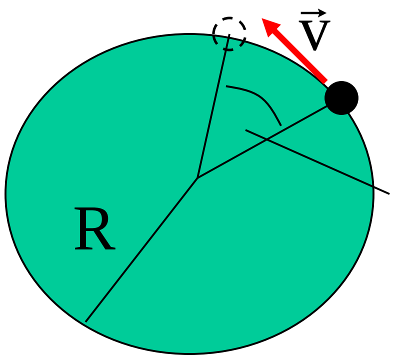
Lecture 3 & 4

Rotational & rolling motion

Angular momentum

http://fizipedia.bme.hu/index.php/Preengineering_course

2. Rotation of a rigid object around a fix axis



Θ: angular displacement [rad]

(rotating disc with radius of R)

Def.: average angular velocity [1/s]

$$\omega_{ave} = \frac{\Delta\Theta}{\Delta t} = \frac{\Theta(t_2) - \Theta(t_1)}{t_2 - t_1}$$

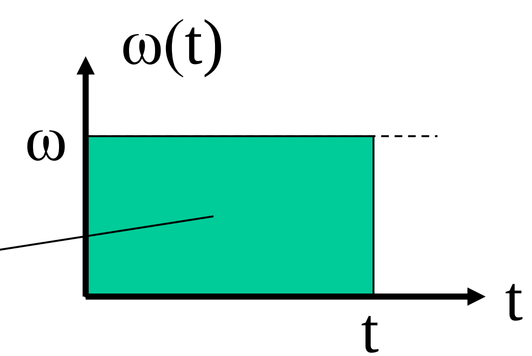
$$\omega = \frac{v}{R}$$

If ω=const.

$$\omega = \frac{\Theta(t) - \Theta_o}{t}$$

Angular position: $\Theta(t)$

Angular displacement: $\Theta(t) - \Theta_o = \omega t$



If ω≠const.

Def.: instantaneous angular velocity

$$\omega(t) = \lim_{\Delta t \to 0} \frac{\Theta(t+\Delta t) - \Theta(t)}{\Delta t} = \frac{d\Theta}{dt}$$

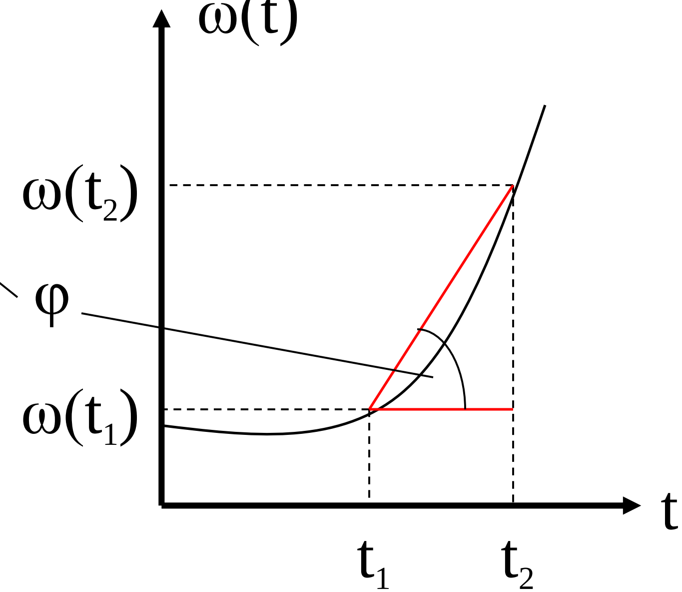
Def.: average angular acceleration [1/s²]

$$\alpha_{ave} = \frac{\Delta\omega}{\Delta t} = \frac{\omega(t_2) - \omega(t_1)}{t_2 - t_1} = \mathrm{tg}\varphi$$

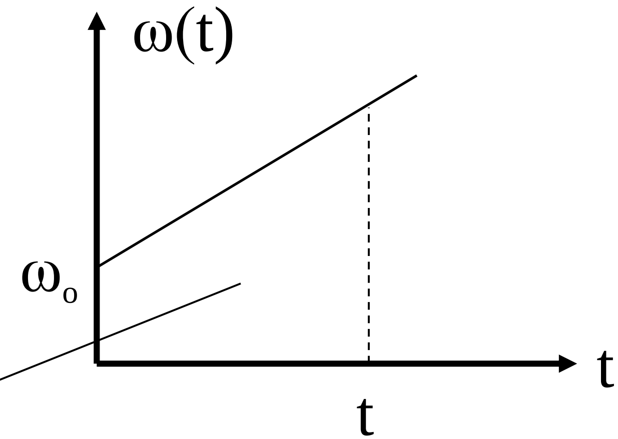
Def.: inst. angular acceleration

$$\alpha(t) = \lim_{\Delta t \to 0} \frac{\omega(t+\Delta t) - \omega(t)}{\Delta t} = \frac{d\omega}{dt}$$

$$\alpha(t) = \frac{a_t}{R}$$



If α=const.

