Quantum Computing Architectures (BME) / Quantum bits in solids (ELTE)
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Control questions, exercises (v3)

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The exam will consist of a written and an oral part. This file contains a few control questions and exercises, similar to those that will appear in the written part of the exam. Of course, these or similar ones might appear also in the oral part. In the oral part, we will ask about a specific topic that was covered in the lectures; for example, describe electron-phonon interaction and its consequences with respect to spin-qubit dynamics; or, describe how to do single-qubit gates on a transmon qubit, etc.

I. QUANTUM BITS

1. List the three Pauli matrices. Determine their eigenvalues and normalized eigenstates.
2. What is the unitary matrix representing the Hadamard gate? What is the result of the Hadamard gate acting on the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? What is the result of the Hadamard gate acting on the state $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?
3. What is the unitary matrix representing a two-qubit $\sqrt{\text{SWAP}}$ gate in the basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$? What is the result of the $\sqrt{\text{SWAP}}$ gate acting on the state $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$?
4. Determine the polarization vectors associated to the following three states: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$.
5. Consider the single-qubit state $\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$. When we measure the qubit, what is the probability of measuring 1? What is the state of the qubit after the measurement?
6. Consider the two-qubit state $\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|10\rangle + \frac{1}{\sqrt{3}}|01\rangle$. When we measure the first qubit, what is the probability of measuring 0? And that of measuring 1? What is the state of the system after measuring 0? And after measuring 1?

II. CONTROL OF QUANTUM SYSTEMS

1. Let H be an N-dimensional time independent Hamiltonian, with known energy eigenvalues E_n and eigenstates ψ_n fulfilling $H\psi_n = E_n\psi_n$. Assume that the system is initialized in the state ψ_i at $t = 0$. Express the time evolution of this state, $\psi(t)$, using E_n, ψ_n and ψ_i.
2. A quantum system is described by the time-dependent Hamiltonian $H(t)$. Write out the time-dependent Schrödinger equation. Perform a time-dependent unitary transformation $U(t)$ on the time-dependent Schrödinger equation. Express the quantity playing the role of the Hamiltonian in the transformed equation, using $H(t)$ and $U(t)$.
3. Consider the Hamiltonian

$$H = H_0 + H_1, \tag{1}$$

where

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \text{ and } H_1 = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \tag{2}$$

with $0 < t \ll \Delta$. Express the lower-energy eigenstate using first-order perturbation theory, and the lower energy eigenvalue using second-order perturbation theory.

4. In single-electron spin resonance, how does the Rabi time depend on the external homogeneous magnetic field B_0? How does the Rabi time depend on the amplitude $B_{\rm ac}$ of the ac magnetic field?

5. Consider the simplest model of two electrons in a double quantum dot, with two spin states and a single orbital in each dot. What is the dimension of the two-electron Hilbert space? Specify the unitary transformation between the product basis and the singlet-triplet basis. What is the dimension of the three-electron Hilbert space?

6. Write out the Jaynes-Cummings Hamiltonian. Sketch the energy level diagram for the resonant but uncoupled case. Within the same diagram, sketch the coupling matrix elements. How does the coupling matrix element change as we climb up the Jaynes-Cummings ladder?

7. Sketch the energy level diagram of the Jaynes-Cummings model, as well as the coupling matrix elements, in the situation where it is used to perform dispersive readout. What is the dispersive shift of the resonator if the qubit is in its ground state? What is the dispersive shift if the qubit is in its excited state?

8. A harmonic oscillator can mediate interaction between two qubits. Assume that this system is described by the two-qubit Jaynes-Cummings Hamiltonian discussed at the lecture. Set the resonator eigenfrequency to 9.5 GHz, and qubit Larmor frequencies to 10 GHz, and the coupling strength to $g/2\pi = 50$ MHz. Sketch and label the 7 energy levels that play a role in the sqrt-of-iswap gate mediated by virtual photon exchange. Estimate the time scale of that gate.

III. QUBITS BASED ON THE ELECTRON SPIN

1. In GaAs, where the effective mass of the conduction-band electrons is $m \approx 0.063 m_e$, we make a quantum dot with an orbital level spacing of 1 meV. Estimate the spatial extension of the electron occupying the ground-state orbital of this quantum dot. Estimate the charging energy, i.e., the Coulomb repulsion energy between two electrons occupying this dot.

