Chapter 3. Curving

- *General relativity describes only tiny effects, right?*
- *What does "curvature of spacetime" mean?*
- *What tools can I use to visualize spacetime curvature?*
- *Do people at different r-coordinates near a black hole* age *differently? If so, can they* feel *the slowing down/speeding up of their aging?*
- *What is the "event horizon," and what weird things happen there?*
- *Do funnel diagrams describe the gravity field of a black hole?*

CHAPTER

3

Curving

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In my talk ... I remarked that one couldn't keep saying "gravitationally completely collapsed object" over and over. One needed a shorter descriptive phrase. "How about black hole?" asked someone in the audience. I had been searching for just the right term for months, mulling it over in bed, in the bathtub, in my car, wherever I had quiet moments. Suddenly this name seemed exactly right. ... I decided to be casual about the term "black hole," dropping it into [a later] lecture and the written version as if it were an old family friend. Would it catch on? Indeed it did. By now every schoolchild has heard the term.

—John Archibald Wheeler with Kenneth Ford

3.1 ■ THE SCHWARZSCHILD METRIC

Spherically symmetric massive center of attraction?
The Schwarzschild metric describes the curved, empty spacetime around it.

Einstein to Schwarzschild: "splendid."

In late 1915, within a month of the publication of Einstein's general theory of relativity and just a few months before his own death from a battle-related illness, Karl Schwarzschild (1873-1916) derived from Einstein's field equations the metric for spacetime surrounding the spherically symmetric black hole. Einstein wrote to him, "I had not expected that the exact solution to the problem could be formulated. Your analytic treatment of the problem appears to me splendid."

Orbits stay in a plane.

An isolated satellite zooms around a spherically symmetric massive body. After a few orbits we discover that the satellite's motion stays confined to the initial plane determined by the satellite's position, its direction of motion, and the center of the attracting body. Why? The reason is simple: symmetry! With

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Box 1. Metric in Polar Coordinates for Flat Spacetime

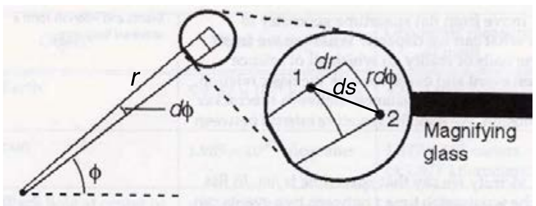

FIGURE 1 Spatial separation between two points in polar coordinates.

The metric for flat spacetime is:

$$d\tau^2 = dt^2 - ds^2 \qquad \text{(flat spacetime)} \qquad (1)$$

where ds is the spatial separation, expressed in Cartesian coordinates as

$$ds^2 = dx^2 + dy^2 \qquad \text{(flat space)} \qquad (2)$$

We look for a similar ds expression for two adjacent events numbered 1 and 2, events separated by polar coordinate increments *dr* and $d\phi$ (Figure 1).

Draw little arcs of different radii through events 1 and 2 to form a tiny box, shown in the magnified inset. The squared spatial separation between events 1 and 2 is—approximately—the sum of the squares of two adjacent sides of the little box. For a differential $d\phi$, we are entitled to express the differential space separation between event 1 and event 2 by the formula

$$ds^2 = dr^2 + r^2 d\phi^2 \qquad \text{(flat space)} \qquad (3)$$

This squared spatial separation is the space part of the squared wristwatch time differential for flat spacetime

$$d\tau^2 = dt^2 - dr^2 - r^2 d\phi^2 \qquad \text{(flat spacetime)} \qquad (4)$$

This derivation is valid only when $d\phi$ is small—vanishingly small in the calculus sense—so that the differential segment of arc $rd\phi$ is indistinguishable from a straight line. There is no such limitation to differentials for rectangular Cartesian space coordinates in flat spacetime, so each d for differential in (2) can be expanded to Δ, as it was in Section 1.10.

From Einstein's general relativity equations, Schwarzschild derived a generalization of (4) that goes beyond flat spacetime and describes curved spacetime in the vicinity of a spherically symmetric (thus non-spinning) uncharged black hole.

respect to this initial plane there is no distinction between "up out of" and "down out of" the plane, so the satellite cannot choose either and must remain in that plane. The limitation of isolated particle and light motion to a single plane greatly simplifies our analysis of physical events in this book.

We use polar coordinates (r, ϕ) for the black hole (Box 1), because polar coordinates reflect its symmetry on a plane through the black hole's center; Cartesian coordinates (x, y) do not.

Schwarzschild timelike metric

Think of two adjacent events that lie on our equatorial r, ϕ plane through the center of the black hole. These events have differential coordinate separations dt, dr, and $d\phi$. The **Schwarzschild metric** gives us the invariant $d\tau$ between this pair of events:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \dfrac{2M}{r}\right)} - r^2 d\phi^2 \qquad \text{(timelike)} \qquad (5)$$

$$(-\infty < t < \infty \quad \text{and} \quad 0 < r < \infty \quad \text{and} \quad 0 \le \phi < 2\pi)$$

Equation (5) is the *timelike* form of the Schwarzschild metric, whose left side gives us the invariant *differential wristwatch time* $d\tau$ of a free stone that moves between a pair of adjacent events for which the magnitude of the first term on the right side is greater than the magnitude of the last two terms. In contrast, think of a pair of events for which the magnitude of the last two terms on the right predominate. Then the invariant *differential ruler distance* $d\sigma$ between these events is given by the *spacelike* form of the Schwarzschild metric:

Schwarzschild spacelike metric

$$d\sigma^2 = -d\tau^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \dfrac{2M}{r}\right)} + r^2 d\phi^2 \quad \text{(spacelike)} \tag{6}$$

$$(-\infty < t < \infty \quad \text{and} \quad (0 < r < \infty \quad \text{and} \quad 0 \leq \phi < 2\pi)$$

Neither a stone nor a light flash can move between an adjacent pair of events with spacelike separation. Instead, the separation $d\sigma$ represents a differential ruler distance between two events. To make use of global metrics (5) and (6), we need to define carefully the meaning of global coordinates t, r, and ϕ. Section 3.2 shows how to measure mass in meters, so that $2M/r$ becomes unitless, as it must in order to subtract it from the unitless number one in the expression $(1 - 2M/r)$.

Comment 1. Terminology: global metric
We refer to either expression (5) or (6) as a *global metric*. Professional general relativists call these expressions *line elements*; they reserve the term *metric* for the collection of coefficients of the differentials—such as $(1 - 2M/r)$, the coefficient of dt^2. We find the term *metric* to be simple, short, and clear; so in this book we use a slightly-deviant terminology and call an expression like (5) or (6) the **global metric**.

Meaning of "global metric"

DEFINITION 1. Invariant (general relativity)
Section 1.2 defined an *invariant* in special relativity as a quantity that has the same value when calculated using different *local inertial* coordinates. An **invariant** in general relativity is a quantity that has the same value when calculated using different *global* coordinate systems. Equations (5) and (6) calculate invariants $d\tau$ and $d\sigma$, respectively, using Schwarzschild global coordinates. Box 3 in Section 7.5 shows that an infinite number of global coordinate systems exist for the non-spinning black hole (indeed, for any isolated black hole). Calculation of $d\tau$ using any of these global coordinate systems delivers the same—the invariant!—value of $d\tau$ given by metrics (5) and (6).

Definition: **invariant** in general relativity

Two coefficients in the Schwarzschild metric contain the expression $(1 - 2M/r)$, which goes to zero when $r \to 2M$, thus sending the first metric coefficient to zero on the right side of the metric and the magnitude of the second coefficient to infinity. This warns us about trouble at $r = 2M$, which we describe below. To the global spacetime surface at $r = 2M$ we assign the name **event horizon**, for reasons that will become clear in later sections.

Event horizon

It is important to realize how rare and wonderful is the Schwarzschild metric. Einstein's set of field equations is nonlinear and can be solved in simple form only for physical systems with considerable symmetry.

Simple global metrics are rare.

Schwarzschild used the symmetry of an isolated spherical non-spinning center of attraction in the derivation of his metric. This symmetry is broken—and no simple global metric exists—when we place a black hole on every street corner, although in principle a computer can provide a numerical solution of Einstein's field equations for any distribution of mass/energy/pressure. It is a measure of the scarcity of physical systems with simple metrics that almost fifty years passed before Roy Kerr found a (relatively!) simple metric for a spinning black hole in 1963 (Chapters 17 through 21).

Schwarzschild description of spacetime is **complete.**

Further investigation shows that the Schwarzschild metric plus the connectedness ("topology") of the region provides a *complete* description of spacetime external to any isolated spherically symmetric, uncharged massive body—and everywhere around such a black hole except at its central singularity (at $r = 0$), where spacetime curvature becomes infinite and general relativity fails. *Every* feature of spacetime around this kind of black hole is described or implied by the Schwarzschild metric. This one expression tells it all!

QUERY 1. Flat spacetime as $r \to \infty$

Show that as $r \to \infty$ Schwarzschild metric (5) becomes metric (4) for flat spacetime.

Ways in which the Schwarzschild metric makes sense:

We will derive the Schwarzschild metric in Chapter 22. Even now, however, we should not accept it uncritically. Here we check three ways in which it makes sense.

1. Depends only on r coordinate.

First, the expression $(1 - 2M/r)$ that appears in both the dt term and the dr term depends only on the r coordinate, not on the angle ϕ. How come? Because we are dealing with a spherically symmetric body, an object for which there is no way to tell one side from the other side or the top from the bottom. This impossibility is reflected in the absence of any direction-dependent coefficient in the metric.

2. Goes to inertial metric for zero M.

Second, the Schwarzschild metric uses coordinates that clearly show spacetime is flat when $M \to 0$, that is when there is *no* center of attraction. In this limit, the Schwarzschild metric (5) goes smoothly into the inertial metric (4) for flat spacetime.

3. Goes to local inertial metric for large r.

Third, even when M is nonzero the Schwarzschild metric (5) reduces to a local flat spacetime metric (4) very far from the black hole. The expression $(1 - 2M/r) \to 1$ when $r \to \infty$.

Timelike and spacelike Schwarzschild metrics (5) and (6) describe the spacetime *external* to *any* isolated spherically symmetric, uncharged massive body. They apply with high precision to spacetime *outside* a slowly revolving massive object such as Earth or an ordinary star like our Sun. Think of a stone moving outside such an object; it makes no difference what the coordinates are inside the attracting spherical body because the stone never gets there; before it can, it collides with the surface—in the short term, our

Schwarzschild metric applies only outside the surface.

Box 2. More About the Black Hole

John Archibald Wheeler adopted the term "black hole" in 1967 (initial quote), but the concept itself is old. As early as 1783, John Michell argued that light must "be attracted in the same manner as all other bodies" and therefore, if the attracting center is sufficiently massive and sufficiently compact, "all light emitted from such a body would be made to return toward it." Pierre-Simon Laplace came to the same conclusion independently in 1795 and went on to reason that "it is therefore possible that the greatest luminous bodies in the universe are on this very account invisible."

Michell and Laplace used Isaac Newton's "action-at-a-distance" theory of gravity in analyzing the escape of light from, or its capture by, an already-existing compact object. But is such a static compact object possible? In 1939, J. Robert Oppenheimer and Hartland Snyder published the first detailed treatment of gravitational collapse within the framework of Einstein's theory of gravitation. Their paper predicts the central features of a non-spinning black hole.

Ongoing theoretical study has shown that the black hole is the result of natural physical processes. A nonsymmetric collapsing system is not necessarily blown apart by its instabilities but can quickly—in a few seconds measured on a remote clock!—radiate away its turbulence as gravitational waves and settle down into a stable structure.

An uncharged spherically symmetric black hole is completely described by the Schwarzschild metric (plus the spacetime topology), which was derived from Einstein's field equations by Karl Schwarzschild and published in 1916. The energy of such a non-spinning black hole cannot be milked for use outside its event horizon. For this reason, a non-spinning black hole deserves the name "dead black hole."

In contrast to the non-spinning dead black hole, the typical black hole, like the typical star, has a spin, sometimes a large spin. The energy stored in this spin, moreover, is available for doing work: for driving jets of matter and for propelling a spaceship. In consequence, the spinning black hole deserves and receives the name "live black hole."

The spinning black hole—or any spinning mass—drags everything in its vicinity around with it, including spacetime (Chapters 17 through 21). Near Earth this dragging is a small effect. Theory predicts that, near a rapidly-spinning black hole, such effects can be large, even irresistible, dragging along nearby spaceships no matter how powerful their rockets.

Black holes appear to be divided roughly into two groups, depending on their source: Those that result from the collapse of a single star have several times the mass our Sun. Others formed near the centers of galaxies can be monsters with millions—even billions—of times the mass of our Sun. These black holes may even shape the evolution of galaxies.

In 1963 Roy P. Kerr derived a metric for an uncharged spinning black hole. In 1967 Robert H. Boyer and Richard W. Lindquist devised a simple and convenient global coordinate system for the spinning black hole. In 2000 Chris Doran published the global coordinate system for a spinning black hole that we use in this book. In 1965 Ezra Theodore Newman and others solved the Einstein equations for the spacetime geometry around an *electrically charged* spinning black hole.

Subsequent research shows that for a steady-state black hole of specified mass, charge, and angular momentum, Kerr-Newman geometry is the *most general* solution to Einstein's field equations. The variety, detail, and beauty of everything that forms or falls into a black hole disappears—at least according to classical (non-quantum) physics—leaving only mass, charge, and angular momentum. John Wheeler summarized this finding in the phrase, "The black hole has no hair," which is known as the **no-hair theorem**.