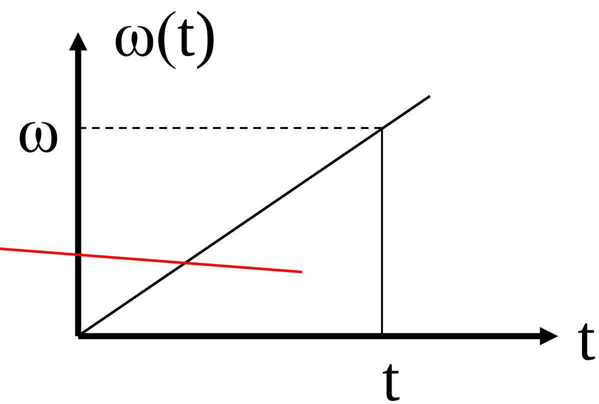


$$\omega(t)=\omega_o+\alpha t$$

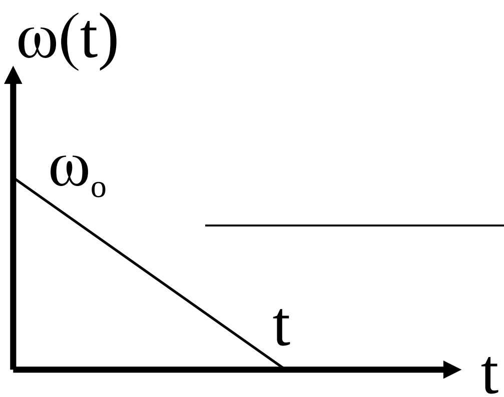
Angular displacement:

$$\Theta = \omega_o t + \frac{1}{2}\alpha t^2$$

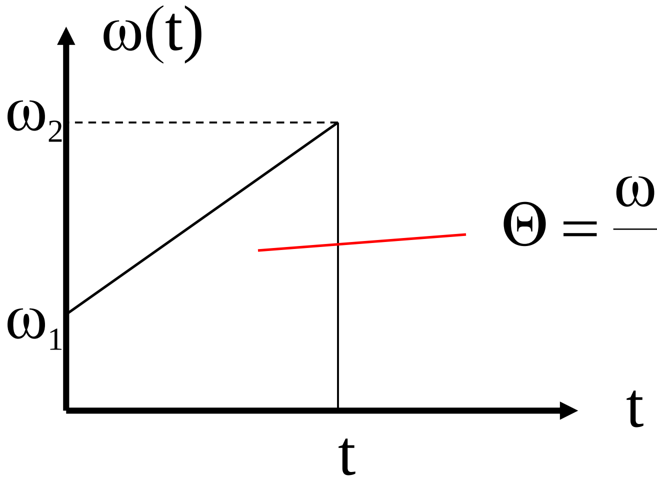
If $\omega_o=0$



$$\Theta = \frac{1}{2}\alpha t^2 = \frac{\omega t}{2} = \frac{\omega^2}{2\alpha}$$



$$\alpha \rightarrow |\alpha| \ , \ \omega \rightarrow \omega_o$$



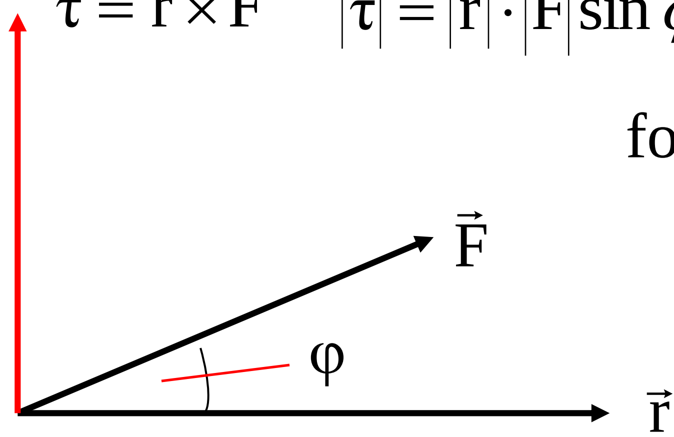
$$\Theta = \frac{\omega_1 + \omega_2}{2} t = \frac{\omega_2^2 - \omega_1^2}{2\alpha}$$