2. At the lecture, we have discussed how to use spin-to-charge conversion and charge sensing to measure the spin relaxation time $T_{\rm spin}$. There are four characteristic energy scales in this experiment, one related to the spin relaxation time ($\hbar/T_{\rm spin}$), one related to the tunnel rate between the quantum dot and the electron reservoir ($\hbar\Gamma$), the Zeeman splitting ($\hbar\omega_L \equiv g^* \mu_B B$), and the thermal energy scale ($k_B T$). What condition(s) among these energy scales should be satisfied to make the experiment work?

3. Consider the gate-voltage-dependent conductance of the QPC appearing on slide 11 of Lecture 3. The QPC is tuned the the working point denoted by the cross. Imagine that a single electron appears at a distance of 200 nm from the QPC. How much does the conductance change due to the appearance of the electron?

4. Estimate the tunnel rate Γ from the measurement data shown in slide 12 of Lecture 3.

5. Estimate the temperature of the experiment from the data set shown in slide 15 of Lecture 3. It is allowed to use the material parameters of GaAs for the estimate.

6. Write out the 6×6 Hamiltonian matrix of the two-electron sector of the two-site Hubbard model, whose energy spectrum is shown in slide 17 of Lecture 3. Do include the magnetic field in the Hamiltonian. To obtain the energy spectrum shown there, we used $t_H = 0.05\,U$ and $g\mu_B B = 0.1\,U$. Assuming $U = 1$ meV, what is the S-T_0 energy splitting at zero detuning $\epsilon = 0$? What is the Larmor frequency of this singlet-triplet qubit at the zero-detuning point? What is the magnetic field value (in Teslas) corresponding to this parameter set?

IV. COHERENT CONTROL OF ELECTRON SPINS

1. An infinitely thin wire carries a current of 1 mA. Calculate the magnetic field induced by this current at a distance of 50 nm from the wire.

2. Consider the measurement cycle on slide 7 of Lecture 4. Estimate the characteristic scale of the maximum dc current achievable by periodically repeating this cycle.

3. By numerical diagonalization of the two-site Hubbard Hamiltonian in a magnetic field, reproduce the two spectra shown in slide 8 of Lecture 4. On that slide, $\beta = g^* \mu_B B$. What is the value of the B-field on the top right figure on slide 8 of Lecture 4, if $U = 1$ meV and $g^* = 0.4$?

4. Consider a single phosphorus atom (P). It has a spin-1/2 nuclear spin. The gyromagnetic ratio of this nuclear spin is $\gamma_n \approx 17\,\mathrm{MHz/T}$. Calculate the thermal population of the ground and excited states of this nuclear spin at magnetic field $B = 1$ T and temperature $T = 100$ mK.

5. Consider the transverse magnetic-field profile $B_\perp(x, y)$ shown in slide 10 of Lecture 4. Assume that we do a μ-EDSR experiment in such a profile, in a GaAs 2DEG, by driving the system along the x axis with a homogeneous ac electric field, in a circularly symmetric quantum dot with orbital level spacing $\hbar\omega_0 = 1$ meV. Estimate the amplitude of the ac electric-field drive required to produce spin Rabi oscillations with Rabi time $T = 2\pi/\Omega = 1\,\mu$s.

6. Consider an electron in a two-dimensional electron gas, subject to Rashba spin-orbit interaction:

$$H = \frac{p_x^2 + p_y^2}{2m} + \alpha(\sigma_y p_x - \sigma_x p_y). \tag{3}$$

   Assume that the electron is occupying a plane-wave state moving along x, with momentum vector $\boldsymbol{k} = (k_x, 0)$. How do the energy eigenvalues depend on k_x? How do the energy eigenstates depend on k_x?

7. Derive the SOI-EDSR Rabi frequency, following the first-order perturbative calculation outlined in slides 17 and 18 of Lecture 4. Note that in the last step leading to $H_{E,q}$, a first-order series expansion in the small parameter $\omega_L/\omega_0 \ll 1$ was made. How many terms are there in the first-order perturbative expressions of the dressed qubit basis states $|\overline{0_x 0_y \uparrow}\rangle$ and $|\overline{0_x 0_y \downarrow}\rangle$?