Sun can be thought of as in equilibrium. The more compact the massive body, however, the larger the external region the stone can explore. Our Sun's surface is 696 000 kilometers from its center. A cool white dwarf with the mass of our Sun has a surface r-coordinate of about 5000 kilometers, roughly that of Earth. The Schwarzschild metric describes spacetime geometry in the region external to that r-coordinate. A neutron star with the mass of our Sun has a surface r-coordinate of about 10 kilometers—the size of a typical city—so the stone can come even closer and still be "outside," that is, in the region described correctly by the Schwarzschild metric (if the neutron star is not spinning too fast).

Box 3. Singularities: Fictitious or Real?

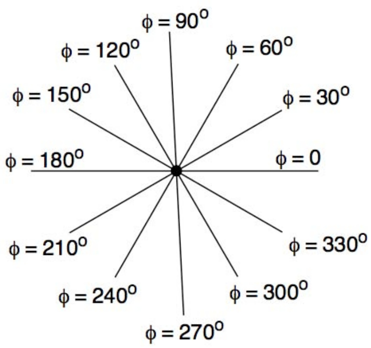

FIGURE 2 Polar coordinates on a flat Euclidean surface have a *coordinate singularity* at the center. Obviously $r = 0$ there, but what is its value of ϕ? That singularity, however, is fictitious because there is no *space* singularity at that point.

How do we know that the blow-up of the term $dr^2/(1 - 2M/r)$ at $r = 2M$ in the Schwarzschild metric does *not* signal a physical singularity? Why is this blow-up no threat to an observer falling through the event horizon—other than its one-way nature and the gradually-increasing tidal forces she feels as she descends? Einstein and others initially thought that the Schwarzschild coordinate singularity at the event horizon had a physical reality, but it does not.

Similarly, how do we know that the blow-up of the term $(1-2M/r)dt^2$ at $r = 0$ is lethal to all comers? How can we understand the difference between the two discontinuities in Schwarzschild coordinates?

Draw an analogy to the polar coordinate system (r, ϕ) on a flat Euclidean surface (Figure 2). The radial coordinate of the origin is clearly $r = 0$, but what is the polar angle ϕ there? *Answer:* The origin is singular in angle ϕ. *Proof:* Start at the right on the horizontal axis with label $\phi = 0$; move leftward along this axis and through the origin at $r = 0$. At this origin the axis label suddenly flips to $\phi = 180°$. There is a discontinuity of ϕ at the origin. The *coordinate* ϕ violates the requirements of uniqueness and smoothness.

The problem here is *not* Euclidean space, it is our silly (r, ϕ) coordinate system. In contrast, Cartesian coordinates $x = r\cos\phi$ and $y = r\sin\phi$ are perfectly unique and continuous at all points on the flat surface, including the origin.

Is there some way to show that there is no physical singularity at the event horizon of a non-spinning black hole? Yes, by finding a coordinate system which is perfectly smooth at the event horizon, in the same way that Cartesian coordinates in Euclidean space are perfectly smooth at the origin. In Chapter 7 we develop what we call *global rain coordinates*. At the event horizon no term blows up in the metric expressed in global rain coordinates. Global rain coordinates assign unique labels to each event and are smooth and continuous at the event horizon and all the way down to (but not including) $r = 0$.

What about the location at the center of a black hole? No coordinate system can be smooth at $r = 0$, because the so-called **Riemann curvature** is infinite there. The Riemann curvature, discovered in the 1860s by mathematician Bernhard Riemann, has a value at every spacetime event that is independent of the coordinate system. The Riemann curvature is infinite only at a physical cusp or singularity, such as the black hole singularity at $r = 0$. In contrast, the Riemann curvature is finite at $r = 2M$.

Schwarzschild describes **all** spacetime around the black hole outside the singularity.

A wonderful thing about a black hole is that it has no physical surface and no matter with which to collide. A stone can explore *all* of spacetime (except at $r = 0$) without bumping into a surface—since there is no surface at all.

Objection 1. *How can a black hole have "no matter with which to collide"? If it isn't made of matter, what is it made of? What happened to the star or group of stars that collapsed to form the black hole? Basically, how can something have mass without being made of matter?*

We think that everything that collapses into the black hole is effectively still there in some form, inducing the curvature of surrounding spacetime. This mass is crushed into a singularity at the center—along with the probe we sent in to explore it. How do we know this? We don't. What can "crushed to a singularity" possibly *mean*? We don't know. Startling? Crazy? Absurd? Welcome to general relativity!

Objection 2. *The global metric comes from Einstein's equations, which you say we will derive in Chapter 22. In the meantime you give us only global metrics. Why should we believe you, and why are you keeping the fundamental equations from us?*

Einstein's equations are most economically expressed in advanced mathematics such as *tensors*, and deriving a global metric from them is a bit tricky. In contrast, the global metric expresses itself in differentials, the working mathematics of most technical professions, and leads directly to measurable quantities: wristwatch time and ruler distance. We choose to start with the directly useful.

Next we examine the meaning of mass in units of length, so that the expression $1 - 2M/r$ in both the first and second term in the metric coefficients can have the same units, namely no units at all.

3.2 ■ MASS IN UNITS OF LENGTH

Want to reduce clutter in the metric? Then measure mass in meters!

The description of spacetime near any gravitating body is simplest when we express the mass M of that body in spatial units—in meters or kilometers. This section derives the conversion factor between, for example, kilograms and meters.

Measure mass in meters.

Earlier we wanted to measure space and time in the same unit (Section 1.2), so we used the conversion factor c, the speed of light. Conversion from kilograms to meters is not so simple. Nevertheless, here too Nature provides a conversion factor, a combination of the speed of light and Newton's **universal gravitation constant G**.

Newton's theory of gravitation predicts that the gravitational force between two spherically symmetric masses $M_{\rm kg}$ and $m_{\rm kg}$ is proportional to the product of these masses and inversely proportional to the square of the Euclidean distance r between their centers:

$$F_{\rm Newtons} = -\frac{GM_{\rm kg}m_{\rm kg}}{r^2} \qquad \text{(Newton, conventional units)} \tag{7}$$

In this equation G is the "constant of proportionality," whose units depend on the units with which mass and spatial separation are measured. The numerical value of G in conventional units is:

$$G = 6.67 \times 10^{-11} \frac{\text{meter}^3}{\text{kilogram second}^2} \tag{8}$$

Numerical values of G

Divide G by the square of the speed of light c^2 to find the conversion factor that translates the conventional unit of mass, the kilogram, into what we have already chosen to be the natural unit, the meter:

$$\frac{G}{c^2} = \frac{6.67 \times 10^{-11} \frac{\text{meter}^3}{\text{kilogram second}^2}}{8.9876 \times 10^{16} \frac{\text{meter}^2}{\text{second}^2}} \qquad (9)$$
$$= 7.42 \times 10^{-28} \frac{\text{meter}}{\text{kilogram}}$$

Now convert from mass M_{kg} measured in conventional units of kilograms to mass M in meters by multiplication with this conversion factor:

$$M \equiv \frac{G}{c^2} M_{\text{kg}} = \left(7.42 \times 10^{-28} \frac{\text{meter}}{\text{kilogram}}\right) M_{\text{kg}} \qquad (10)$$

Mass in meters unclutters equations.

Why make this conversion? Because it allows us to get rid of the symbols G and c^2 that otherwise clutter up our equations.

Table 1 displays in both kilograms and meters the masses of Earth, Sun, the huge spinning black hole at the center of our galaxy, and the mass of an even larger black hole in a nearby galaxy. For each of these objects the global r-coordinate of the event horizon is twice its mass in meters. To express their masses in meters cuts planets and stars down to size!

Objection 3. *This is nuts! Stars and planets are not the same as space. No twisting or turning on your part can make mass and distance the same. How can you possibly propose to measure mass in units of meters?*

True, mass is not the same as spatial separation. Neither is time the same as space: The separation between clock ticks is different from meterstick lengths! Nevertheless, we have learned to use the conversion factor c to measure both time and space in the same unit: light-years of spatial separation and years of time, for example, or meters of spatial coordinate separation and meters of light-travel time. Payoff? The result simplifies our equations.

There are two primary birthplaces for black holes: The first is the collapse of a single star, which produces a black hole with mass equal to a modest multiple of the mass of our Sun. The second birthplace is accumulation in a galaxy, which produces a black hole with mass equal thousands to billions of the mass of our Sun. Typically, a small galaxy contains a smaller black hole, for example 50,000 times the mass of our Sun, while a large black hole, such as the last entry in Table 1, has a mass billions of times the mass of our Sun.

Objection 4. *You are being totally inconsistent about mass! In Chapter 1 we heard about the mass m of a stone; there you said nothing about mass in units of length. Now you define M with length units. Make up your mind!*

TABLE 1 Masses of some astronomical objects.

Object	Mass in kilograms	Geometric measure of mass	Equatorial r-coordinate
Earth	5.9742×10^{24} kilograms	4.44×10^{-3} meters or 0.444 centimeters	6.371×10^{6} meters or 6371 kilometers
Sun	1.989×10^{30} kilograms	1.477×10^{3} meters or 1.477 kilometers	6.960×10^{8} meters or 696 000 kilometers
Black hole at center of our galaxy	8×10^{36} kilograms (4×10^{6} Sun masses)	6×10^{9} meters	
Black hole in galaxy NGC 4889	4.2×10^{40} kilograms (21×10^{9} Sun masses)	3.1×10^{13} meters	

Excellent point. The difference between the mass M of a center of attraction and the mass m of a stone is important. First, a stone is a "free particle . . . whose mass warps spacetime too little to be measured" (inside the back cover). Second, most often we combine the stone's mass m with another quantity in such a way that the result is a unitless ratio—for example E/m—by choosing the *same* unit in numerator and denominator. It does not matter which unit we use—joules or kilograms or electron-volts or the mass of the proton—as long as we use the *same* unit in numerator and denominator.

In contrast to the stone, the mass of a star or black hole *does* curve and warp spacetime. In this book the capital letter M *always* signals this fact. Here too we can arrange things so that M appears in a unitless ratio, such as $2M/r$, in which case M and r must have the same unit, which we choose to be meters.

Objection 5. *Okay, terrific, and this gives me a great idea: Why not simplify things even more by using unitless spacetime coordinates. Divide the Schwarzschild metric through by M^2, then define dimensionless coordinates $\tau^* \equiv \tau/M$ and $t^* \equiv t/M$ and $r^* \equiv r/M$. Here the asterisk (*) reminds us that we are using dimensionless coordinates. Now the timelike Schwarzschild metric takes the simplest possible form:*

$$d\tau^{*2} = \left(1 - \frac{2}{r^*}\right) dt^{*2} - \frac{dr^{*2}}{\left(1 - \frac{2}{r^*}\right)} - r^{*2} d\phi^2 \qquad (11)$$

(unitless coordinates)

This notation has two big advantages: First, our equations are no longer cluttered with the symbol M, just as we have already eliminated from our equations the clutter of constants G and c. Second, metric (11) applies automatically to ***all*** *black holes, of whatever mass M.*

Box 4. "Our Little Jugged Apocalypse"

We tend to think of a black hole as a large object, especially the "monster" at the center of our galaxy (Table 1). But the word *large* invites the question, "Large compared to what?" The diameter of the black hole in our galaxy is about 10^{-6} light year. Our galaxy, a typical one, is some 10^5 light years in diameter. Any object a factor 10^{-11} the size of a galaxy must be considered a relative dot in the galactic scheme of things. Its relatively small size allows us to call the black hole our "little jugged apocalypse," a phrase the writer John Updike uses to describe the view into the portal of a front-loading clothes-washing machine. Conveniently, spacetime curvature increases from zero far from the isolated black hole to an unlimited value at its singularity. This makes the black hole a useful example to teach large swaths—but not all—of general relativity.

Originally we used your idea for a few chapters, but then returned these chapters to our current notation, which has several advantages: (1) Keeping the M allows us to check units in every equation. An equation can be wrong if the units are correct, but it is *always* wrong if the units are incorrect! (2) We can return to flat spacetime and special relativity simply by letting $M \to 0$; a second useful check. (3) We prefer to be continually reminded of the concrete *heft*—the observed massiveness—of astronomical objects: stars and black holes. For these reasons we choose to retain coordinates in units of length and the explicit symbol M in our equations.