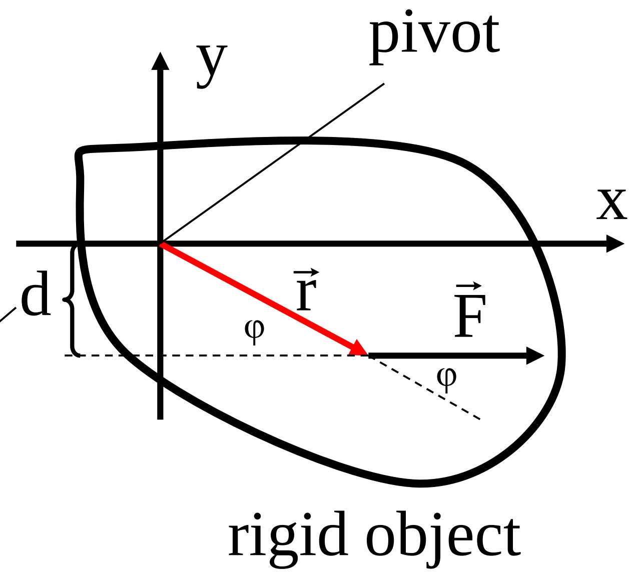
Def.: torque [Nm] $\vec{\tau} = \vec{r} \times \vec{F}$

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad |\vec{\tau}| = |\vec{r}| \cdot |\vec{F}| \sin\varphi = F \cdot d$$