8. An unreadable formula on the blackboard encodes the spatial dependence of an inhomogeneous magnetic field in the vicinity of a quantum dot:

$$\boldsymbol{B}(x, y, z) = (\beta ?, 0, B_0), \tag{4}$$

   where ? is either x or y. Which one is it? Assume that the lecturer did no mistake.

9. Consider a single electron in a quantum dot defined electrostatically in a two-dimensional electron gas. Consider the (i) Rabi frequency due to ac magnetic excitation (electron spin resonance), (ii) Rabi frequency in electrically driven spin resonance due to Rashba spin-orbit interaction. Specify how the above quantities change (one by one), if the lateral size of the quantum dot is increased.

V. INFORMATION LOSS MECHANISMS FOR ELECTRON SPINS

1. The speed of longitudinal acoustic phonons in GaAs is $v_{\mathrm{LA}} \approx 5000\,\mathrm{m/s}$. Calculate the wave length of a phonon that is emitted by a spin qubit at an external magnetic field of 1 Tesla.

2. A longitudinal compression wave (a classical wave analogous to a longitudinal acoustic phonon) is propagating in a crystal, with a wave length of $1\,\mu$m, and with a strain amplitude of 1%. This creates an effective electrostatic potential for the conduction-band electrons, due to the deformation-potential mechanism. Assume that the deformation-potential constant for the conduction band of this material is $\Xi = 10$ eV. What is the peak value of the effective electric field that is felt by the conduction-band electrons, created by this compression wave?

3. An unreadable formula on the blackboard shows the spin relaxation rate of an electron in a quantum dot, due to spin-orbit interaction and spontaneous phonon emission, as

$$\frac{1}{T_1} = \frac{1}{6\pi} \frac{\alpha^2 \omega_L^7}{\rho v_L^7 \omega_0^? \hbar^?}, \tag{5}$$

   where the question marks denote unreadable integers. Recall the meaning of each quantity of the formula, and use dimensional analysis to determine the values of the two unreadable exponents.

4. Consider a single electron in a quantum dot defined electrostatically in a two-dimensional electron gas. Consider the zero-temperature spin relaxation rate Γ_1 due to Rashba spin-orbit interaction and spontaneous phonon emission. Specify how Γ_1 changes when the lateral size of the quantum dot is increased.

5. Do the exercises listed in slide 13 of lecture 5.

6. We have discussed how to utilize the Ramsey experiment based on electron spin resonance to measure the inhomogeneous spin dephasing time T_2^* in GaAs. Assume that the Overhauser field in our quantum dot has x, y, z components with standard deviation $\sigma = 1$ mT. Assume also that our goal is to use a Ramsey experiment to determine T_2^* in an external homogeneous field of 100 mT. Are there any constraints on the ac magnetic field amplitude we have to apply in this experiment?

7. In a certain quantum dot, which is hosted in a material where every atomic nucleus carries a nuclear spin, the inhomogeneous dephasing time of the spin of the electron occupying the dot is $T_2^* = 10$ ns. Assume that we can make an identical quantum dot in which 99% of the nuclear spins are eliminated. In this dot, $T_2^* =$? Hint: The average distance covered by a one-dimensional N-step random walk is $\sqrt{N}$.

VI. SUPERCONDUCTING QUBITS: BASIC ARCHITECTURES

1. The London penetration depth of Aluminium is $\lambda = 4.5$ nm. Estimate the density of the superconducting charge carriers in this material. See slide 4 of lecture 6, and recall that $m^* = 2m_\mathrm{e}$ and $e^* = 2e$.

2. Estimate the value of the critical magnetic field of niobium at temperature 4.2 K.

3. What is the frequency and the wavelength of the radiation coming from a Josephson junction biased with a dc voltage of 1 mV?

4. Consider an overdamped Josephson junction, as shown on slide 15 of lecture 6. The junction is current-biased, and we measure the source-drain voltage. Assume that the current is slowly ramped up from $I(t = 0) = 0$ to $I(t = 3T) = 3I_c$, then ramped back to $I(t = 9T) = -3I_c$, and then to $I(t = 12T) = 0$. Here, T is some long time window, e.g., $T = 1$ minute. The current change is linear in time in all three sections. Plot the measured voltage as a function of time.