Newton's gravity with mass in meters

How does Newton's law of gravitation change when we express mass in meters? Think of a stone of mass m_{kg} near a center of attraction of mass M_{kg}. Rewrite Newton's second law of motion ($F = ma$) for this case, using the gravitational force equation (7), with $m_{\text{kg}} g_{\text{conv}}$ on the left, where g_{conv} is the local acceleration of gravity. The stone's mass m_{kg} cancels from both sides of the resulting equation. A minus sign signals that the acceleration is in the decreasing r direction.

$$g_{\text{conv}} = -\frac{GM_{\text{kg}}}{r^2} \qquad \text{(Newton, conventional units)} \qquad (12)$$

Now divide both sides of (12) by c^2 so as to obtain the conversion factor of equation (9). We can then write

$$g \equiv \frac{g_{\text{conv}}}{c^2} = -\frac{M}{r^2} \qquad \text{(Newton, mass in meters)} \qquad (13)$$

Newton's g_{Earth} with mass in meters

Remember that this is an equation of Newton's mechanics, not an equation of general relativity. The quantities M and r both have the unit meter, so g has the unit meter^{-1}. Substitute into (13) the values of M_{Earth} and r_{Earth} from inside the front cover to obtain the value for the acceleration of gravity g_{Earth} at Earth's surface in units of inverse meters:

$$g_{\text{Earth}} = -\frac{M_{\text{Earth}}}{r^2_{\text{Earth}}} = -1.09 \times 10^{-16}\ \text{meter}^{-1} \qquad \text{(Newton, mass in meters)} \qquad (14)$$

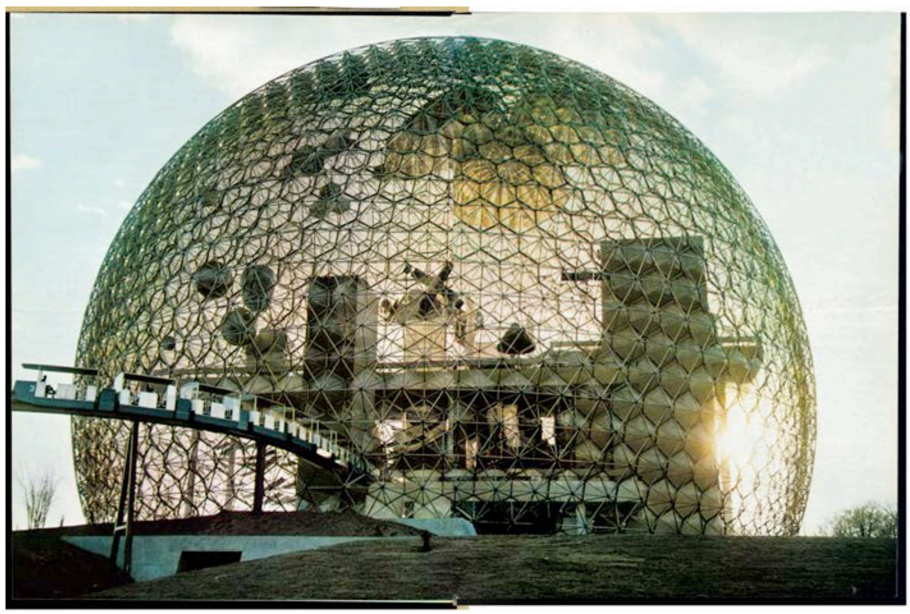

FIGURE 3 US Pavilion "geodesic dome" designed by R. Buckminster Fuller for the 1967 International and Universal Exposition in Montreal. Place a clock at every intersection of rods on the outer surface of this sphere to create a small model of our imaginary nested spherical shells concentric to a black hole. Image courtesy of the Estate of R. Buckminster Fuller.

Does this numerical value seem small? It is the same acceleration we are used to, just expressed in different units. To jump from a high place on Earth is dangerous, whatever units you use to describe your motion!

Next we continue the explanation of Schwarzschild metrics (5) and (6) with a definition of the global radial coordinate r in these equations.

3.3 ■ THE GLOBAL SCHWARZSCHILD *r*-COORDINATE

Measure the r-coordinate while avoiding the trap in the center

Why Schwarzschild global coordinates?

Section 2.5 asked, "Does the black hole care what global coordinate system we use in deriving our global spacetime metric?" and answered, "Not at all!" General relativity allows us to use *any global coordinate system whatsoever,* subject only to some requirements of smoothness and uniqueness (Section 5.9). *Next question:* Since Schwarzschild had (almost) complete freedom to choose his global coordinates t, r, and ϕ, why did he choose the particular coordinates that appear in (5) and (6)? *Next answer:* Schwarzschild's global coordinates take advantage of the spherical symmetry of a non-spinning black hole. When these coordinates are submitted to Einstein's equations, they return metrics that are (relatively!) simple. In this and the following section we introduce and describe Schwarzschild global coordinates.

Spherical shell of rods and clocks

Start with Schwarzschild's r coordinate: Take the center of attraction to be a black hole with the same mass as our Sun. In imagination, build around it a spherical shell of rods fitted together in an open mesh (Figure 3). On this shell mount a clock at every intersection of these rods. The rods and clocks of such a collection of shells provides one system of coordinates to determine the location of events that occur outside the event horizon.

We cannot measure r-coordinate directly.

How shall we define the size of the sphere formed by this latticework shell? Shall we measure directly the radial separation between the sphere's surface to its center? That won't do. Yes, in imagination we can stand on the shell. Yes, we can lower a plumb bob on a "string." But for a black hole, any string, any tape measure, any steel wire—whatever its strength—is relentlessly torn apart by the unlimited pull the black hole exerts on any object that dips close enough to its center. And even for Earth or Sun, the surface itself keeps us from lowering our plumb bob directly to the center.

Derive r-coordinate from measurement of circumference.

Therefore try another method to define the size of the spherical shell. Instead of lowering a tape measure from the shell, run a tape measure around it in a great circle. The measured distance so obtained is the *circumference* of the sphere. Divide this circumference by $2\pi = 6.283185...$ to obtain a distance that would be the directly-measured r-coordinate of the sphere *if* the space inside it were flat. But that space is *not flat,* as we shall see. Yet this procedure yields the most useful known measure of the size of the spherical shell.

The "radius" of a spherical object obtained by this method of measuring has acquired the name r-**coordinate,** because it is no genuine Euclidean radius. We call it also the **reduced circumference,** to remind us that it is derived ("reduced") from the circumference:

Definition: r-**coordinate**

$$\begin{aligned} r\text{-coordinate} &\equiv \text{reduced circumference} \\ &\equiv \frac{\text{measured circumference}}{2\pi} \end{aligned} \tag{15}$$

We sometimes use the expression Schwarzschild-r, which labels the global coordinate system of which r is a member. From now on we try not to use the word "radius" for the r-coordinate, because it can confuse results for flat spacetime with results for curved spacetime.

During construction of each shell the contractor stamps the value of its r-coordinate on it for all to see.

?

Objection 6. *Aha, gottcha! To define the r-coordinate in (15), you measure the length of the entire circumference of a spherical shell. Near a massive black hole, this circumference could be hundreds of kilometers long. Yet from the beginning you say, "Report every measurement using a local inertial frame." Near a black hole a local inertial frame is tiny compared with the length of this circumference. You do not follow your own rules for measurement.*

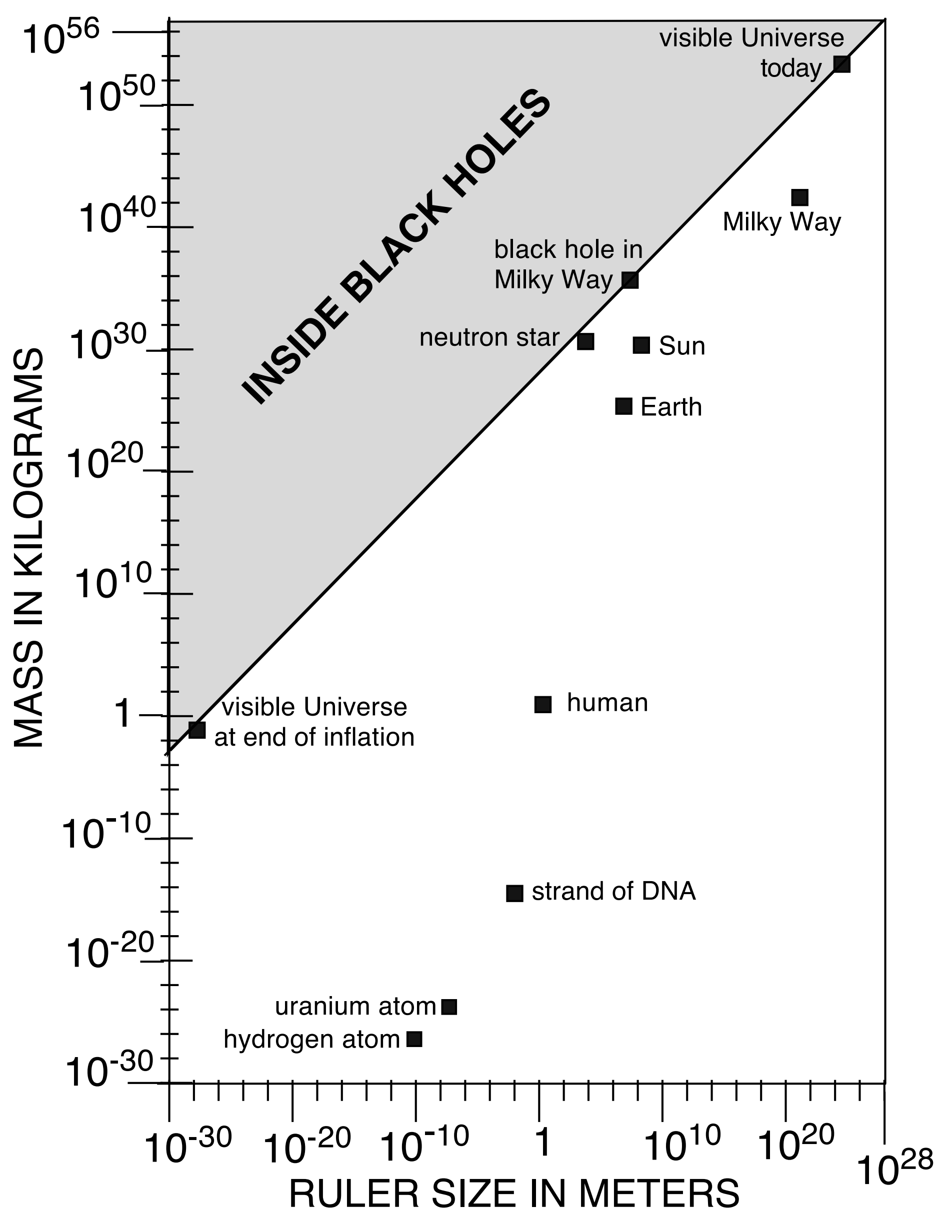

FIGURE 4 The scale of some objects described by physics. Objects close to the diagonal line are those for which correct predictions require general relativity. See Box 5. Figure adapted from the textbook *Gravity* by James Hartle.

!

Guilty as charged! We failed to spell out the process: Use a whole string of overlapping local inertial frames parked around the circumference of the

Box 5. When is General Relativity Necessary?

When is general relativity required to describe and predict accurately the behavior of structures and phenomena in our Universe? See Figure 4.

ORDINARY STAR. An ordinary star like our Sun does not require general relativity to account for its development, structure, or physical properties. Like all massive centers of attraction, however, it does deflect and focus passing light in ways accounted for by general relativity (Chapter 13).

WHITE DWARF. A white dwarf is the burned out cinder of an ordinary star, with a mass approximately equal to that of our Sun and r-coordinate of its surface comparable to that of Earth. General relativity is not required to account for the structure of the white dwarf but is needed to predict stability, especially near the so-called **Chandrasekhar limit** of mass—about 2.4 times the mass of our Sun—above which the white dwarf is doomed to collapse.

NEUTRON STAR. A neutron star can result from the collapse of a white dwarf star. Its mass is approximately that of our Sun with an r-coordinate of its surface about 10 kilometers, the size of a city. General relativity significantly affects the structure and oscillations of the neutron star. Emission of gravitational waves (Chapter 16) may damp out non-radial vibrations.

BLACK HOLE. "The physics of black holes calls on Einstein's description of gravity from beginning to end." (Misner, Thorne, and Wheeler)

GRAVITY WAVES. We have observed gravitational radiation predicted by general relativity.

THE UNIVERSE. Models of the Universe as a single structure employ general relativity (Chapters 14 and 15). It seems increasingly likely that general relativity correctly accounts for non-quantum features of the Universe, but it remains possible that general relativity fails over these immense spans of spacetime and must be replaced by a more general theory.

spherical shell, then define the circumference to be the summed measured distances across each of these local inertial frames. In practice this procedure is awkward, but in principle it avoids your otherwise valid objection.

Directly-measured separation between nested shells is **greater** than the difference in their r-values.

Think of building two concentric shells, a lower shell of reduced circumference r_L and a higher shell of reduced circumference r_H, such that the difference in reduced circumference $r_\mathrm{H} - r_\mathrm{L}$ equals 100 meters. Stand on the higher shell and lower a plumb bob, and for the first time measure directly the radial separation perpendicularly from the higher shell to the lower one. Will we measure a 100-meter radial separation between our two shells? We would if space were flat. But outside a massive body space is *not* flat. The relation between global differential dr and measured radial differential $d\sigma$ comes from the spacelike version of the Schwarzschild metric (6) with $dt = d\phi = 0$.

$$d\sigma = \frac{dr}{\left(1 - \dfrac{2M}{r}\right)^{1/2}} \qquad \text{(radial shell separation, } dt = d\phi = 0\text{)} \qquad (16)$$

We note immediately that for the radial shell separation $d\sigma$ to be a real quantity, we must have $r > 2M$; otherwise the square root in the denominator has an imaginary value. This is an indication that shells can be built only outside the event horizon (Section 6.7).

Outside the event horizon, the magnitude of the denominator on the right side of (16) is always less than one. Hence Schwarzschild geometry tells us that every radial differential increment $d\sigma$ is *greater* than the corresponding differential increment dr of the reduced circumference. Therefore the summed (integrated) measureable distance between our two shells is greater than 100 meters, even though their circumferences differ by exactly $2\pi\times100$ meters. This discrepancy between measured separation and difference in global r-coordinate provides striking evidence for the curvature of space. See Sample Problem 1.

Small effect near our Sun

Built around our Sun, the r-coordinate of the inner shell cannot be less than that of our Sun's surface, 695 980 kilometers. Around this inner shell we erect a second one—again in imagination—of r-coordinate 1 kilometer greater: 695 981 kilometers. The directly-measured distance between the two would be not 1 kilometer, but 2 millimeters *more* than 1 kilometer.

Get closer to the center.

How can we get closer to the center of a stellar object with mass equal to that of our Sun—but still remain external to the surface of that object? A white dwarf and a neutron star each has roughly the same mass as our Sun, but each is much smaller than our Sun. So we can—in principle—conduct a more sensitive test of the nonflatness of space much closer to the centers of these objects while staying external to them (Box 5). The effects of the curvature of space are much greater near the surface of a white dwarf than near the surface of our Sun—and greater still near the surface of a neutron star.