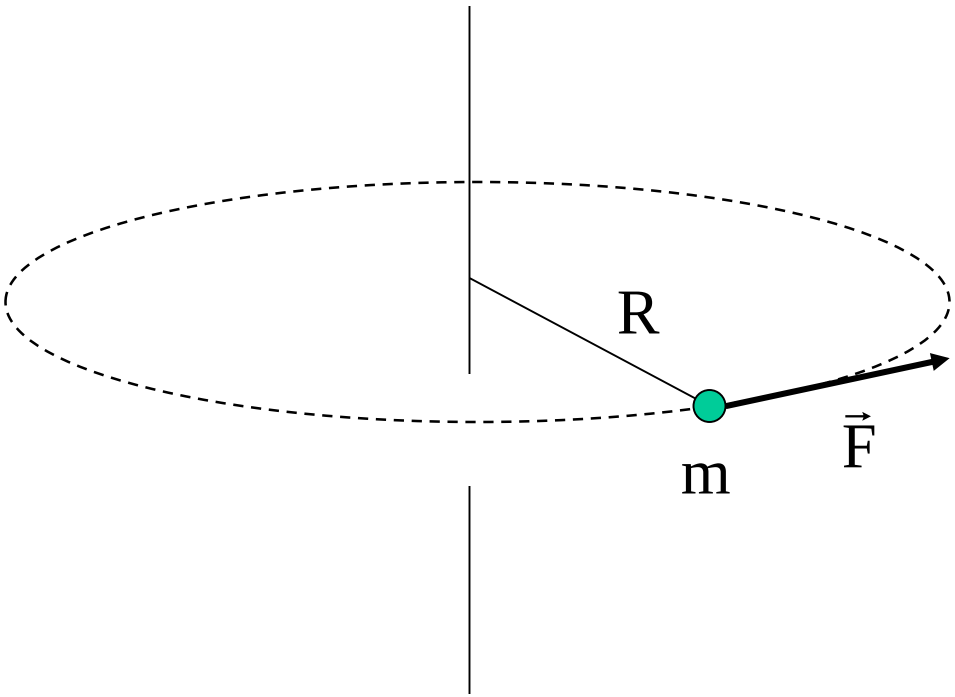
force

lever arm





Newton's 2nd law for rotating rigid object



$$F = ma$$

$$FR = mRa$$

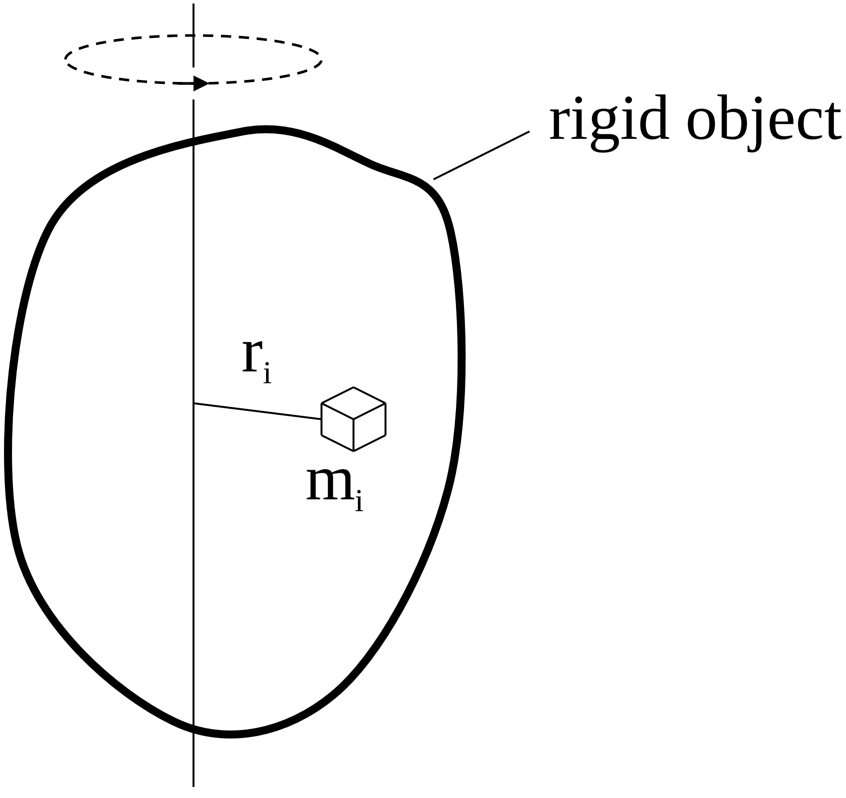
$$a_t = \alpha R$$

$$\tau = \underbrace{mR^2}\alpha$$

I: moment of inertia

$$\boxed{\tau = I\alpha}$$

$$\langle F = ma \rangle$$

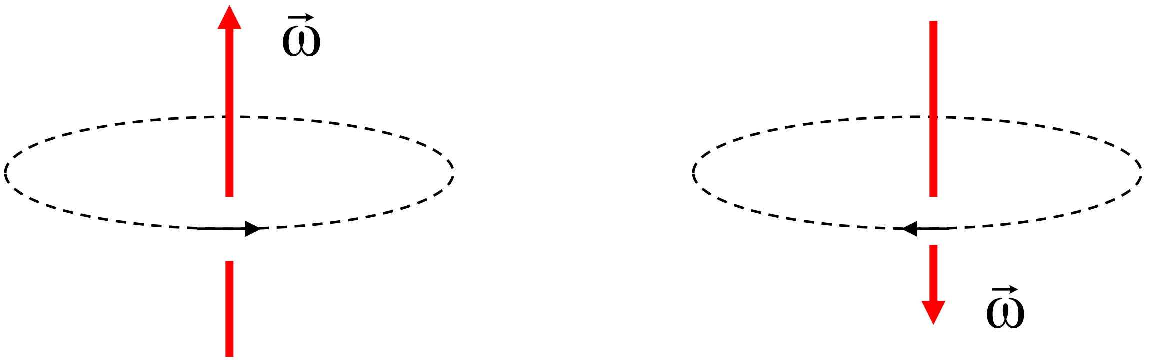


Moment of inertia

$$I = \sum_i m_i r_i^2$$

Parallel axis theorem: $I = I_o + ms^2$

Direction

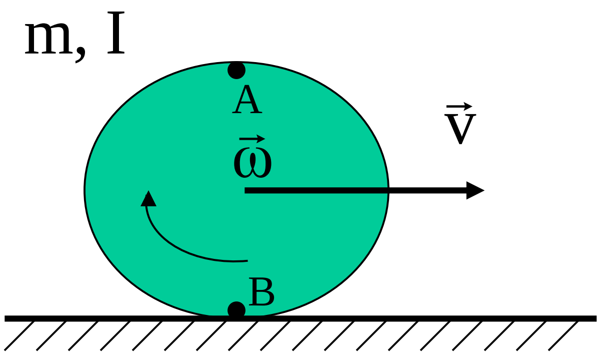


Kinetic energy: $E_k = \frac{1}{2} I\omega^2$

Work: $W = \tau \cdot \Theta$ $W = \int_{\Theta_1}^{\Theta_2} \tau(\Theta) d\Theta$

