extent of an inertial frame in curved spacetime

Flat spacetime is the arena in which special relativity describes Nature. The power of special relativity applies strictly only in an inertial frame—or in each one of a collection of overlapping inertial frames in uniform relative motion. In every inertial frame, by definition, a free stone released from rest remains at rest and a free stone launched with a given velocity maintains the magnitude and direction of that velocity.

Limits on size of local inertial frames? We need general relativity.

If it were possible to embrace the Universe with a single inertial frame, then special relativity would describe our Universe, and we would not need general relativity. But we *do* need general relativity, precisely because typically an inertial frame is inertial in only a limited region of space and time. Near a center of attraction, every inertial frame must be **local.** An inertial frame can be set up, for example, inside a sufficiently small "container," such as (a) an unpowered rocket ship in orbit around Earth or Sun, or (b) an elevator on Earth whose cables have been cut, or (c) an unpowered rocket ship in interstellar space. In each such inertial frame, for a limited extent of space and time, we find no evidence of gravity.

Inertial frame cannot be too large, because . . .

Well, *almost* no evidence. Every inertial enclosure in which we ride near Earth cannot be too large or fall for too long a frame time without some unavoidable change in relative motion between a pair of free stones in the enclosure. Why? Because each one of a pair of widely separated stones within a large enclosed space is affected differently by the nonuniform gravitational field of Earth—as Newton would say. For example, two stones released from rest side by side are both attracted toward the center of Earth, so they move closer together as measured inside a falling long narrow horizontal railway coach (Figure 8, left panel). Their motion toward one another has nothing to do with gravitational attraction between these stones, which is entirely negligible.

As another example, think of two stones released from rest far apart vertically, one directly above the other in a long narrow vertical falling railway coach (Figure 8, right panel). For vertical separation, their gravitational accelerations toward Earth are both in the same direction. However, the stone nearer Earth is more strongly attracted to Earth, so gradually leaves the other stone behind, according to Newton's analysis. As a result, viewed from inside the coach the two stones move farther apart. *Conclusion:* The large enclosure is not an inertial frame.

. . . tidal accelerations occur in large frames.

A rider in either railway car such as those shown in Figure 8 sees the pair of horizontally-separated stones accelerate *toward* one another and a pair of vertically-separated stones accelerate *away* from one another. These relative motions earn the name **tidal accelerations,** because they arise from the same kind of nonuniform gravitational field that accounts for ocean tides on Earth—tides due to the field of the Moon, which is stronger on the side of Earth nearer the Moon.

Unavoidable tidal accelerations? Then unavoidable spacetime curvature!

As we fall toward the center of attraction, there is no way to avoid the relative—*tidal*—accelerations at different locations in the long railway car. We