?

Objection 7. *Why not define the r-coordinate differently—call it $r_{\rm new}$—in terms of the directly-measured distance between two adjacent shells. For example, we could give the innermost shell at the event horizon the radial coordinate $r_{\rm new} = 2M$, and the next shell $r_{\rm new} = 2M + \Delta\sigma$, where $\Delta\sigma$ is the directly-measured separation between that shell and the innermost shell. And so on. That would eliminate the awkwardness of your quoted results.*

!

You can choose (almost) any global coordinate system you want, but the one you suggest is inconvenient. First, you cannot escape the deviation from Euclidean geometry imposed by curvature; your definition leads to a calculated circumference $2\pi r_{\rm new}$ that is different from the directly-measured one. Second, outside the event horizon your definition is awkward to carry out, since it requires collaboration between observers on different shells. Third, how is your definition applied inside the event horizon, where no shells exist? (Box 7 in Section 7.8 shows how to measure the Schwarzschild reduced circumference r inside the event horizon.) Finally, your definition of $r_{\rm new}$, when submitted with t and ϕ to Einstein's equations, results in a different metric—a more complicated one—which would be more inconvenient to use than the Schwarzschild global metric.

Huge effect near black hole

Turn attention now to a black hole of mass M. Close to it the departure from flatness is much larger than it is anywhere around a white dwarf or a neutron star. Construct an inner shell having an r-coordinate, a reduced circumference, of $3M$. Let an outer shell have an r-coordinate of $4M$. In contrast to these two r-coordinates, defined by measurements around the two shells, the directly-measured radial distance between the two shells is $1.542M$,

Sample Problem 1. "Space Stretching" Near a Black Hole

Here we verify the statement near the end of Section 3.3 that for a black hole of mass M, the directly-measured radial distance calculates as 1.542M between the lower shell at r-coordinate $r_{\mathrm{L}} = 3M$ and the higher shell at r-coordinate $r_{\mathrm{H}} = 4M$. In Euclidean geometry this measured distance would be $1.000M$, but not in curved space!

SOLUTION Equation (16) gives the radial differential $d\sigma$ between shells separated by a differential dr of the global radial coordinate r. The term $2M/r$ changes significantly over the range from $r = 3M$ to $r = 4M$, so our "summation" must be an integral. Integrating (16) from lower r-coordinate $r_{\mathrm{L}} = 3M$ to higher r-coordinate $r_{\mathrm{H}} = 4M$ yields:

$$\sigma = \int_{r_{\mathrm{L}}}^{r_{\mathrm{H}}} \frac{dr}{\left(1 - \dfrac{2M}{r}\right)^{1/2}} = \int_{r_{\mathrm{L}}}^{r_{\mathrm{H}}} \frac{r^{1/2}dr}{(r-2M)^{1/2}} \qquad (17)$$

This integral is not in a common table of integrals, so make the substitution $r = z^2,$ from which $dr = 2zdz$. The resulting integral has the solution:

$$\sigma = \int_{z_{\mathrm{L}}}^{z_{\mathrm{H}}} \frac{2z^2dz}{\left(z^2 - 2M\right)^{1/2}} \qquad (18)$$

$$= \left[z(z^2 - 2M)^{1/2} + 2M \ln\left|z + (z^2 - 2M)^{1/2}\right|\right]_{z_{\mathrm{L}}}^{z_{\mathrm{H}}}$$

Here the symbol ln (spelled "ell" "en") represents the natural logarithm (to the base *e*) and vertical-line brackets indicate absolute value. Substitute the values

$$z_{\mathrm{L}} = (3M)^{1/2} \quad \text{and} \quad z_{\mathrm{H}} = (4M)^{1/2}$$

and recall that for logarithms, $\ln(\mathrm{B}) - \ln(\mathrm{A}) = \ln(\mathrm{B}/\mathrm{A})$. The result is

$$\sigma = 1.542M \qquad \text{(radial, exact)} \qquad (19)$$

Here the symbol σ predicts the *exact* radial separation between these shells measured by the shell observer who uses a short ruler, say one-centimeter long, laid end to end many times to find a total measured distance. This exact result is radically different from $1.000M$ predicted by Euclid.

compared to the Euclidean-geometry figure of $1.000M$ (Sample Problem 1). At this close location, the curvature of space results in measurements quite different from anything that textbook Euclidean geometry would lead us to expect. We call this effect the **stretching of space**.

Objection 8. *WHY is the directly-measured distance between spherical shells greater than the difference in r coordinates between these shells? Is this discrepancy caused by gravitational stretching or compression of the measuring rods?*

No, the quoted result assumes rigid measuring equipment. In practice, of course, a measuring rod held by the upper end will be subject to gravitational stretching (or compression if held by the lower end). Make the rod short enough; then gravitational stretching is unimportant. Now count the number of times the rod has to be moved end to end to cross from one shell to the other.

Objection 9. *Are you refusing to answer my question? What CAUSES the discrepancy, the fact that the directly-measured distance between spherical shells is greater than the difference in r coordinates between these shells? WHY this discrepancy?*

Sample Problem 2. Our Sun Causes Small Curvature

The Schwarzschild metric function $(1 - 2M/r)$ gauges the difference between flat and curved spacetime. How far from the center of our Sun must we be before the resulting curvature becomes extremely small or negligible?

A. As a first example, find the r-coordinate from a point mass with the mass of our Sun ($M \approx 1.5 \times 10^3$ meters) such that the metric function differs from the value one by one part in a million. Compare this r-coordinate to the actual r-coordinate of the surface of our Sun ($r_{\text{Sun}} \approx 7 \times 10^8$ meters).

B. As a second example, find the radial r-coordinate from our Sun such that the metric function differs from the value one by one part in 100 million. Compare the value of this r-coordinate with the average r-coordinate of Earth's orbit ($r \approx 1.5 \times 10^{11}$ meters).

SOLUTIONS

A. We want $(1 - 2M/r) \approx 1 - 10^{-6}$, which yields

$$\begin{aligned} r \approx \frac{2M}{10^{-6}} &= 2 \times 1.5 \times 10^3 \times 10^6 \text{meters} \\ &= 3 \times 10^9 \text{meters} \end{aligned} \qquad (20)$$

This r-coordinate is approximately four times the r-coordinate of our Sun's surface.

B. In this case we want $(1 - 2M/r) \approx 1 - 10^{-8}$, so

$$\begin{aligned} r \approx \frac{2M}{10^{-8}} &= 2 \times 1.5 \times 10^3 \times 10^8 \text{ meters} \\ &= 3 \times 10^{11} \text{ meters} \end{aligned} \qquad (21)$$

which is approximately twice the r-coordinate of Earth's orbit.

A deep question! Fundamentally, this discrepancy shatters the notion of Euclidean space. We are faced with a weird measured result, which we can summarize with the statement, "Mass stretches space." Your question "Why?" is not a scientific question, and science cannot answer it. We know only observed results and their derivation from general relativity. Does the following satisfy you? *Space stretching causes the discrepancy!* Section 3.8 exhibits one way to visualize this stretching.

3.4 ■ THE GLOBAL SCHWARZSCHILD *t*-COORDINATE

Freeze global space coordinates; examine the warped t-coordinate.

To describe orbits, we need curvature of spaceTIME.

It is not enough to know the results of curvature on the r-coordinate alone. To appreciate how the grip of spacetime tells planets how to move requires us to understand how curvature affects the global t-coordinate as well. The coordinate differential dt appears on the right side of the Schwarzschild metric. Basically, Schwarzschild's definition of the t-coordinate was arbitrary, like the definition of every global coordinate.

Relation between $d\tau$ and dt

How does Schwarzschild coordinate differential dt relate to the differential wristwatch time $d\tau$ between two successive events that occur at at fixed r- and ϕ-coordinates? The coordinate differentials dr and $d\phi$ are both equal to zero for that pair of events. Then the interval between ticks is the wristwatch time derived from metric (5), that is:

$$d\tau = \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad \text{(stationary clock: } dr = d\phi = 0) \qquad (22)$$

Equation (22) shows that far from a black hole ($r \rightarrow \infty$), Schwarzschild-t coincides with the time of a shell clock located there. This is an important, but accidental, convenience of Schwarzschild's choice of global t-coordinate. It is not true for the metrics of many other global coordinate systems for the non-spinning black hole.

Slogan: "A clock at smaller r runs slower."

Now look at the prediction of equation (22) closer to a black hole—but still outside the event horizon. There the Schwarzschild coordinate differential dt will be *larger* than the differential wristwatch time $d\tau$ measured by a clock at rest on the shell at that r-coordinate. Smaller wristwatch time $d\tau$ between two standard events leads to the useful but somewhat imprecise slogan, *A clock closer to a center of attraction runs slower* (see Section 4.3).

Schwarzschild: complete description

We have now carefully defined each of the Schwarzschild global coordinates and displayed the resulting global metric handed to us by Einstein's equations, including the range of global coordinates given in equations (5) and (6). This combination—plus its connectedness (topology)–provides a *complete* description of spacetime near the isolated non-spinning black hole. These tools alone are sufficient to determine every (classical, that is non-quantum) observable property of spacetime in this region.

?

Objection 10. *Hold it! You gave us separate Sections 3.3 and 3.4 on two global coordinates, Schwarzschild-r and Schwarzschild-t, respectively. Why no section on the third global coordinate, Schwarzschild-ϕ?*

!

Good question. In answer, compare metric (4) for flat spacetime in Box 1 with the Schwarzschild metric (5) for curved spacetime. The last term is the same in both equations: $-r^2 d\phi^2$. Typical in relativity, the t-coordinate gives us the most trouble and the r-coordinate less trouble. In the non-spinning black hole metrics used in this book, the angle ϕ gives no trouble at all, due to the angular symmetry. For the spinning black hole (Chapters 17 through 21), however, even this angle becomes a troublemaker!

3.5 ■ CONSTRUCTING THE GLOBAL SCHWARZSCHILD MAP OF EVENTS

Read a road map, but don't drive on it!

"Think globally; measure locally."

In this book we choose to make every measurement and observation in a local inertial frame. But that does not suffice to describe the relation between events far from one another in the vicinity of the black hole. Suppose we know the stone's energy and momentum measured in one local inertial frame through which it passes. How can we predict the stone's energy and momentum in a second local inertial frame far from the first?

This prediction requires (a) knowledge of the stone's initial location in global coordinates, (b) analysis of the global worldline of the stone between widely-separated local frames, and (c) conversion of a piece of the global trajectory to local inertial coordinates in the remote inertial frame. This section begins that process, which we summarize with the slogan *"Think globally, measure locally."*

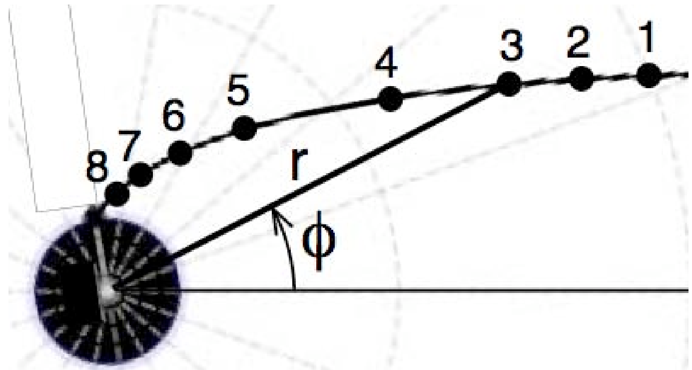

FIGURE 5 *Schwarzschild map* of the trajectory of a free stone that falls into a black hole. As it falls, it emits (numbered) flashes equally separated in time on its wristwatch. However, these flash emissions are not equally spaced along the Schwarzschild map trajectory. Each numbered event also has its Schwarzschild-t. NO ONE observes directly the entire trajectory shown on this map. *Question:* Why are numbered emission events closer together near both ends of the trajectory than in the middle of the trajectory? The answer for events 1 through 3 should be simple. The answer for events 5 through 8 appears in Section 6.5.

Global Schwarzschild coordinates locate events around a black hole similar to the way in which latitude and longitude locate places on Earth's surface (Section 2.3). A global map of Earth is nothing but a rule that assigns unique coordinates to each *point* on its surface.

The **spacetime map** assigns coordinates to every event.

By analogy, we speak of a **spacetime map**, which is nothing but a rule that assigns unique coordinates to each *event* in the region described by that map. This section describes the construction and uses of the Schwarzschild spacetime map, a task that we personalize as the work of an archivist.

Schwarzschild mapmaker

Think of Schwarzschild coordinates as an accounting system, a bookkeeping device, a spreadsheet, a tabulating mechanism, an international language, a space-and-time database created by an archivist who records every event and all motions in the entire spacetime region exterior to the surface of the Earth or Moon or Sun—or anywhere around a black hole except exactly at its center. We personify the supervisor of this record as the **Schwarzschild mapmaker**. The Schwarzschild mapmaker receives reports of actual measurements made by local shell and other inertial observers, then converts and combines them into a comprehensive description of events (in Schwarzschild coordinates) that spans spacetime around a black hole. The mapmaker makes no measurements himself and does not analyze measurements. He is a data-handler, pure and simple.

Mapmaker: the central coordinator.

The Schwarzschild mapmaker (or his equivalent) is absolutely necessary for a complete description of the motion of stones and light signals around a black hole. He has the central coordinating role in describing globally all the events that take place outside the event horizon of the black hole. He collates

Box 6. The Metric as Spacetime Micrometer

FIGURE 6 The micrometer caliper measures directly a tiny distance or thickness, bypassing x and y coordinates. The watch measures directly the invariant wristwatch time between two events, bypassing separate global coordinate increments. (Photo by Per Torphammar.)