Inst. power: $P = \tau \cdot \omega$

Rolling motion and angular momentum



$$E_k = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

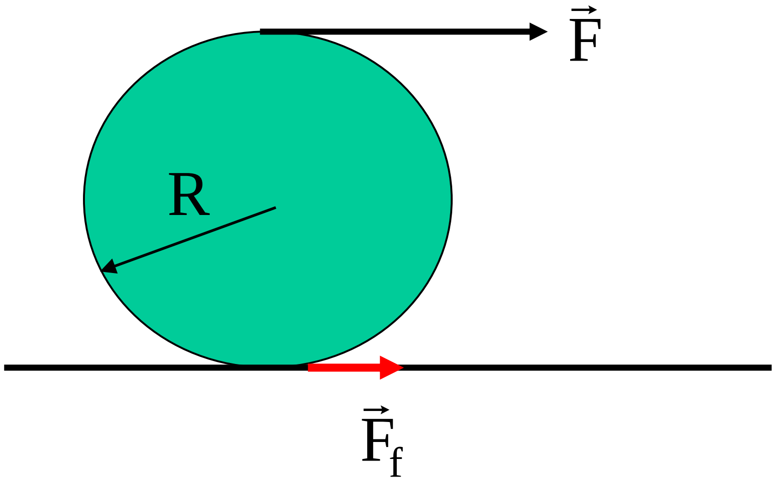
$$\omega = \frac{v}{R}$$

$$E_k = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2$$

$v_B = 0$

$v_A = 2v$

Example: rolling motion (without slipping)



I. $F + F_f = ma$

$(\tau = I\alpha)$

II. $(F - F_f)R = I\alpha = I\dfrac{a}{R}$

Uniform solid disc: $I = \dfrac{1}{2}mR^2$

Frictional force: $F_f \le F_{f,\,max.}$

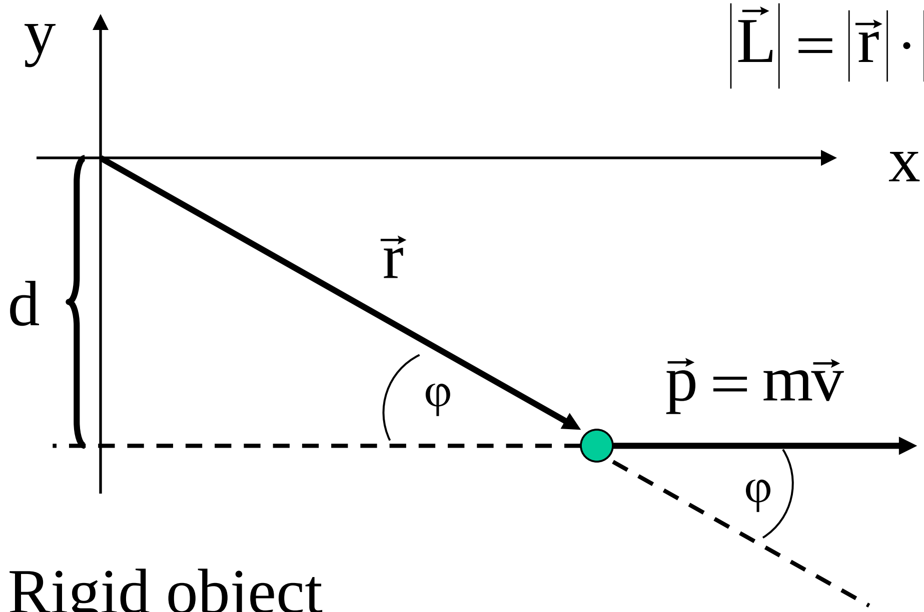
$$a = \frac{4}{3}\frac{F}{m}$$

$$F_f = \frac{1}{3}F$$

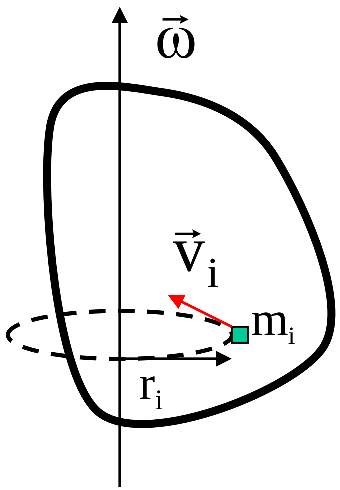
Def.: angular momentum – point mass

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\left|\vec{L}\right| = \left|\vec{r}\right| \cdot \left|\vec{p}\right| \cdot \sin(\varphi) = mvr\sin(\varphi) = mvd$$



Rigid object



$$L_i = m_i v_i r_i = m_i r_i^2 \omega \qquad (v_i = \omega r_i)$$

$$L = \sum_i m_i v_i r_i = \sum_i m_i r_i^2 \omega = I\omega$$

Conservation of angular momentum

$$\tau = I\alpha$$

↓

$$\tau = I\alpha = I\frac{\Delta\omega}{\Delta t} = I\frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{I\omega_2 - I\omega_1}{\Delta t} = \frac{L_2 - L_1}{\Delta t}$$