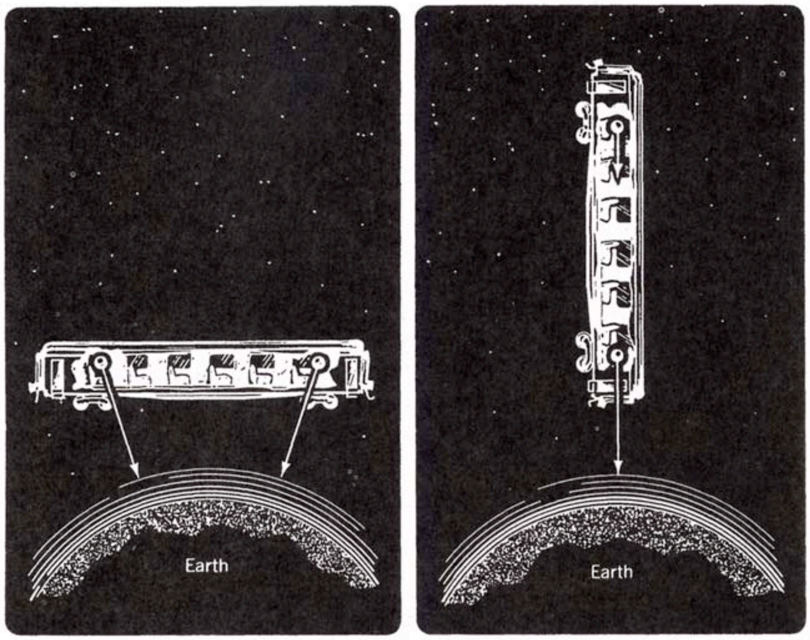

FIGURE 8 Einstein's old-fashioned railway coach in free fall, showing relative accelerations of a pair of free stones, as described by Newton (not to scale). *Left panel:* Two horizontally separated free stones are both attracted toward the center of Earth, so as viewed by someone who rides in the falling horizontal railway car, this pair of stones accelerate *toward* one another. *Right panel:* A free stone nearer Earth has a greater acceleration than that of a free stone farther from Earth. As viewed by someone who rides in the falling vertical railway car, this pair of free stones accelerate *away* from one another. We call these relative accelerations **tidal accelerations**.

can do nothing to eliminate tidal accelerations completely. These relative accelerations are central indicators of the **curvature of spacetime.**

Even though we cannot completely eliminate tidal accelerations near a center of gravitational attraction, we can often reduce them sufficiently so that they do not affect the results of a local measurement that takes place entirely in that frame.

Make every measurement in a local inertial frame.

Conclusion: Almost everywhere in the Universe we can set up a *local* inertial frame in which to carry out a measurement. Throughout this book we *choose* to make every observation and measurement and carry out every experiment in a local inertial frame. This leads to one of the key ideas in this book (see back cover):

> **We *choose* to report every measurement and observation using an inertial frame—a *local* inertial frame in curved spacetime.**

But the local inertial frame tells only part of the story. How can we analyze a pair of events widely separated near the Earth, near the Sun, or near

General relativity: patchwork quilt of inertial frames.

a neutron star—events too far apart to be enclosed in a single inertial frame? For example, how do we describe the motion of a comet whose orbit completely encircles the Sun, with an orbital period of many years? The comet passes through a whole series of local inertial frames, but cannot be tracked using a single global inertial frame—which does not exist. Special relativity has reached its limit! To describe motion that oversteps a single local inertial frame, we must turn to a theory of curved spacetime such as Einstein's general relativity—his **Theory of Gravitation**—that we start in Chapter 3, Curving.

Comment 8. Which way does wristwatch time flow?
In your everyday life, time flows out of what you call your past, into what you call your future. We label this direction **the arrow of time**. But equation (37) contains only squared differentials, which allows wristwatch time lapse to be negative—to run backward—instead of forward along your worldline. So why does your life flow in only one direction—from past to future on your wristwatch? A subtle question! We do not answer it here. In this book we simply assume one-way flow of wristwatch time along any worldline. This assumption will lead us on an exciting journey!

Fuller Explanation: *Spacetime Physics,* Chapter 2, Falling Free, and Chapter 9, Gravity: Curved Spacetime in Action.

1.12 ■ GENERAL RELATIVITY: OUR CURRENT TOOLKIT

Ready for a theory of curved spacetime.

General relativity: amazing predictive power

The remainder of this book introduces Einstein's general theory of relativity, currently our most powerful toolkit for understanding gravitational effects. You will be astonished at the range of observations that general relativity describes and correctly predicts, among them gravitational waves, space dragging, the power of quasars, deflection and time delay of light passing a center of attraction, the tiny precession of the orbit of planet Mercury, the focusing of light by astronomical objects, and the existence of gravitational waves. It even makes some predictions about the fate of the Universe.

General relativity faces extension or revision.

In spite of its immense power, Einstein's general relativity has some inadequacies. General relativity is incompatible with quantum mechanics that describes the structure of atoms. Sooner or later a more fundamental theory is sure to replace general relativity and surmount its limits.

What makes up 96% of the Universe?

We now have strong evidence that so-called "baryonic matter"—everything we can see and touch on Earth (including ourselves) and everything we currently see in the heavens—constitutes only about four percent of the *stuff* that affects the expansion of the Universe. What makes up the remaining 96 percent? Current theories of **cosmology**—the study of the history and evolution of the Universe (Chapter 15)—examine this question using general relativity. But an alternative possibility is that general relativity itself requires modification at these huge scales of distance and time.