What *is* the metric? What is it *good for*? Think of a **micrometer caliper** (Figure 6), a device used by metalworkers and other practical workers to measure a small distance. The micrometer caliper translates turns of a calibrated screw on the cylinder into the directly-measured distance across the gap between the flat ends of the little cylinders in the upper right corner of the figure. The worker *owns* the micrometer; the worker *chooses* which distance to measure with the micrometer caliper.

The metric is our "four-dimensional micrometer" that translates global coordinate separations between an adjacent pair of events into the measurable wristwatch time lapse or ruler distance between those events. You *own* the metric. You *choose* the events whose separation you wish to measure with the metric.

1. **One possible choice:** Two sequential ticks of a clock bolted to a spherical shell. Then $dr = d\phi = 0$ and the wristwatch time $d\tau$ is the time lapse read directly on the shell clock.
2. **A second possible choice:** Events with the same global t-coordinate that occur at the two ends of a stick held at rest radially between two adjacent shells, so that $dt = d\phi = 0$. Then the ruler distance $d\sigma$ is the directly-measured length of the stick—equation (16).
3. **A third possible choice:** Two sequential ticks on the wristwatch of a stone in free fall along a radial trajectory. Then $d\phi = 0$ and $d\tau$ is read directly on the wristwatch.

And so on. There are an infinite number of event-pairs near one another that you can choose for measurement using your four-dimensional micrometer—the metric.

Assembling many micrometer caliper measurements can in principle describe the geometry of space. Assembling many wristwatch and ruler measurements can in principle describe the geometry of spacetime: "The metric completely specifies local spacetime and gravitational effects within the global region in which it applies." (Inside back cover.)

What advice will the "old spacetime machinist" give to her younger colleague about the practical use of the metric? She might share the following pointers:

1. Focus on *events* and the separation between each pair of events, not fuzzy concepts like "time" or "location."
2. Do not confuse results from one pair of events with results from another pair of events.
3. Whenever possible, choose two adjacent events for which the increment of one or more map coordinates is zero.
4. Whenever possible, identify the wristwatch time or ruler distance with some observer's direct measurement.
5. When a light flash moves directly from one event to another event, the wristwatch time *and* the ruler distance between those events are both zero: $d\tau = d\sigma = 0$.

data from many local observers and combines them in various ways, for example drawing a global map such as the one plotted in Figure 5.

The Schwarzschild mapmaker can be located anywhere. How does he learn of events in his dominion? Like a taxi dispatcher, he uses radio to keep track of moving stones, light flashes, and in addition locates explosions and other events of interest, perhaps as follows:

Stamped on each spherical shell is its map r-coordinate; we mark different locations around the shell with different values of ϕ. At each location place a recording clock that reads the Schwarzschild-t (Box 6). Each clock radios to

Box 7. Where does the event horizon come from?

The event horizon—that one-way spacetime surface that lets light and stones pass inward but forbids them to cross outward—is a surprise. Who could have predicted it? Answer: Nobody did.

Newton readily predicts the gravitational consequences of a point mass, telling us immediately the initial acceleration of a stone released from rest at any r-coordinate. Twice the attracting mass, twice the stone's acceleration at that r-value; a million times the attracting mass, a million times the stone's acceleration. Newton's theory of gravity is *linear* in mass.

Not so for Einstein's general relativity, which is relentlessly *nonlinear*. In general relativity not only mass but also energy and pressure curve spacetime. A star of twice the mass typically has increased internal pressure, resulting in more than twice the gravitational effects at the same r-coordinate outside its surface. For an ordinary star the added effect of pressure is negligible; for a neutron star the added effect of pressure is important; for a black hole the added effect of pressure is catastrophic.

When a neutron star, for example, steals mass from a normal companion star, the pressure near its center increases, along with the added matter. The net result is greater than that due to the added matter alone. At a certain point, this process "runs away," resulting in collapse into a black hole.

Linear effects mean proportional response in phenomena. Nonlinear effects lead to entirely new phenomena. For the non-spinning black hole, a major outcome of nonlinearity is the event horizon. Near to the *spinning* black hole (Chapters 17 through 21), the nonlinearity of Einstein's theory leads to an even more complex geometry of spacetime and consequent radical, unexpected physical effects.

Mapmaker: top level, bureaucrat

the mapmaker the nature of an event that occurs next to it, along with its global coordinates (t, r, ϕ). After inevitable transmission delays due to the finite speed of light, the mapmaker at the control center assembles a global Schwarzschild map that gives coordinates and description of every measurement and observation. Our mapmaker acts as a top-level bureaucrat.

Using the Schwarzschild map

No one lives on a road map, but we use it to describe the territory and to plan our trip. Similarly, coordinates r, ϕ, and t are simply labels on a spacetime map. These coordinates uniquely locate events in the entire spacetime region outside the surface of any spherically symmetric gravitating body or anywhere around a black hole except on its singularity. The Schwarzschild map guides our navigation near a black hole, in the same way that an arbitrary set of global coordinates—made into maps—guides our travels on Earth's surface.

Map coordinate difference $\neq$ measured length or time lapse.

But never forget: In most cases Schwarzschild map coordinate separations are *not* what any local inertial observer measures directly.

Advice: It is best never to confuse a global map coordinate separation with the local inertial frame measurement of a distance or time lapse. More details in Chapter 5.

Objection 11. *Stop giving me second-hand ideas! I want* ***reality.*** *Your concept of a Schwarzschild map is nothing but an analogy to the inevitable distortions in geography when Earth's spherical surface is squashed onto a flat map. Where is the true representation of curved spacetime, corresponding to the true spherical map of Earth's surface?*

Early in the history of sea travel, mapmakers thought the world was flat. An ancient sea captain acquainted with Euclid's plane geometry (and also the much later calculus differential notation of Leibniz!) would puzzle over the metric for differential distance ds on Earth's surface, equation (3) in Section 2.3:

$$ds^2 = R^2 \cos^2 \lambda \, d\phi^2 + R^2 d\lambda^2 \qquad \text{(space metric: Earth's surface)} \quad (23)$$

The ancient sea captain asks, "What is R?" (r-coordinate of the Earth's surface). "What are λ and ϕ?" (angles of latitude and longitude). "Why does differential distance ds depend on latitude λ?" (convergence at the poles of lines of constant longitude). "Where is the edge?" (There is no edge.) Who is responsible for the captain's perplexity about a curved surface? Not Nature; not Mother Earth. Neither is Nature responsible for our perplexity about curved spacetime. Everything will be crystal clear as soon as we can visualize four-dimensional curved spacetime. But we do not know anyone who can do this; *we* certainly cannot! So we compromise, we do our best to live with our limitations and to develop intuition from the analogy to curved surfaces in space, such as the partial visualization of Schwarzschild geometry in the following sections.

Black holes just didn't "smell right"

During the 1920s and into the 1930s, the world's most renowned experts on general relativity were Albert Einstein and the British astrophysicist Arthur Eddington. Others understood relativity, but Einstein and Eddington set the intellectual tone of the subject. And, while a few others were willing to take black holes seriously, Einstein and Eddington were not. Black holes just didn't "smell right"; they were outrageously bizarre; they violated Einstein's and Eddington's intuitions about how our Universe ought to behave . . . We are so accustomed to the idea of black holes today that it is hard not to ask, "How could Einstein be so dumb? How could he leave out the very thing, implosion, that makes black holes?" Such a reaction displays our ignorance of the mindset of nearly everybody in the 1920s and 1930s . . . Nobody realized that a sufficiently compact object must *implode, and that the implosion will produce a black hole.*

—Kip Thorne

3.6 ■ THE SPACETIME SLICE

Do the best we can to visualize curved spacetime

This section introduces a method of visualizing curved spacetime—called the **spacetime slice**—that we use repeatedly throughout the book. Every such visualization of curved spacetime is partial and incomplete—it does not tell all!—but can carry us some of the way toward intuitive understanding of spacetime curvature.

DEFINITION 2. Spacetime slice

A **spacetime slice**—which we usually just call a **slice**—is a two-dimensional spacetime surface on which we plot two global coordinates of all events that lie on that surface and that have equal

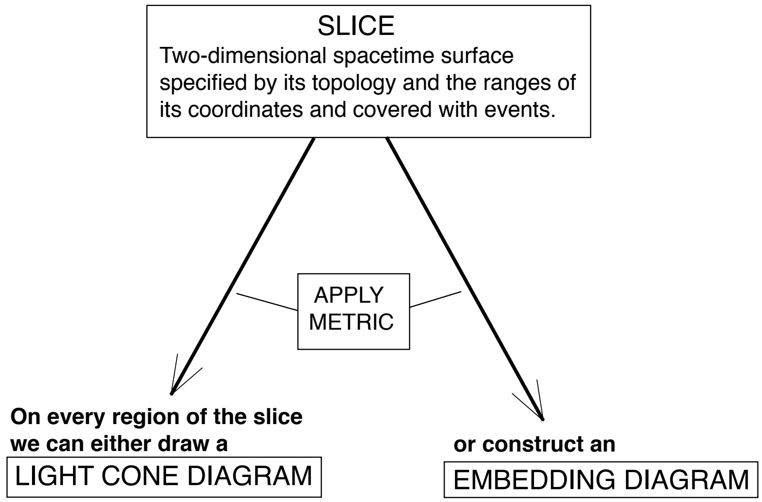

FIGURE 7 *Preview:* When we apply the global metric to a slice, then on every region of the slice we can either draw a light cone diagram or construct an embedding diagram.

Definition: **spacetime slice**

values for all other global coordinates. We indicate a slice with square brackets; the three alternative slices for our Schwarzschild global coordinates are $[r, \phi]$, $[r, t]$, and $[\phi, t]$. Our definition of *slice* includes its range of coordinates and its connectedness (topology). The slice—even when populated with events—does not use the metric, so *a spacetime slice carries no information whatsoever about spacetime curvature.* This feature makes the slice useful in both special and general relativity.

On every region of every slice: light cone diagram or embedding diagram

The following remarkable property of the spacetime slice will illuminate the remainder of this book: *When we apply the global metric to a spacetime slice, then on every region of every slice we can either draw worldlines or set up an embedding diagram.* Figure 7 previews the content of the following sections.

What does "every region" of the slice mean in the caption to Figure 7? For the non-spinning black hole the regions are outside and inside the event horizon. Section 3.7 shows that light cones can be drawn on both regions for the $[r, t]$ slice. Section 3.9 shows that outside the event horizon the $[r, \phi]$ slice is an embedding diagram.

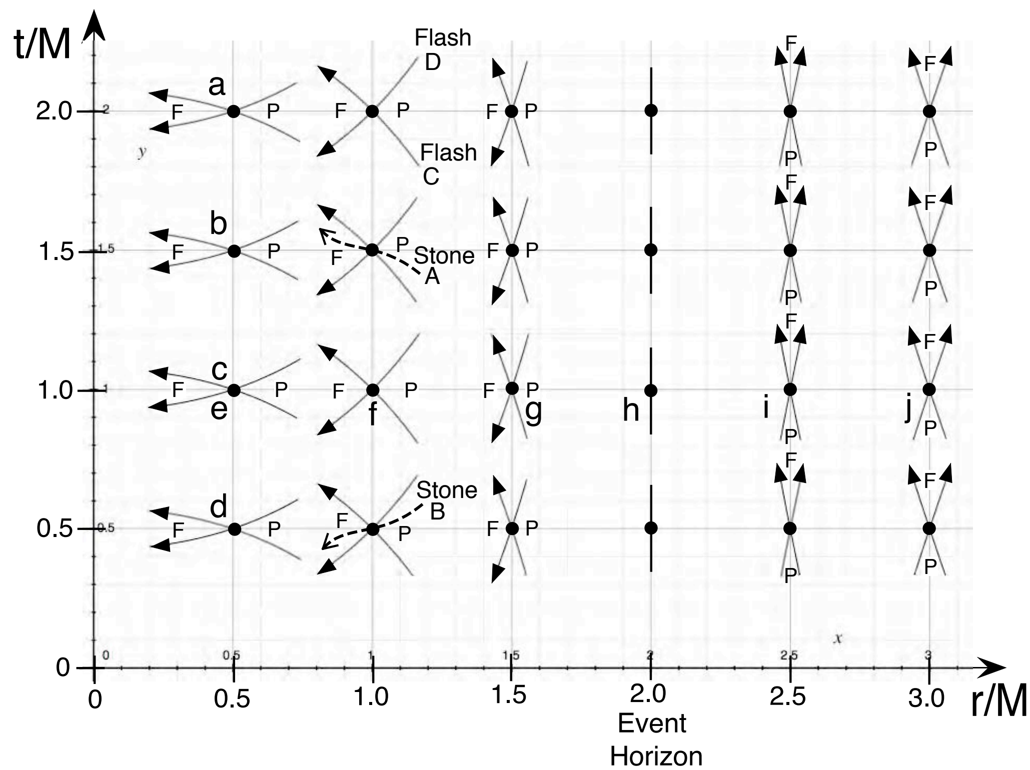

FIGURE 8 Schwarzschild light cone diagram on an $[r, t]$ slice, constructed from segments of light worldlines from equation (26), showing future (F) and past (P) of each event (filled dots). At each r-coordinate the light cone can be moved up or down vertically without change of shape, as shown.

3.7 ■ LIGHT CONE DIAGRAM ON AN [r,t] SLICE

The global t-coordinate can run backward along a worldline!

On an $[r, t]$ slice. . .