5. Consider the RCSJ model of a Josephson junction. Assume that the junction resistance is infinite, $R = \infty$, the criticial current is $I_c = 100\,\mu$A, and the junction capacitance is $C = 10$ pF. Calculate the equilibrium position and the eigenfrequency of the phase particle without current bias, $I = 0$. Estimate the eigenfrequency at low current bias, $I = 0.1I_c$.

6. Consider a flux qubit defined in an rf SQUID. Assume that the loop encloses an area $A = 2\mu m \times 2\mu m$, the junction has a critical current $I_c = 500nA$, an inductance of $L = 5\,$pH and we apply a flux bias of (a) $\Phi_\mathrm{ext} = 0$, (b) $\Phi_\mathrm{ext} = \Phi_0/4$, (c) $\Phi_\mathrm{ext} = \Phi_0/2$. Determine the phase values that minimize the potential energy of the circuit. How strong an external magnetic field should we use to reach those flux values (a), (b), (c), assuming that the magnetic field is homogeneous outside the superconductor and points perpendicular to the device plane?

7. Use the parameters given above for the flux qubit. What is the amplitude of ac magnetic field required to do Rabi oscillations with a Rabi period of 10 ns? The coupling term in case of flux qubits can be written as $I\delta\Phi$, where $\delta\Phi$ is the flux modulation due to the external ac field. Suppose that we are far from the degeneracy point, and $I \sim I_c$!

8. Consider the Josephson junction with the geometry and parameters specified at the bottom of slide 7 of lecture 7. How large the junction should be, to enter the quantum regime (charge regime)? How cold should the device be to thermalize in the quantum ground state with a large probability ≈ 0.95?

9. Calculate the Josepshon inductance for a junction with critical current $I_c = 1\,\mu$A! How big should be the charging energy of the qubit for a qubit frequency of 5 GHz?

10. Calculate the following commutators: $\left[\hat{\delta}, \hat{N}\right]$ and $\left[\hat{\Phi}, \hat{Q}\right]$!

11. Start from the following Hamiltonian:

$$\hat{H} = E_c \sum_N (N - N_g)^2 |N\rangle \langle N| - \frac{E_J}{2} \sum_N |N\rangle \langle N+1| + |N+1\rangle \langle N|$$

Suppose that $N_g = 1/2 + \Delta_g$. Derive the qubit Hamiltonian (restrict the Hilbert space to the lowest two states):

$$H = E_C \Delta_g \sigma_z - \frac{E_J}{2} \sigma_x.$$

What are the eigenenergies?

12. Suppose a transmon qubit with $E_J/E_c = 50$ and $\omega_{01} = 5\,\text{GHz}$. When performing a Rabi oscillation one uses a sine function which is turned on for a time τ (rectangular window function). If too short pulses are applied, the Rabi oscillation will also excite the $1-2$ transition. How long the pulse can be, such that the $1-2$ is excited with a 95% smaller amplitude?

13. For an underdamped Jospehson junction, the switching current can be smaller than the critical current. The reduction of the critical current can be calculated by the following formula:

$$< \Delta I_c >= I_c \left(\frac{k_B T}{2E_{J0}} \ln \frac{\omega_p \Delta t}{2\pi} \right)^{2/3}$$

Here, Δt is the time spent sweeping the applied current through the dense part of the distribution of observed critical current values. Suppose Δt is 2 seconds, $\omega_p \sim 10^{10}$ 1/s. How much the critical current is reduced for $k_B T/E_{J0} = 0.05$

14. We make a $\lambda/2$ electromagnetic resonator with a fundamental mode at a frequency of 10 GHz. Assume that we make it from a superconducting material with *effective dielectric constant* of 6. How long should be the resonator?

15. How low temperature is needed for a resonator with resonance frequency $\omega_r = 5\,\text{GHz}$ to have an average photon number $< n > \leq 0.05$? What should be the temperature for a qubit with $\omega_q = 5\,\text{GHz}$ to have an excited state population smaller than 0.05?