Theoretical research into quantum gravity is active; so are experimental tests looking for violations of general relativity, experiments whose outcomes

In the meantime, general relativity is a powerful toolkit.

might guide a new synthesis. Meanwhile, Einstein's general relativity is highly successful and increasingly important as an everyday toolkit. The conceptual issues it raises (and often satisfies) are profound and are likely to be part of any future modification. Welcome to this deep, powerful, and intellectually delicious subject!

Comment 9. Truth in labeling: "Newton" and "Einstein"

Throughout this book we talk about Newton and Einstein as if each were responsible for the current form of his ideas. This is false: Newton published nothing about kinetic energy; Einstein did not believe in the existence of black holes. Hundreds of people have contributed—and continue to contribute—to the ongoing evolution and refinement of ideas created by these giants. We do not intend to slight past or living workers in the field. Rather, we use "Newton" and "Einstein" as labels to indicate which of their worlds we are discussing at any point in the text.

?

Objection 3. *You have told me a lot of weird stuff in this chapter, but I am interested in truth and reality. Do moving clocks* ***really*** *run slow? Are clocks synchronized in one frame* ***really*** *unsynchronized in a relatively-moving frame? Give me the truth about* ***reality****!*

!

Truth and *reality* are mighty words indeed, but in both special and general relativity they are distractions; we strongly suggest that you avoid them as you study these subjects. Why? Because they direct your attention away from the key question that relativity is designed to answer: *What does this inertial observer measure and report?* Ask THAT question and you are ready for general relativity!

Fuller Explanation: *Spacetime Physics,* Chapter 9, Gravity: Curved Spacetime in Action

> *Now Besso has departed from this strange world a little ahead of me. That means nothing. We who believe in physics, know that the distinction between past, present and future is only a stubbornly persistent illusion.*
>
> —Albert Einstein, 21 March 1955, in a letter to Michele Besso's family; Einstein died 18 April 1955.

Comment 10. Chapter preview and summary

This book does not provide formal chapter previews or summaries. To preview the material, read the section titles and questions on the left hand initial page of each chapter, then skim through the marginal comments. Do the same to summarize material and to recall it at a later date.

1.13 ■ EXERCISES

1. Answer to challenge problem in Sample Problem 3:

Event B cannot cause either Event A or Event C because it occurs *after* those events in the given frame. The temporal order of events with a timelike relation will not change, no matter from what frame they are observed: See Section 2.6, entitled "The Difference between Space and Spacetime."

2. Spatial Separation I

Two firecrackers explode at the same place in the laboratory and are separated by a time of 3 seconds as measured on a laboratory clock.

A. What is the spatial distance between these two events in a rocket in which the events are separated in time by 5 seconds as measured on rocket clocks?

B. What is the relative speed v_{rel} between rocket and laboratory frames?

3. Spatial Separation II

Two firecrackers explode in a laboratory with a time difference of 4 seconds and a space separation of 5 light-seconds, both space and time measured with equipment at rest in the laboratory. What is the distance between these two events in a rocket in which they occur at the same time?

4. Maximum wristwatch time

Show that equation (18) corresponds to a maximum, not a minimum, of total wristwatch time of the stone, equation (17), as it travels across two adjacent segments of its worldline.

5. Super Cosmic Rays

The Pierre Auger Observatory is an array of cosmic ray detectors lying on the vast plain *Pampa Amarilla* (yellow prairie) in western Argentina, just east of the Andes Mountains. The purpose of the observatory is to study cosmic rays of the highest energies. The highest energy cosmic ray detected had an energy of 3×10^{20} electron-volts.

A. A regulation tennis ball has a mass of 57 grams. If this tennis ball is given a kinetic energy of 3×10^{20} electron volts, how fast will it move, in meters per second? (*Hint:* Try Newton's mechanics.)

B. Suppose a proton has the energy 3×10^{20} electron-volts. How long would it take this proton to cross our galaxy (take the galaxy diameter to be 10^5 light-years) as measured on the proton's wristwatch? Give your answer in seconds.