We can learn a lot about predictions of the Schwarzschild metric by plotting light cones. To derive the worldline of a light flash in r, t coordinates, set $d\tau = 0$ and $d\phi = 0$ in (5). The result is:

$$0 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \qquad \text{(light, and } d\phi = 0\text{)} \qquad (24)$$

Which leads to the equation:

$$\frac{dt}{dr} = \pm \frac{r}{r - 2M} \qquad \text{(light, radial motion)} \qquad (25)$$

Light cones

Integrate this to find the equation for the worldline of a light flash:

Box 8. A White Hole?

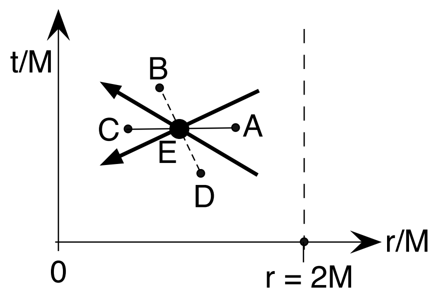

FIGURE 9 Schematic of a light cone inside the event horizon in Schwarzschild global coordinates.

Inside the event horizon, do a stone and light flash really move only toward smaller r? And does Figure 8 correctly represent this? Why do the light cones not open upward in this figure, as they do in flat spacetime and also outside the event horizon?

To answer these questions, assume that the worldline of the stone passes through event E, the intersection of the light cone worldlines in Figure 9. Then determine what worldlines through E are possible between A and C (solid line) or between D and B (dashed line). The metric tells us how the stone's wristwatch advances along its constant-ϕ worldline, From (24), it reads

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \quad (27)$$

Wristwatch time in (27) is real, therefore physical, only if the right side is positive. You can show that along a worldline connecting events D and B (dashed line), the wristwatch time is imaginary. In contrast, you can show that along a worldline that connects events A and C (solid line), wristwatch time is real. *First conclusion:* Worldlines of stones that pass through event E can pass only from either the A region to the C region or from the C region to the A region. No stone worldline through event E can connect events B and D.

Next question: In which direction does the stone move between events A and C inside the event horizon? Arrows on the light cone imply that the motion is from A to C, namely to smaller r. But all differentials in (27) are squared: The metric allows motion in either direction.

We now show that motion to larger r cannot occur inside the event horizon. This means that the solution of the metric that allows motion to larger r inside the event horizon is an extraneous solution and does not correspond to the workings of Nature.

Suppose that the stone moves to larger r, from event C to event A, in which case the light cone arrows in Figure 9 would point to the right. That means that at an earlier wristwatch time the stone was at C. Now draw a light cone that crosses at event C. Then there is a still earlier event to the left of C through which the stone passed. Repeat this process until we reach $r = 0$, from which this stone must have emerged. The result is what we call a **white hole**. A white hole spews stones and light outward from its singularity, the opposite of a black hole.

Do white holes exist in Nature? We have not detected any. And if they should temporarily form, how could they possibly survive, since their central feature is to empty themselves into surrounding spacetime? The method we use here is called *reductio ad absurdum*, reduction to an absurd result.

Final conclusion: Arrows on the light cones inside the event horizon in Figure 9 point in the physically correct direction, which funnels stones and light toward the singularity. The corresponding light cones in Figure 8 do the same.

$$t - t_1 = \pm\left(r - r_1 + 2M \ln\left|\frac{r/M - 2}{r_1/M - 2}\right|\right) \qquad \text{(light, radial motion)} \quad (26)$$

where (r_1, t_1) are initial coordinates of the light flash. Figure 8 plots the resulting light cone diagram for many different values of (r_1, t_1).

Trouble at the event horizon

Figure 8 tells us a lot about physical predictions of the Schwarzschild metric. The light cone of an event tells us the past (P) and future (F) of that event. Note, first, that at the event horizon light does not change r-coordinate on this slice. Second, inside the event horizon everything moves to smaller r. The light cone corrals possible worldlines of a stone that passes through that event—such as worldlines for Stone A and Stone B in the plot. Note, third, that the t-coordinate runs backward along worldlines B and D.

Objection 12. *How can light be stuck at the event horizon, moving neither inward or outward?*

Figure 8 tells us that near the event horizon the t-coordinate changes very rapidly along a light ray, while the r coordinate changes very little. This is a problem with the Schwarzschild t coordinate that obscures observed results. We can say that the Schwarzschild t-coordinate is *diseased*, does not correctly predict observations. Chapters 6 and 7 analyze and overcome this global coordinate difficulty and show that light can fall to smaller r, but not move to larger r inside the event horizon.

Objection 13. *Oops! How can time run backward along a worldline, such as that of Stone B in Figure 8? Its arrow tends downward with respect to the t/M axis.*

Careful! Never use the word "time" by itself (Section 2.7). Only the global t-coordinate runs backward along worldlines B and D in Figure 9. Global coordinates are (almost) totally arbitrary; we choose them freely, so we cannot trust them to tell us what we will observe. Only the left side of the metric does that, for example giving us wristwatch time between two events. The wristwatch time is positive as the stone progresses along worldline B in Figure 8; and along the worldline of *every* light flash the wristwatch time is zero. Box 8 shows that the motion of both light and stones must be to smaller r inside the event horizon.

Objection 14. *Aha! I've caught you in a serious contradiction. Inside the horizon the worldline of the stone in Figure 8 is flatter than that of light. That is, the stone traverses a greater span of r coordinate per unit time than light does. The stone moves faster than light! Let's see you wiggle out of that one!*

Again you use the word "time" incorrectly and compound the error by changing r rather than moving a distance. Global coordinates are arbitrary—our choice!—and global coordinate separations are not measured quantities. This arbitrariness combines with spacetime curvature to create the distortions plotted in Figure 8. Different global coordinates give different distortions—see the same plot with different global coordinates in Figure 5, Section 7.6. For *every* global coordinate system dr/dt inside the event horizon does not *measure* the velocity of anything. We favor measurement and observation on a local flat patch, where special relativity rules. Chapter 5 has a lot more on this subject.

3.8 ■ INSIDE THE EVENT HORIZON: A LIGHT CONE DIAGRAM ON AN [r,ϕ] SLICE

Inside the event horizon, Schwarzschild-r is timelike!

On an $[r,\phi]$ slice. . .

To continue our attempt to visualize curved spacetime around a black hole, we plot light cones on an $[r,\phi]$ slice. Light plots on this slice require that $d\tau = 0$ and $dt = 0$. With these conditions, (5) becomes

$$0 = -\left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \qquad \text{(light, and } dt = 0) \tag{28}$$

So the trajectory of light on the $[r,\phi]$ slice satisfies the equation:

$$\frac{d\phi}{dr} = \pm\frac{1}{r^{1/2}(2M - r)^{1/2}} \qquad \text{(light, } dt = 0) \tag{29}$$

Light cones inside the event horizon

The left side of (29) is real only if $r \leq 2M$, namely at or inside the event horizon. Whoops: *The only region on the* $[r,\phi]$ *slice on which we can draw worldlines is inside the event horizon.* So what is going on *outside* the event horizon? Section 3.9 answers this question; here we plot light cones on the $[r,\phi]$ slice inside the event horizon. To integrate (29), use the substitution:

$$r = 2Mz^2 \qquad \text{so} \qquad dr = 4Mzdz \tag{30}$$

With this substitution, (29) becomes:

$$\frac{d\phi}{dz} = \pm\frac{4z}{(2z^2)^{1/2}(2 - 2z^2)^{1/2}} = \pm\frac{2}{(1 - z^2)^{1/2}} \qquad \text{(light, } dt = 0) \tag{31}$$

Integrate this to obtain:

$$\phi - \phi_1 = \pm 2\int_{z_1}^{z} \frac{dz}{(1 - z^2)^{1/2}} = \pm 2\left[\arcsin z - \arcsin z_1\right] \tag{32}$$

Substitute back from (30) to yield the integral of (29):

$$\phi - \phi_1 = \pm 2\left[\arcsin\left(\frac{r}{2M}\right)^{1/2} - \arcsin\left(\frac{r_1}{2M}\right)^{1/2}\right] \tag{33}$$

$$\text{(light, } 0 < r \leq 2M,\ 0 \leq \phi < 2\pi)$$

Light cones sprout from events at the filled dots (r_1, ϕ_1) in Figure 10. Equation (33) does not give real results for $r > 2M$. However, as r approaches $r_1 = 2M$ from below, the magnitude of the slopes of $d\phi/dr$ in (29) increases without limit, leading to the vertical lines at $r = 2M$ in the figure.

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Objection 15. *Wait a minute! I thought we could draw light cones only on a diagram with one space axis and one time axis. Figure 10 plots light cones using two space coordinates,* r *and* ϕ*!*

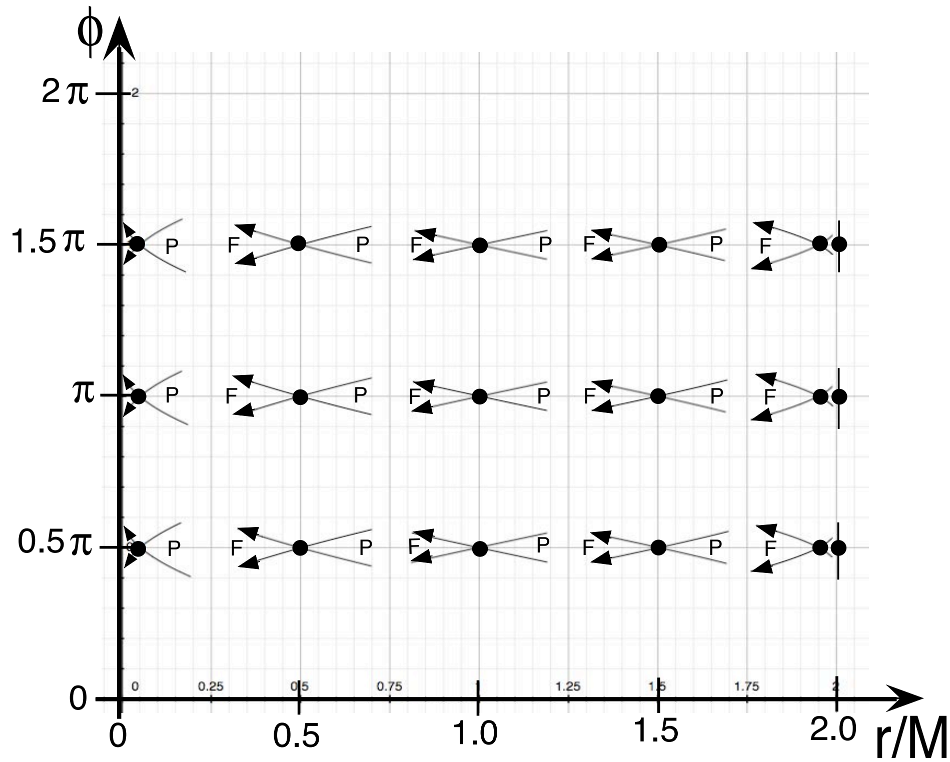

FIGURE 10 Light cones for different events (filled dots) on an $[r, \phi]$ slice inside and at the event horizon, showing the past (P) and future (F) of each event. Each light cone can be moved vertically, as shown. At $r = 2M$ the light moves neither inward nor outward, hence the vertical line. Because of the cyclic nature of ϕ, namely $\phi + 2\pi = \phi$, this diagram can be rolled up as a cylinder, on which the $\phi = 0$ axis and the $\phi = 2\pi$ line coincide.

Never assume that global coordinate separations in t, r, or ϕ tell us anything about space and time *measurements*. We favor measurement in a local inertial frame, using local coordinates—*not* global coordinates. Later we show that inside the event horizon the Schwarzschild r coordinate behaves like a time (and the Schwarzschild t coordinate behaves like a distance). So Figure 10 does describe the motion of light. The light cones in the figure fulfill one of their basic functions: For each event they divide spacetime into the past (P), the future (F), and the absolute elsewhere.

3.9 ■ OUTSIDE THE EVENT HORIZON: AN EMBEDDING DIAGRAM ON AN [r,ϕ] SLICE

Freeze Schwarzschild-t; examine stretched space.

On an $[r,\phi]$ slice: embedding diagram outside the event horizon

Equation (29) tells us that we cannot draw light cones on the $[r,\phi]$ slice outside the event horizon. Figure 7 predicts an alternative way to visualize curved spacetime: an **embedding diagram**. Figure 12 shows the world's most famous embedding diagram, the funnel whose form we now explain and derive.

We add a third dimension.

Think of the $[r,\phi]$ slice outside of the event horizon as an initially horizontal rubber sheet. Here's how we create the embedding diagram: Anchor a ring at $r = 2M$ on the original flat slice, then for $r > 2M$ pull the rubber sheet upward, perpendicular to that flat surface, in such a way that the curve with $d\phi = 0$, called $Z(r)$, satisfies the equation

$$d\sigma^2 = \frac{dr^2}{\left(1 - \dfrac{2M}{r}\right)} \qquad \text{(embedded surface profile)} \qquad (34)$$

Figure 11 illustrates the resulting construction. From this figure:

$$d\sigma^2 = dZ^2 + dr^2 \qquad (35)$$

From equations (34) and (35):

$$dZ^2 = d\sigma^2 - dr^2 = \frac{2M}{r - 2M}dr^2 \qquad (36)$$

Take the square root of both sides of (36) and integrate the result from the lower limit at $r = 2M$:

$$Z(r) = \pm\,(2M)^{1/2}\int_{2M}^{r}\frac{dr}{(r-2M)^{1/2}} = 2^{3/2}M^{1/2}(r-2M)^{1/2} \qquad (37)$$

Parabaloid funnel

We choose the plus sign for the final expression on the right of (37) for convenience of drawing. Square both sides of (37) to obtain an equation of the form $Z^2 = Ar + B$; this shows that the funnel profile is a parabola. Rotate this curve around the vertical line $r = 0$ to create the surfaces in Figures 12 and 13. This funnel surface, with its parabola profile, is called a **paraboloid of revolution**. It is sometimes called a **gravity well** or **Flamm's paraboloid** after Ludwig Flamm, the first to identify it in 1916.