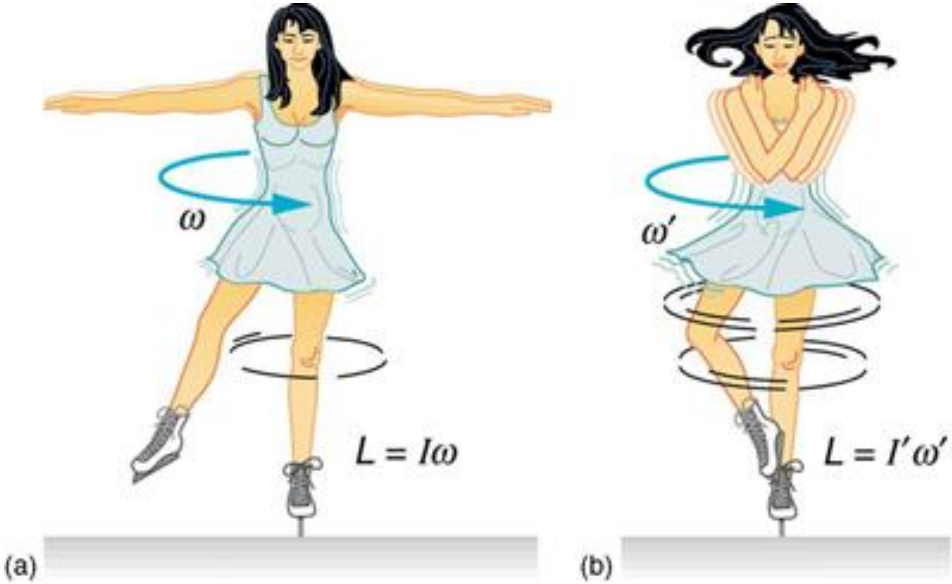
↓

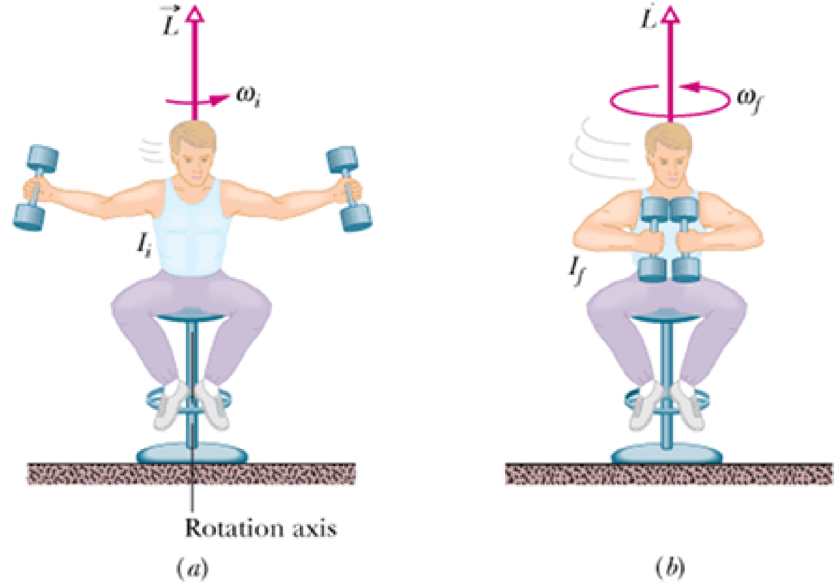
$$\tau = \frac{\Delta L}{\Delta t} \quad \rightarrow \quad \tau = \frac{dL}{dt}$$

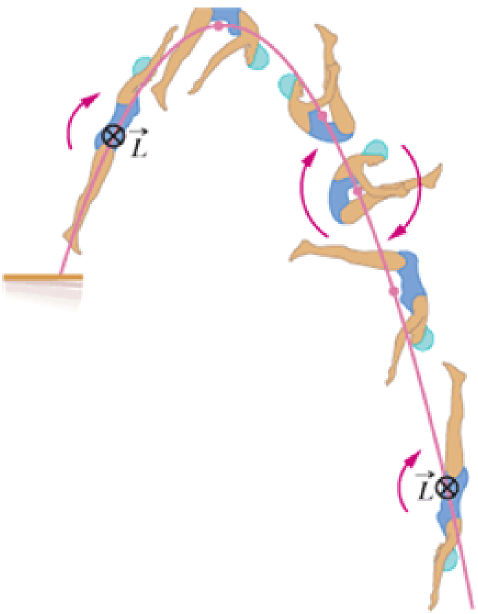
⇒ Conservation of angular momentum:

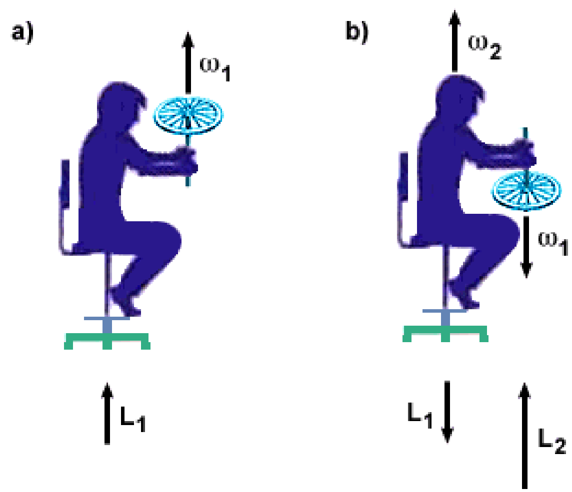
⇒ If $\vec{\tau}_{net} = 0 \Rightarrow \vec{L} = \text{const.}$

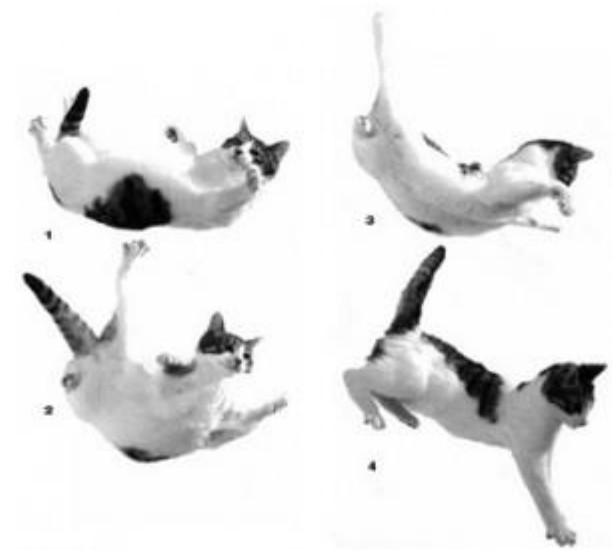
Conservation of angular momentum: examples