C. What is the diameter of the galaxy measured in the rest frame of the proton?

6. Mass-Energy Conversion

A. How much mass does a 100-watt bulb dissipate (in heat and light) in one year?

B. Pedaling a bicycle at full throttle, you generate approximately one-half horsepower of *useful* power. (1 horsepower = 746 watts). The human body is about 25 percent efficient; that is, 25 percent of the food burned can be converted to useful work. How long a time will you have to ride your bicycle in order to lose 1 kilogram by direct conversion of mass to energy? Express your answer in years. (One year = 3.16×10^7 seconds.) How can weight-reducing gymnasiums stay in business? What is misleading about the way this exercise is phrased?

C. One kilogram of hydrogen combines chemically with 8 kilograms of oxygen to form water; about 10^8 joules of energy is released. A very good chemical balance is able to detect a fractional change in mass of 1 part in 10^8. By what factor is this sensitivity more than enough—or insufficient—to detect the fractional change of mass in this reaction?

7. Departure from Newton

Use equations (33) and (34) to check the limits of Newtonian's expression for kinetic energy:

A. An asteroid that falls from rest at a great distance reaches Earth's surface with a speed of 10 kilometers/second (if we neglect atmospheric resistance). By what percent is Newton's prediction for kinetic energy in error for this asteroid?

B. At what speed does the all-speed expression for kinetic energy (34) yield a kinetic energy that differs from Newton's prediction—embodied in equation (33)—by one percent? ten percent? fifty percent? seventy-five percent? one hundred percent? Use the percentage expression $100 \times [KE - (KE)_{\text{Newton}}]/KE$, where KE is the relativistic expression for kinetic energy.

8. Units and Conversions

A. Show that the speed of a stone in an inertial frame (as a fraction of the speed of light) is given by the expression

$$v_{\text{inertial}} = \left(\frac{ds}{dt}\right)_{\text{inertial}} = \left(\frac{p}{E}\right)_{\text{inertial}} \tag{45}$$

B. What speed v does (45) predict when the mass of the particle is zero, as is the case for a flash of light? Is this result the one you expect?

C. The mass and energy of particles in beams from accelerators is often expressed in GeV, that is billions of electron-volts. Journal articles describing these measurements refer to particle momentum in units of GeV/c. Explain.

9. The Pressure of Light

A flash of light has zero mass. Use equation (40), in conventional units, to answer the following questions.

A. You can feel on your hand an object with the weight of 1 gram mass. Shine a laser beam downward on a black block of wood that you hold in your hand. You detect an increased force as if the block of wood had increased its mass by one gram. What power does the laser beam deliver, in watts?

B. The block of wood described in part A absorbs the energy of the laser beam. Will the block burst into flame?

10. Derivation of the Expression for Momentum

A. Carry out the derivation of the relativistic expression for momentum described in Section 1.8. Lay out this derivation in a series of numbered steps that parallel those for the derivation of the energy in Section 1.7.

B. Write an expression for p in conventional units.

11. Verifying energy-momentum transformation equations

Derive transformation equations (43) and (44) using the procedure outlined just before these equations.

12. Newtonian transformation

Show that for Newton, where all velocities are small compared to the speed of light, the Lorentz transformation equations (41) reduce to the familiar Galilean transformation equations and lead to the universality of time.

13. The Photon

NOTE: Exercises 13 through 18 are related to one another.

A. A photon is a quantum of light, a particle with zero mass. Apply equation (39) for a photon moving only in the $\pm x$-direction. Show that in this conversion to light, $p_{\mathrm{x}} \rightarrow \pm E$.

B. Write down the Lorentz transformation equations (43) and (44) for a photon moving in the positive x-direction.

C. Write down the Lorentz transformation equations (43) and (44) for a photon moving in the negative x-direction.

D. Show that *it does not matter* what units you use for E in your photon Lorentz transformation equations, as long as the units for each occurrence of E are the same.

14. One-Dimensional Doppler Equations

A mongrel equation (neither classical nor quantum-mechanical) connects the quantum energy E of a single photon with the frequency f of a classical electromagnetic wave. In conventional units, this equation is:

$$E_{\text{conv}} = hf_{\text{conv}} \qquad \text{(photon, conventional units)} \qquad (46)$$

where f_{conv} is the frequency in oscillations per second and h is Planck's constant. In mks units, E_{conv} has the units joules, and **Planck's constant** h has the numerical value $h = 6.63 \times 10^{-34}$ joule-second.