Spacetime only on funnel surface

The vertical dimension in Figures 11, 12, and 13 is an artificial construct; it is *not* a dimension of spacetime. *We ourselves added this third Euclidean space dimension to help visualize Schwarzschild geometry.* Only the embedded surface represents physical spacetime where objects and people can exist. An observer posted on this paraboloidal surface is bound to stay on that surface, not because he is physically limited in any way, but because locations off the surface in these diagrams simply do not exist in physical spacetime.

The embedding diagram in Figure 13 illustrates some analytical results derived earlier in this chapter. For example:

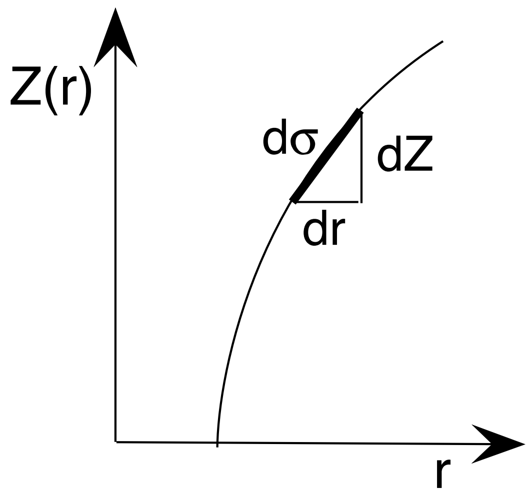

FIGURE 11 Constructing the radial profile of the funnel in Figures 12 and 13.

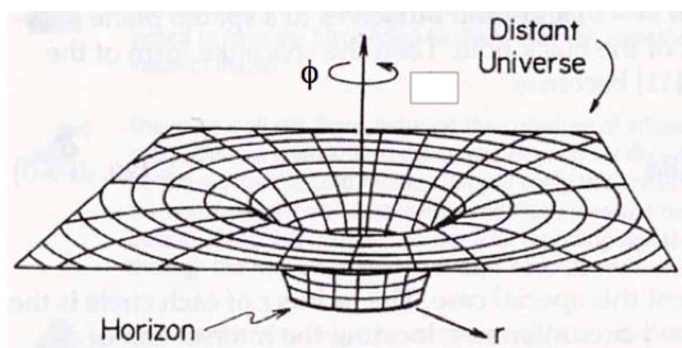

FIGURE 12 Space geometry visualized by distorting a slice through the center of a black hole, the result “embedded” in a three-dimensional Euclidean perspective. Adjacent circles represent adjacent shells. WE add the vertical dimension to show that the radial differential distance $d\sigma$ is greater than the differential dr (see Figure 13). Space stretching appears as a “bending” of the plane downwards into the shape of a funnel. At the throat of the funnel, where its slope is vertical, the r-coordinate is $r = 2M$.

1. Along the radial direction, $d\sigma$ is greater than dr, as equation (35) implies and Figure 12 illustrates.
2. The ratio $d\sigma/dr$ increases without limit as the radial coordinate decreases toward the critical value $r = 2M$ (vertical slope of the paraboloid at the throat of the funnel).

Picturing analytical results

3. The observer constrained to the paraboloid surface cannot directly measure the r-coordinate of any shell. He derives this r-coordinate—the “reduced circumference”—indirectly by measuring the circumference of the shell and dividing this circumference by 2π (Section 3.3).

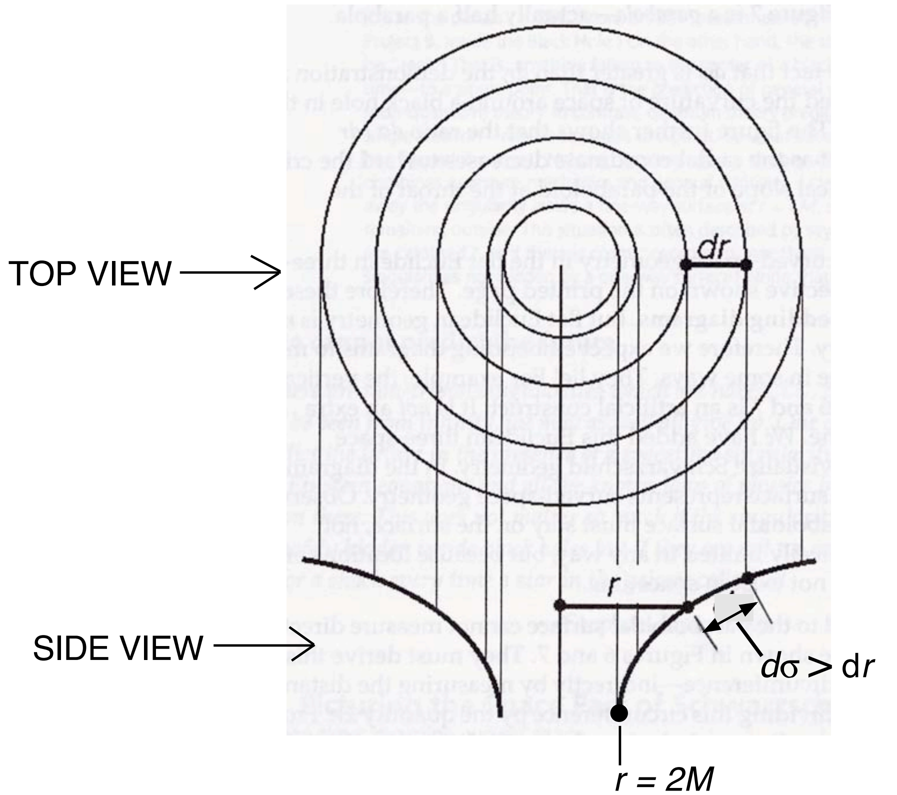

FIGURE 13 Projections of the embedding diagram of Figure 12. The thick curves in the side view are parabolas. WE choose the vertical coordinate for these curves in such a way that the increment along a parabola corresponds to the radial increment $d\sigma$ measured directly by the shell observer. A shell observer can exist only on the paraboloidal surface (shown edge-on as the thick curve). He can measure $d\sigma$ directly but not r or dr. He derives the r coordinate (*"reduced circumference"*) of a given circle by measuring its circumference and dividing by 2π. Then dr is the *computed* difference between the reduced circumferences of adjacent circles; *no* shell observer measures dr directly.

4. In contrast, the observer *can* measure the distance—call it $\sigma_{1,2}$—between adjacent shells. He finds that this directly-measured distance is greater than the difference of their r-coordinates: $\sigma_{1,2} > r_2 - r_1$.

QUERY 2. Spacelike relation of adjacent events on an embedded surface

A. Explain how on an embedded surface every adjacent pair of events—separated by differential global coordinates—has a spacelike relation to each other.

B. Argue that the answer to the question, "Can a worldline (Definition 9, Section 1.5) lie on an embedding diagram?" is a resounding "NO!"

In Query 1 you show that every pair of adjacent events on an embedded surface has a spacelike relation to one another ($d\sigma^2 > 0$). In contrast, a stone *must* move between timelike events along its worldline ($d\tau^2 > 0$). Therefore a stone *cannot* move on an embedded surface. Even light—which moves along a lightlike trajectory ($d\tau = 0$)—cannot move on an embedded surface. Hence an embedding diagram cannot display motion at all.

Adjacent events on an embedding diagram have a spacelike relation.

Objection 16. *In a science museum I see steel balls rolling around in a metal funnel. Is this the same as the funnel in Figure 13?*

No. The motion of these balls approximate Newtonian orbits provided the depth at each funnel radius is proportional to the inverse of the radius, which mimics the Newtonian potential energy. This is unrelated to the general relativistic distortion of space near a center of gravitational attraction. The cross section curve in Figure 13 is a parabola.

Comment 2. Terminology: "Except on the singularity."
Neither the Schwarzschild metric, nor any other global metric we use, is valid on the singularity of a black hole. On a singularity, by definition, spacetime curvature increases without limit, so general relativity is not valid there. In all the global coordinates we use, the non-spinning black hole has a point singularity. The spinning black hole has a ring singularity in our global coordinates (Chapter 18). We authors get tired of using—and you get tired of reading—the steady refrain "except at the singularity." So from now on that idea will mostly "go without saying." We will repeat the phrase occasionally, as a reminder, but—please!—mentally insert the phrase "except at the singularity" into every discussion of global coordinates around a black hole.

Objection 17. *So in summary, the space outside the event horizon of the non-spinning black hole has the shape of a funnel, right? I certainly see that funnel in textbooks and popular articles about general relativity.*

Here is the correct statement: "The global metric in Schwarzschild coordinates leads to a funnel embedding diagram for $r > 2M$." *Notice:* This statement describes a consequence of using Schwarzschild global coordinates. But it is not the consequence in *every* global coordinate system. Chapter 7 introduces a global coordinate system —Painlevé-Gullstrand (which we call global rain coordinates)—whose global metric leads to an embedding diagram that is **flat everywhere**, inside as well as outside the event horizon (Box 5, Section 7.6). The key idea here is that **curvature is a property of spacetime**, not of either global space coordinates alone or the global t-coordinate alone. Light cone plots and embedding diagrams help us to visualize features of curved spacetime, but no single diagram fully represents curved **spacetime.** Sorry!

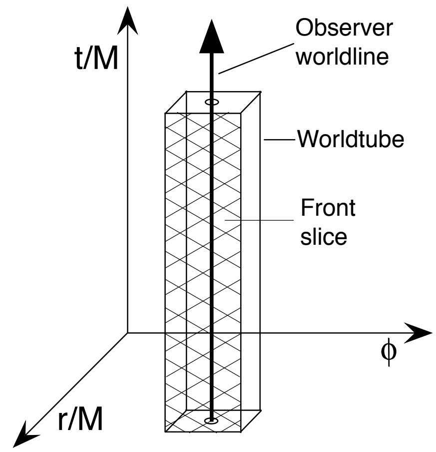

FIGURE 14 A worldtube surrounding an observer at rest in $(\phi, r/M)$ coordinates. This worldtube is bounded with slices, one of which is shaded. How "fat" the worldtube can be and still keep the the local frame of the observer inertial depends on the local spacetime curvature and the sensitivity to tides of the experiment we want to conduct.

3.10 ■ ROOM AND WORLDTUBE

Drill a hole through spacetime.

We are used to the idea of experimenting or carrying out an observation in a room. A **room** is a physical enclosure, such as (1) a laboratory, (2) a powered or unpowered spaceship, or (3) an elevator with or without its supporting cables.

Definition: **room**

DEFINITION 3. Room

A **room** is a physical enclosure of fixed spatial dimensions in which we make measurements and observations over an extended period of time.

Thus far our room is empty; we have not yet installed the rods and clocks that allow us to record and analyze events (Figure 4, Section 5.7). However, even if the room is stationary in global r and ϕ coordinates, it changes its global t-coordinate. As it does so, the room sweeps out what we call a **worldtube** in global coordinates. Figure 14 shows the worldtube of a room at rest in r and ϕ coordinates surrounding the worldline of an observer at rest in the room.

Definition: **worldtube**

DEFINITION 4. Worldtube

A **worldtube** is a bundle of worldlines of objects at rest in a room and worldlines of the structural components of that room. Think of a worldtube as sheathing the worldline of an observer at work in the room. Sometimes, but not always, we choose to bound the worldtube with spacetime slices, as in Figure 14.

Worldtube plot typically curves.

The plot of the worldtube need not be straight, since it bounds the observer's worldline, which typically curves in global coordinates. Figure 15 shows a worldtube inside the event horizon.

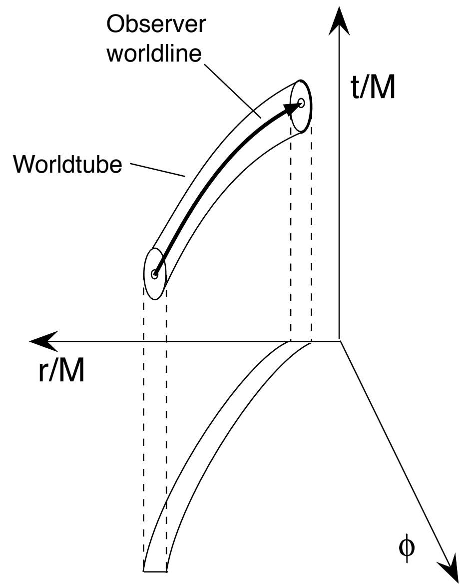

FIGURE 15 A worldtube inside the event horizon. The cross section of this particular worldtube is not rectangular; its sides are not slices in Schwarzschild coordinates. A horizontal or near-horizontal worldline is permitted inside the event horizon; see Figure 8.

In this book we prefer to make every measurement in a local inertial frame. In curved spacetime inertial frames are limited in spacetime extent. Viewed locally, each experiment takes place inside a room of limited space dimension and during a limited time lapse on clocks installed and synchronized in that room. Viewed globally, every experiment takes place within a limited segment of a worldtube.

?

Objection 18. *You keep saying, "In this book we prefer to make every measurement in a local inertial frame." Is this necessary? Could you describe general relativity without using local inertial frames at all?*

!

Yes. The timelike global metric (5) delivers, on its left side, the observed wristwatch time between two events differentially close to one another. You can integrate this differential along the worldline of a stone, for example, to find the wristwatch time between two events widely separated along this worldline. A similar distant spatial separation derives from the spacelike global metric (6). All of physics hangs on events, so all of (classical, non-quantum) physics can be analyzed without local inertial frames. Our preference for measurement in local inertial frames, where special relativity rules, is a matter of taste, clarity, and convenience for us and the reader.

