Superconducting nanostructures

Literature:

• Y.V. Nazarov, Y.M. Blanter: Quantum transport (Cambridge University Press, 2009)

• S. Datta, P.F. Bagwell, M.P. Anantram: Scattering Theory of Transport for Mesoscopic Superconductors (http://docs.lib.purdue.edu/ecetr/107/)

• R. Cron Ph.D. thesis (Atomic contacts: a Test-Bed for Mesoscopic Physics, CEA Saclay)

• J. Cserti: Superconducting mesoscopic systems (lecture notes)

"Survival kit for superconductivity"

•Below certain temperature (T_C, the SC transition temperature) the electrical resistance of certain materials becomes zero.

•Some conventional superconductors: Nb (9.26K), Pb (7.19K), V (5.3K), Ta (4.48K), Hg (4.15K), Sn (3.72K), In (3.41K), Al (1.2K), Zn (0.85K), Ti (0.39 K), W (0.015K) (See http://hyperphysics.phy-astr.gsu.edu/hbase/solids/scond.html).

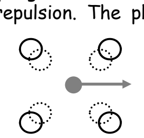
•Superconductors are ideal diamagnets, weak magnetic fileds do not penetrate the bulk of the superconductor (Meissner effect). High magnetic field destroys superconductivity. The critical field, H_C varies between ~10^{-4} T (W) to ~0.2 T (Nb).

•There is a narrow layer at the boundary, where the external magnetic field decreses exponentially to zero. Characteristic length: λ (penetration depth).

•In the SC state the specific heat depends exponentially on the temperature.

•Microscopic BCS theory (Bardeen, Cooper, Schrieffer):

- •the electron-phonon coupling can introduce an attractive interaction between the electrons which may overcome Coulomb repulsion. The phonon mediated attraction is a local interaction, $V_{e\text{-}ph}=-(2\lambda/\nu)\delta(r_1-r_2)$.

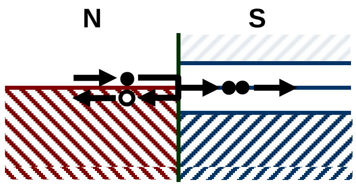


Naive picture: an electron moving in the lattice attracts the ions, which will than attract the next electron passing by.

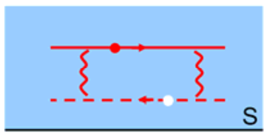
- •The ground state of two electrons with attraction is a bound state with $E=-2\Delta$, where $\Delta=\hbar\omega_D\exp(-1/\lambda)$ is the superconducting energy gap. ($\Delta(T=0)\approx1.76k_BT_C$, approaching T_C it vanishes by $(T_C-T)^{1/2}$.) In the SC state bound states of electron pairs with **k**$\downarrow$ and **-k**$\uparrow$ are formed (Cooper pairs)

- •The superconducting order parameter is a complex number with the absolute value equal to the gap, and the phase ϕ.

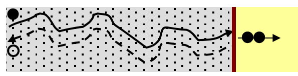
Andreev reflection



At a **normal metal – superconductor (NS) interface** the Andreev reflection is the basic process for charge conversion. An incident up-spin electron with energy $E=\varepsilon_F+\varepsilon$ *drags* with it a down-spin electron of energy $E=\varepsilon_F-\varepsilon$ to form a Cooper pair in the superconductor leaving behind a hole in the spin-down band.



In a bulk superconductor repeated Andreev reflections lead to the modification of the energy eigenstates, and the formation of the superconducting condensate with Cooper pairs.



In a normal metal coupled to a superconductor Andreev reflections introduce superconducting correlations, these are called **proximity effects**.

The process of Andreev reflection can be described by the Bogoliubov – de Gennes (BdG) equations. The BdG equations could be formally derived from the BCS theory of superconductivity.

"Electron – hole description" of superconductivity

Superconductivity is described by the BCS mean field Hamiltonian:

$$\hat{H}_S = \sum_k \xi_k \left(c^+_{k\uparrow}c_{k\uparrow} + c^+_{k\downarrow}c_{k\downarrow}\right) + \underbrace{\Delta c^+_{k\uparrow}c^+_{-k\downarrow} + \Delta^* c_{-k\downarrow}c_{k\uparrow}}_{\text{pairing}}, \qquad \Delta = V\left\langle c_{-k\downarrow}c_{k\uparrow}\right\rangle$$

V describes the attractive interaction, ξ is measured from the Fermi energy.

This hamiltonian is strange in the sense, that the pairing terms do not conserve the particle number. Usually the Hamiltonians contain c^+c -type terms. This structure can be recovered with the transformation:

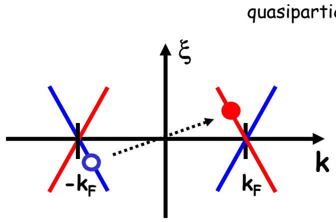
$$e_k = c_{k\uparrow}, \quad h_k = c^+_{-k\downarrow}$$

The "h^+" operator annihilates an electron, i.e. it creates a hole. The spin label is not necessary any more as the operators "e" stand for spin up electrons and the operators "h" stand for spin down "holes".

Let us first look at the normal state Hamiltonian (Δ=0) with the new set of operators:

$$\hat{H}_N = \sum_k \xi_k \left(e^+_k e_k - h^+_k h_k\right) + \sum_k \xi_k \qquad (h_k h^+_k + h^+_k h_k = 1)$$

A -k, down spin electron state is converted to k down spin hole state. The quasiparticle energy of the hole state is opposite to that of the electron state.