A. Substitute (46) into your transformation equations for the photon, and replace γ_{rel} in those equations with its definition $(1 - v_{\text{rel}}^2)^{-1/2}$. Planck's constant disappears from the resulting equations between frequency f_{lab} in the laboratory frame and frequency f_{rocket} in the rocket frame:

$$f_{\text{lab}} = \left[\frac{1 \pm v_{\text{rel}}}{1 \mp v_{\text{rel}})}\right]^{1/2} f_{\text{rocket}} \qquad (\ \pm x, \text{ light}) \qquad (47)$$

$$f_{\text{rocket}} = \left[\frac{1 \mp v_{\text{rel}}}{1 \pm v_{\text{rel}}}\right]^{1/2} f_{\text{lab}} \qquad (\ \pm x, \text{ light}) \qquad (48)$$

These are the **one-dimensional Doppler equations** for light moving in either direction along the x-axis.

B. The relation between frequency f_{conv} and wavelength λ_{conv} for a classical plane wave in an inertial frame, in conventional units

$$f_{\text{conv}}\lambda_{\text{conv}} = c \qquad \text{(classical plane wave)} \qquad (49)$$

Rewrite equations (47) and (48) for the relation between laboratory wavelength λ_{lab} and rocket wavelength λ_{rocket}.

15. Speed-Control Beacon

An advanced civilization sets up a beacon on a planet near the crowded center of our galaxy and asks travelers approaching directly or receding directly from the beacon to use the Doppler shift to measure their speed relative to the beacon, with a speed limit at $v = 0.2$ relative to that beacon. The beacon emits light of a single *proper* wavelength λ_0, that is, the wavelength measured in the rest frame of the beacon. Four index colors are:

$$\begin{aligned} \lambda_{\mathrm{red}} &= 680 \times 10^{-9}\mathrm{meter} = 680 \text{ nanometers} \\ \lambda_{\mathrm{yellow}} &= 580 \times 10^{-9}\mathrm{meter} = 580 \text{ nanometers} \\ \lambda_{\mathrm{green}} &= 525 \times 10^{-9}\mathrm{meter} = 525 \text{ nanometers} \\ \lambda_{\mathrm{blue}} &= 475 \times 10^{-9}\mathrm{meter} = 475 \text{ nanometers} \end{aligned} \qquad (50)$$

A. Choose the beacon proper wavelength λ_0 so that a ship approaching at half the speed limit, $v = 0.1$, sees green light. What is the proper wavelength λ_0 of the beacon beam? What color do you see when you stand next to the beacon?

B. As your spaceship moves directly toward the beacon described in Part A, you see the beacon light to be blue. What is your speed relative to the beacon? Is this below the speed limit?

C. In which direction, toward or away from the beacon, are you traveling when you see the beacon to be red? What is your speed relative to the beacon? Is this below the speed limit?

16. Radar

An advanced civilization uses radar to help enforce the speed limit in the crowded center of our galaxy. Radar relies on the fact that with respect to its rest frame a spaceship reflects a signal back with a frequency equal to the incoming frequency measured in its frame.

A. Show that a radar signal of frequency f_0 at the source is received back from a directly approaching ship with the reflected frequency f_{reflect} given by the expression:

$$f_{\mathrm{reflect}} = \frac{1+v}{1-v} f_0 \qquad \text{(radar)} \qquad (51)$$

where v is the speed of the spaceship with respect to the signal source.

B. What is the wavelength $\lambda_{\mathrm{reflect}}$ of the signal reflected back from a spaceship approaching at the speed limit of $v = 0.2$?

C. The highway speed of a car is very much less than the speed of light. Use the approximation formula inside the front cover to find the following approximate expression for $f_{\mathrm{reflect}} - f_0$:

$$f_{\mathrm{reflect}} - f_0 \approx 2 v f_0 \qquad \text{(highway radar)} \qquad (52)$$

The US state of Massachusetts Highway Patrol uses radar with microwave frequency $f_0 = 10.525 \times 10^9$ cycles/second. By how many cycles/second is the reflected beam shifted in frequency when reflected from a car approaching at 100 kilometers/hour = 27.8 meters/second?