3.11 ■ EXERCISES

1. Measured Distance Between Spherical Shells

A black hole has mass $M = 5$ kilometers, a little more than three times that of our Sun. Two concentric spherical shells surround this black hole. The **L**ower shell has map r-coordinate r_{L}; the **H**igher shell has map r-coordinate $r_{\text{H}} = r_{\text{L}} + \Delta r$. Assume that $\Delta r = 1$ meter and consider the following four cases:

(a) $r_{\text{L}} = 50$ kilometers
(b) $r_{\text{L}} = 15$ kilometers
(c) $r_{\text{L}} = 10.1$ kilometers
(d) $r_{\text{L}} = 10.01$ kilometers
(e) $r_{\text{L}} = 10.001$ kilometers

A. For each case (a) through (e), use (16) to make an estimate of the radial separation σ measured directly by a shell observer. Keep three significant digits for your estimate.
B. Next, in each case (a) through (e) use the result of Sample Problem 1 in Section 3.3 to find the exact distance between shells measured directly by a shell observer. Keep three significant digits for your result.
C. How do your estimates and exact results compare, to three significant digits, for each of the five cases? Give a criterion for the condition under which the estimate of part A will be a good approximation of the exact result of part B.

2. Grazing our Sun

Verify the statement in Section 3.4 concerning two spherical shells around our Sun. The lower shell, of reduced circumference $r_{\text{L}} = 695\,980$ kilometers, just grazes the surface of our Sun. The higher shell is of reduced circumference one kilometer greater, namely $r_{\text{H}} = 695\,981$ kilometers. Verify the prediction that the directly-measured distance between these shells will be 2 millimeters more than 1 kilometer. *Hint:* Use the approximation inside the front cover. (Outbursts and flares leap thousands of kilometers up from Sun's roiling surface, so this exercise is unrealistic—even if we could build these shells!)

3. Many Shells?

The President of the Black Hole Construction Company is waiting in your office when you arrive. He is waxing wroth. ("Tell Roth to wax [him] for awhile."— Groucho Marx)

"You are bankrupting me!" he shouts. "We signed a contract that I would build spherical shells centered on Black Hole Alpha, the shells to be 1 meter apart extending down to the event horizon. But we have already constructed the total number we thought would be required and are nowhere near finished. We are running out of materials and money!"

"Calm down a minute." you reply. "Black Hole Alpha has an event horizon r-coordinate $r = 2M = 10$ kilometers $= 10\,000$ meters. You agreed to build 1000 spherical shells starting at reduced circumference $r = 10\,001$ meters, then $r = 10\,002$ meters, then $r = 10\,003$ meters, and so forth, ending at $r = 11\,000$ meters. So what is the problem?"

"I don't know. Here is our construction method: My worker robot mounts a 1-meter rod vertically (radially) from each completed shell, measures this rod in place to be sure it is exactly 1 meter long, then welds to the top end of this rod the horizontal (tangential) beam of the next spherical shell of larger r-coordinate."

"Ah, then your company is indeed facing a large unnecessary expense," you conclude. "But I think I can tell you how you should construct the shells."

A. Explain to the President of the Black Hole Construction Company what his construction method should have been in order to fulfill his obligation to build 1000 correctly spaced spherical shells. Be specific, but do not be fussy.

B. Substitute the r-coordinate of the innermost shell into equation (16) to make a first estimate of the directly-measured separation between the innermost shell and the second shell, the one with the next-larger r-coordinate.

C. Using the r-coordinate of the second shell, the one just outside the innermost shell, make a second estimate of the directly-measured separation between the innermost shell and the second shell.

D. *Optional.* Use equation (18) to make an exact calculation of the directly-measured separation between the innermost shell and the one just outside it. How does the result of your exact calculation compare with the estimates of Parts B and C?

E. Determine the number of shells that the Black Hole Construction Company would have built if the President had completed the task according to his misunderstood plan.

4. A Dilute Black Hole

Most descriptions of black holes are apocalyptic; you get the impression that black holes are extremely dense objects. Of course a black hole is not dense throughout, because all matter quickly dives to the central crunch point. Still, one can speak of an artificial "average density," defined, say, by the total mass M divided by a spherical Euclidean volume of radius $r = 2M$. In terms of this definition, general relativity does not require that a black hole have a large average density. In this exercise you design a black hole with average density equal to that of the atmosphere you breathe on Earth, roughly 1 kilogram per cubic meter. Carry out all calculations to one-digit accuracy—we want an estimate! *Hint:* Be careful with units, especially when dealing with both conventional and geometric units.

A. From the Euclidean equation for the volume of a sphere

$$V = \frac{4}{3}\pi r^3 \qquad \text{(Euclid)}$$

find an equation for the mass M of air contained in a sphere of radius r, in terms of the density ρ in kilograms/meter^3. Use the conversion factor G/c^2 (Section 3.2) to express this mass in meters. (The volume formula used here is for Euclidean geometry, and we apply it to curved space geometry—so this exercise is only the first step in a more sophisticated analysis.)

B. Let the radius of the Euclidean spherical volume of air be equal to the map r-coordinate of the event horizon of the black hole. Assuming that our designer black hole has the density of air, what is the map r of the event horizon in terms of physical constants and air density?

C. Compare your answer to the radius of our solar system. The mean radius of the orbit of the (former!) planet Pluto is approximately 6×10^{12} meters.

D. How many times the mass of our Sun is the mass of your designer black hole?

5. Astronaut Stretching According to Newton

As you dive feet first radially toward the center of a black hole, you are not physically stress-free and comfortable. True, you detect no overall accelerating "force of gravity." But you do feel a tidal force pulling your feet and head apart and additional forces squeezing your middle from the sides like a high-quality corset. When do these tidal forces become uncomfortable? We have not yet answered this question using general relativity, but Newton is available for consultation, so let's ask him. One-digit accuracy is plenty for numerical estimates in this exercise.

A. Take the derivative with respect to r of the local acceleration g in equation (13) to obtain an expression dg/dr in terms of M and r.

We want to find the radius r_{ouch} at which you begin to feel uncomfortable. What does "uncomfortable" mean? So that we all agree, let us say that you are uncomfortable when your head is pulled upward (relative to your middle) with a force equal to the force of gravity on Earth, $\Delta g = |g_{\text{Earth}}|$, your middle is in a local inertial frame so feels no force, and your feet are pulled downward (again, relative to your middle) with a force equal to the force of gravity on Earth $\Delta g = |g_{\text{Earth}}|$.

B. How massive a black hole do you want to fall into? Suppose $M = 10$ kilometers $= 10\,000$ meters, or about seven times the mass of our Sun. Assume your head and feet are 2 meters apart. Find r_{ouch}, in meters, at which you become uncomfortable according to our criterion. Compare this radius with that of Earth's radius, namely 6.4×10^6meters.

C. Will your discomfort increase or decrease or stay the same as you continue to fall toward the center from this radius?

D. Suppose you fall from rest at infinity. How fast are you going when you reach r_{ouch} according to Newton? Express this speed as a fraction of the speed of light.

E. Take the speed in part D to be constant from that radius to the center and find the corresponding (maximum) time in meters to travel from r_{ouch} to the center, according to Newton. This will be the maximum Newtonian time lapse during which you will be—er—uncomfortable.

F. What is the maximum time of discomfort, according to Newton, expressed in seconds?

Note 1: If you carried the symbol M for the black hole mass through these equations, you found that it canceled out in expressions for the maximum time lapse of discomfort in parts E and F. In other words, your discomfort time is the same for a black hole of *any* mass when you fall from rest at infinity—according to Newton. This equality of discomfort time for all M is also true for the general relativistic analysis.

Note 2: Suppose you drop from rest starting at a great distance from the black hole. Section 7.2 analyzes the wristwatch time lapse from any radius to the center according to general relativity. Section 7.9 examines the general relativistic "ouch time."

6. Black Hole Area Never Decreases

Stephen Hawking discovered that the area of the event horizon of a black hole never decreases, when you calculate this area with the Euclidean formula $A = 4\pi r^2$. Investigate the consequences of this discovery under alternative assumptions described in parts A and B that follow.

Comment 3. Increase disorder
The rule that the area of a black hole's event horizon does not decrease is related in a fundamental way to the statistical law stating that the disorder (the so-called **entropy**) of an isolated physical system does not decrease. See Thorne, *Black Holes and Time Warps,* pages 422–426 and 445-446, and Wheeler, *A Journey into Gravity and Spacetime,* pages 218-222.

Assume that two black holes coalesce. One of the initial black holes has mass M_1 and the other has mass M_2.

A. Assume, first, that the masses of the initial black holes simply add to give the mass of the resulting larger black hole. How does the

r-coordinate of the event horizon of the final black hole relate to the r-coordinates of the event horizons of the initial black holes? How does the area of the event horizon of the final black hole relate to the areas of the event horizons of the initial black holes? Calculate the map r and area of the event horizon of the final black hole for the case where one of the initial black holes has twice the mass of the other one, that is, $M_2 = 2M_1 = 2M$; express your answers as functions of M.

B. Now make a different assumption about the final mass of the combined black hole. Listen to John Wheeler and Ken Ford (*Geons, Black Holes, and Quantum Foam,* pages 300-301) describe the coalescence of two black holes.

> *If two balls of putty collide and stick together, the mass of the new, larger ball is the sum of the masses of the balls that collide. Not so for black holes. If two spinless, uncharged black holes collide and coalesce—and if they get rid of as much energy as they possibly can in the form of gravitational waves as they combine—the* square *of the mass of the new, heavier black hole is the sum of the* squares *of the combining masses. That means that a right triangle with sides scaled to measure the [squares of the] masses of two black holes has a hypotenuse that measures the [square of the] mass of the single black hole they form when they join. Try to picture the incredible tumult of two black holes locked in each other's embrace, each swallowing the other, both churning space and time with gravitational radiation. Then marvel that the simple rule of Pythagoras imposes its order on this ultimate cosmic maelstrom.*

Following this more realistic scenario, find the r-value of the resulting event horizon when black holes of masses M_1 and M_2 coalesce. How does the area of the event horizon of the final black hole relate to the areas of the event horizons of the initial black holes?

C. Do the results of both part A and part B follow Hawking's rule that the event horizon's area of a black hole does not decrease?

D. Assume that the mass lost in the analysis of Part B escapes as gravitational radiation. What is the mass-equivalent of the energy of that gravitational radiation?

7. Zeno's Paradox

Zeno of Elea, Greece, (born about 495 BCE, died about 430 BCE) developed several paradoxes of motion. One of these states that a body in motion starting from Point A can reach a given final Point B only after having traversed half the distance between Point A and Point B. But before traversing this half it must cover half of that half, and so on *ad infinitum.* Consequently the goal can never be reached.

A modern reader, also named Zeno, raises a similar paradox about crossing the event horizon. Zeno refers us to the relation between $d\sigma$ and dr for radial separation:

$$d\sigma = \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}} \qquad (dt = 0, d\phi = 0) \tag{38}$$

Zeno then asserts, "As r approaches $2M$, the denominator on the right hand side of (38) goes to zero, so the distance between adjacent shells becomes infinite. Even at the speed of light, an object cannot travel an infinite distance in a finite time. Therefore nothing can arrive at the event horizon and enter the black hole." Analyze and resolve this modern Zeno's paradox using the following argument or some other method.

> As often happens in relativity, the question is: Who measures what? In order to cross the event horizon, the diving object must pass through every shell outside the event horizon. Each shell observer measures the incremental ruler length $d\sigma$ between his shell and the one below it. Then the observer on that next-lower shell measures the incremental ruler distance between that shell and the one below *it.* By adding up these increments, we can establish a measure of the "summed ruler lengths measured by shell observers from the shell at higher map r_{H} to the shell at lower map r_{L}" through which the object must move to reach the event horizon.
>
> We integrated (38) from one shell to another in Sample Problem 1 in Section 3.3. Let $r_{\text{L}} \to 2M$ in that solution, and show that the resulting distance from r_{H} to r_{L}, the "summed ruler lengths," is finite as measured by the collection of collaborating shell observers. This is true even though the right side of (38) becomes infinite exactly at $r = 2M$.
>
> Will collaborating shell observers conclude among themselves that the in-falling stone reaches the event horizon? The present exercise shows that the "summed ruler lengths" is finite from any shell to the event horizon. However, motion involves not only distance but also time—and in relativity time does not follow common expectations! What can we say about the "summed shell time" for the passage of a diver through the "summed shell distance" calculated above? Chapter 6, Diving, shows that the observer on every shell measures an inertial diver to pass him with non-zero speed, a local shell speed that continues to increase as the diver gets closer and closer to the event horizon. Each shell observer therefore clocks a finite (non-infinite) time for the diver to pass from his shell to the shell below. Take the sum of these finite times—"sum" meaning an integral similar to the integral of equation (38) carried out in Sample Problem 1. When computed, this integral of shell times yields

a finite value for the total time measured by the collection of shell observers past whom the diver passes. Hence the group of shell observers agree among themselves: Someone diving radially passes them all in a finite "summed shell time" and reaches the event horizon. Thank you, Zeno!

3.12 ■ REFERENCES

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Flamm's paraboloid, Figure 11 in Section 3.9: Ludwig Flamm (1916). "Beiträge zur Einstein'schen Gravitationstheorie". Physikalische Zeitschrift Volume 17, pages 448-454.

Some items in Box 5 adapted from Misner, Thorne, and Wheeler, GRAVITATION, W. H. Freeman Company, 1970, San Francisco (now New York), page 671

Quote *Black holes just didn't "smell right"* from Kip Thorne, *Black Holes and Time Warps: Einstein's Outrageous Legacy*, New York, W. W. Norton, 1994, pages 134 and 137.