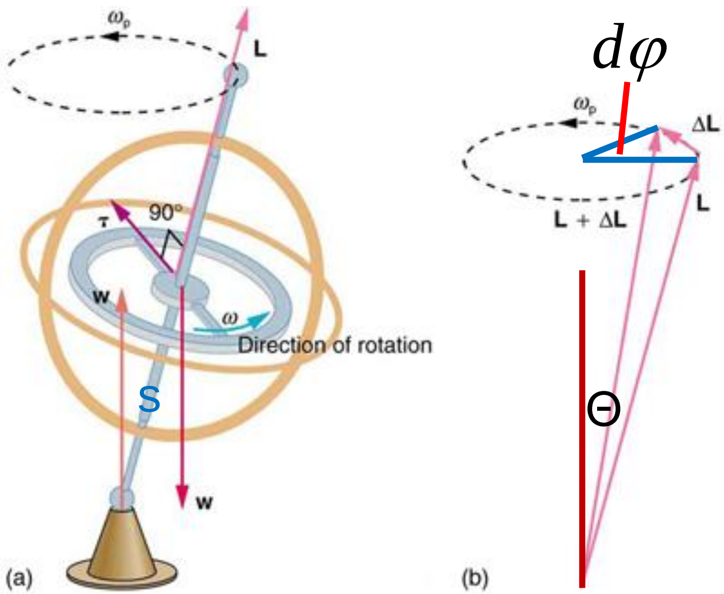


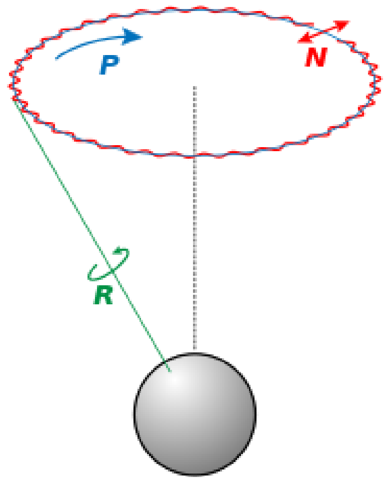


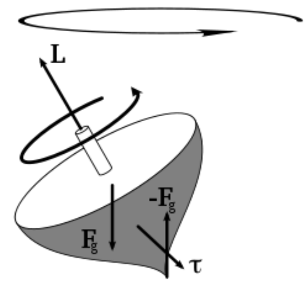


Precession, gyroscope

$$M = \frac{dL}{dt}$$



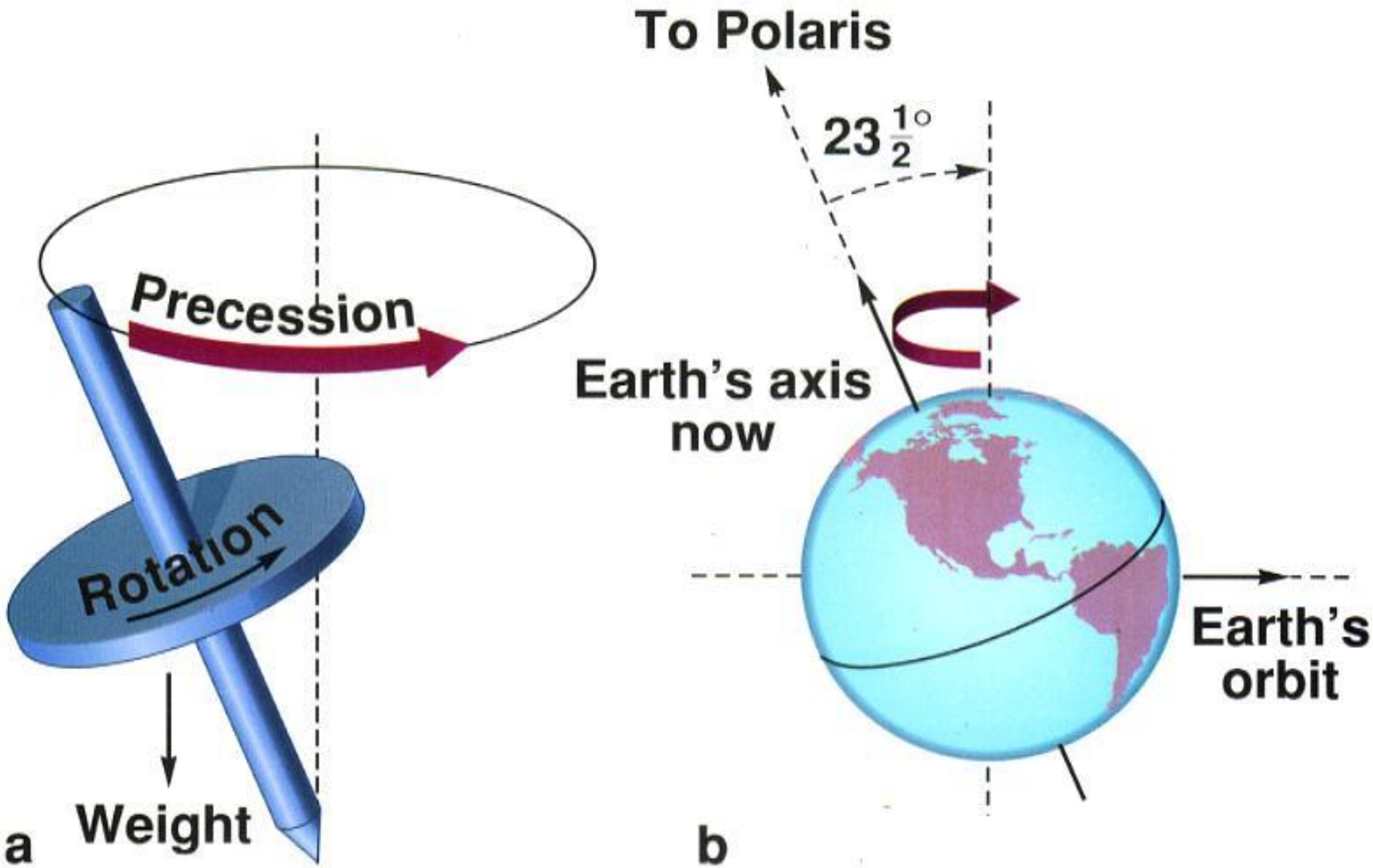




$$d\vec{L} = \vec{M}dt \qquad \left|\vec{M}\right| = mgs\sin\Theta$$

$$\left|d\vec{L}\right| = L\sin\Theta d\varphi = Mdt$$

$$\omega_p = \frac{d\varphi}{dt} = \frac{mgs}{L}$$



The period of a cycle is approximately 26,000 years