17. Two-dimensional Velocity Transformations

An electron moves in the laboratory frame with components of velocity $(v_{\mathrm{x,lab}},\, v_{\mathrm{y,lab}})$ and in the rocket frame with components of velocity $(v_{\mathrm{x,rocket}},\, v_{\mathrm{y,rocket}})$.

A. Use the differential form of the Lorentz transformation equations (42) to relate the velocity components of the electron in laboratory and rocket frames:

$$v_{\mathrm{x,lab}} = \frac{v_{\mathrm{x,rocket}} + v_{\mathrm{rel}}}{1 + v_{\mathrm{rel}} v_{\mathrm{x,rocket}}} \qquad v_{\mathrm{y,lab}} = \frac{v_{\mathrm{y,rocket}}}{\gamma_{\mathrm{rel}}\left(1 + v_{\mathrm{rel}} v_{\mathrm{x,rocket}}\right)} \tag{53}$$

This is called the **Law of Transformation of Velocities**.

B. With a glance at the Lorentz transformation (42) and its inverse (41), make an argument that to derive the inverse of (53), one simply replaces v_{rel} with $-v_{\mathrm{rel}}$ and interchanges lab and rocket labels, leading to:

$$v_{\mathrm{x,rocket}} = \frac{v_{\mathrm{x,lab}} - v_{\mathrm{rel}}}{1 - v_{\mathrm{rel}} v_{\mathrm{x,lab}}} \qquad v_{\mathrm{y,rocket}} = \frac{v_{\mathrm{y,lab}}}{\gamma_{\mathrm{rel}}\left(1 - v_{\mathrm{rel}} v_{\mathrm{x,lab}}\right)} \tag{54}$$

C. Does the law of transformation of velocities allow the electron to move faster than light when observed in the laboratory frame? For example, suppose that in the rocket frame the electron moves in the positive x_{rocket}-direction with velocity $v_{\mathrm{x,rocket}} = 0.75$ and the rocket frame also moves in the same direction with the same relative speed $v_{\mathrm{rel}} = 0.75$. What is the value of the velocity $v_{\mathrm{x,lab}}$ of the electron in the laboratory frame?

D. Suppose two light flashes move with opposite velocities $v_{\mathrm{x,rocket}} = \pm 1$ in the rocket frame. What are the corresponding velocities $v_{\mathrm{x,lab}}$ of the two light flashes in the laboratory frame?

E. Light moves with velocity components $(v_{\mathrm{x,rocket}},\, v_{\mathrm{y,rocket}}, v_{\mathrm{z,rocket}}) = (0,\, -1,\, 0)$ in the rocket frame. *Predict* the magnitude $|v_{\mathrm{lab}}|$ of its velocity measured in the laboratory frame. Does a calculation verify your prediction?

18. Aberration of light

Light that travels in one direction in the laboratory travels in another direction in the rocket frame unless the light moves along the line of relative motion of the two frames. This difference in light travel direction is called **aberration**.

A. Transform the angle of light propagation in two spatial dimensions. Recall that laboratory and rocket x-coordinates lie along the same line, and in each frame measure the angle ψ of light motion with respect to this common forward x-direction. Make the following argument: Light travels with the speed one, which is the hypotenuse of the velocity component triangle. Therefore for light $v_{\mathrm{x,inertial}} \equiv v_{\mathrm{x,inertial}}/1 = \cos\psi$. Show that this argument converts the first of equations (53) to:

$$\cos\psi_{\mathrm{lab}} = \frac{\cos\psi_{\mathrm{rocket}} + v_{\mathrm{rel}}}{1 + v_{\mathrm{rel}}\cos\psi_{\mathrm{rocket}}} \qquad \text{(light)} \qquad (55)$$

B. From equation (39) show that for light tracked in any inertial frame $|p_{\mathrm{inertial}}| = E_{\mathrm{inertial}}$. Hence $p_{\mathrm{x,inertial}}/E_{\mathrm{inertial}} = \cos\psi$ and the first of equations (44) becomes, for light

$$E_{\mathrm{lab}} = E_{\mathrm{rocket}}\gamma_{\mathrm{rel}}\left(1 + v_{\mathrm{rel}}\cos\psi_{\mathrm{rocket}}\right) \qquad \text{(light)} \qquad (56)$$

C. Make an argument that to derive the inverses of (55) and (56), you simply replace v_{rel} with $-v_{\mathrm{rel}}$ and interchange laboratory and rocket labels, to obtain the aberration equations:

$$\cos\psi_{\mathrm{rocket}} = \frac{\cos\psi_{\mathrm{lab}} - v_{\mathrm{rel}}}{1 - v_{\mathrm{rel}}\cos\psi_{\mathrm{lab}}} \qquad \text{(light)} \qquad (57)$$

$$E_{\mathrm{rocket}} = E_{\mathrm{lab}}\gamma_{\mathrm{rel}}\left(1 - v_{\mathrm{rel}}\cos\psi_{\mathrm{lab}}\right) \qquad \text{(light)} \qquad (58)$$

D. A source at rest in the rocket frame emits light uniformly in all directions in that frame. Consider the 50 percent of this light that goes into the forward hemisphere in the rocket frame. Show that in the laboratory frame this light is concentrated in a narrow forward cone of half-angle $\psi_{\mathrm{headlight,lab}}$ given by the following equation:

$$\cos\psi_{\mathrm{headlight,lab}} = v_{\mathrm{rel}} \qquad \text{(headlight effect)} \qquad (59)$$

The transformation that leads to concentration of light in the forward direction is called the **headlight effect**.

19. Cherenkov Radiation

Can an electron move faster than light? No and yes. No, an electron cannot move faster than light *in a vacuum*; yes, it can move faster than light in a medium in which light moves more slowly than its standard speed in a vacuum. P. A. Cherenkov shared the 1958 Nobel Prize for this discovery that an electron emits coherent radiation when it moves faster than light moves in any medium.

What is the minimum kinetic energy that an electron must have to emit Cherenkov radiation while traveling through water, where the speed of light is $v_{\mathrm{light}} \approx 0.75$? Express this kinetic energy as both the fraction (kinetic energy)$/m$ of its mass m and in electron-volts (eV). Type “Cherenkov radiation" into a computer search engine to see images of the blue light due to Cherenkov radiation emitted by a radioactive source in water.

20. Live Forever?

Luc Longtin shouts, "I can live forever! Here is a variation of equation (1): $\Delta\tau^2 = \Delta t^2_{\text{Earth}} - \Delta s^2_{\text{Earth}}$. Relativity allows the possibility that $\Delta\tau \ll \Delta t_{\text{Earth}}$. In the limit, $\Delta\tau \to 0$, so the hour hand on my wristwatch does not move. Eternal life!

"I have decided to ride a 100 kilometer/hour train back and forth my whole life. THEN I will age much more slowly." Comment on Luc's ecstatic claim without criticizing him.

A. When he carries out his travel program, how much younger will 100-year-old Luc be than his stay-at-home twin brother Guy?

B. Suppose Luc rides a spacecraft in orbit around Earth (speed given in Figure 2). In this case, how much younger will 100-year-old Luc be than brother Guy?

C. Suppose Luc manages to extend his life measured in Earth-time by riding on a fast cosmic ray (speed given in Figure 2). When Luc returns to Earth in his old age, it is clear that his brother Guy will no longer be among the living. However, would Luc *experience* his life as much longer than he would have experienced it if he remained on Earth? That is, would he "enjoy a longer life" in some significant sense, for example counting many times the total number of heartbeats experienced by Guy?

1.14 ■ REFERENCES

Initial quote: *The Ultimate Quotable Einstein*, Collected and Edited by Alice Calaprice, 2011, Princeton University Press, page 354.

Final quote: same reference, page 113.

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Quote at the end of Section 1.5: Robert W. Fuller and John A. Wheeler, "Causality and Multiply Connected Space-Time," Physical Review, Volume 128, Number 2, October 15, 1962, pages 919-929.

Newton's first law of motion (Section 1.7) quoted from *The Principia*, A new translation by I. Bernard Cohen and Anne Whitman, University of California Press, 1999, page 416.

Comment 8: Kip S. Thorne's *Black Holes and Time Warps: Einstein's Outrageous Legacy*, W. W. Norton Co., New York, 1994. An introduction that includes the many people who forged the physics of the black hole.