Electron state:
- -e charge
- $|k|>k_F \Rightarrow \xi_k>0$; $|k|<k_F \Rightarrow \xi_k<0$
- sign of k and v~dξ/dk are the same

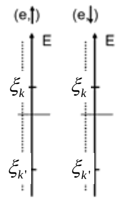
Hole state:
- +e charge
- $|k|>k_F \Rightarrow \xi_k<0$; $|k|<k_F \Rightarrow \xi_k>0$
- sign of k and v~dξ/dk are opposite! (positive k hole state propagates in negative direction!)

Vacuum state, ground state in the normal metal (Δ=0)

Electron description: $(|k|>k_F,\ |k'|<k_F)$

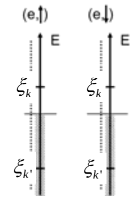
Vacuum state: no electron states are filled

$$c_k|0\rangle_e = 0$$



The ground state of a normal metal (Fermi sphere) is obtained by filling all the electron states below the Fermi energy

$$|G\rangle_N = \prod_{\xi_k<0} c^+_{k\uparrow} c^+_{k\downarrow}|0\rangle_e$$

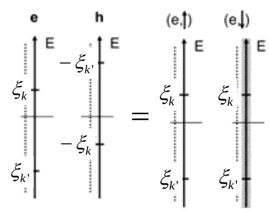


Electron-hole description:

New vacuum state in the electron-hole picture:

$$h_k|0\rangle_{eh} = 0,$$

$$e_k|0\rangle_{eh} = 0$$



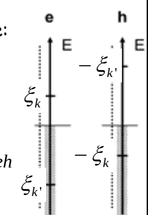
Transformation from the e to the e-h vacuum state:

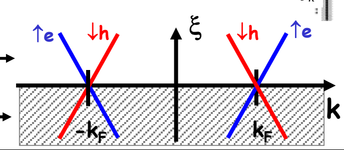
$$|0\rangle_{eh} = \prod_k c^+_{k\downarrow}|0\rangle_e$$

By filling all the states below the Fermi energy both in the electron and the hole band we get the same ground state:

$$|G\rangle_N = \prod_{\xi_k<0} e^+_k h^+_k |0\rangle_{eh}$$

$$|G\rangle_N = \prod_{\xi_k>0} c_{-k\downarrow} \prod_{\xi_k<0} c^+_{k\uparrow}|0\rangle_{eh}$$





Quasiparticles in the superconducting state

The Hamiltonian in the electron-hole representation (the constant $\Sigma\xi_k$ term is omitted) :

$$\hat{H}_S = \sum_k \xi_k\left(e^+_k e_k - h^+_k h_k\right) + \underbrace{\Delta e^+_k h_k + \Delta^* h^+_k e_k}_{\text{Andreev reflection}}$$

In this new representation the particle number is conserved (but the charge is not!). The interaction term corresponds to Andreev reflection, where an electron is converted to a hole travelling on the time-reversed path or vica versa.

In order to diagonalize the Hamiltonian new operators are introduced:

$$\begin{array}{l} \gamma_{k1} = u_k e_k + v_k h_k \\ \gamma_{k0} = -v^*_k e_k + u^*_k h_k \end{array}$$

$$\{\gamma_{k1},\gamma^+_{k1}\} = \{\gamma_{k1},\gamma^+_{k1}\} = 1$$

$$\Updownarrow$$

$$|u_k|^2 + |v_k|^2 = 1$$

$$\{\gamma_{k1},\gamma^+_{k0}\} = 0 \quad \text{(orthogonal states)}$$

$$\hat{H}_S = \sum_k E_k\left(\gamma^+_{k1}\gamma_{k1} - \gamma^+_{k0}\gamma_{k0}\right) + \eta_k\gamma^+_{k1}\gamma_{k0} + \eta^*_k\gamma^+_{k0}\gamma_{k1},$$

$$\begin{cases} E_k = \xi_k\left(|u_k|^2 - |v_k|^2\right) + \Delta u_k v^*_k + \Delta^* u^*_k v_k \\ \eta_k = -2\xi_k u_k v_k + \Delta u_k^2 - \Delta^* v_k^2 \end{cases}$$

The diagonalization condition is η_k=0. Taking u_k as a real number, and imposing E>0:

$$E_k = +\sqrt{\xi_k^2 + |\Delta|^2}$$

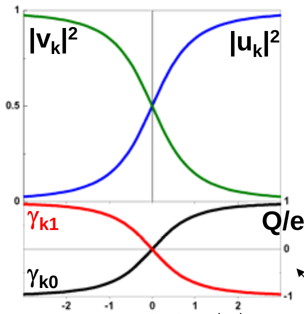
$$u_k = \frac{1}{\sqrt{2}}\left(1+\frac{\xi_k}{|E_k|}\right)^{1/2},\quad |v_k| = \frac{1}{\sqrt{2}}\left(1-\frac{\xi_k}{|E_k|}\right)^{1/2},\quad \frac{v_k}{|v_k|} = +\frac{\Delta}{|\Delta|}$$

The same phase for all k!

The quasiparticles of the superconductor (γ_{k0} and γ_{k1}) are coherent superpositions of electron and hole normal quasiparticles. For ξ_k>>$|\Delta|$ the normal state quasiparticles are recovered. For ξ_k=0 the quasiparticles are an equally weighted superposition of electron and hole normal quasiparticles.

A gap of 2Δ opens in the quasiparticle spectrum!

Superconducting ground state



$$\xi/|\Delta| \sim |k| - k_F$$

Occupation of the electron-hole states: around the Fermi energy the electron states are coupled to time reversed hole staes (Andreev reflection)

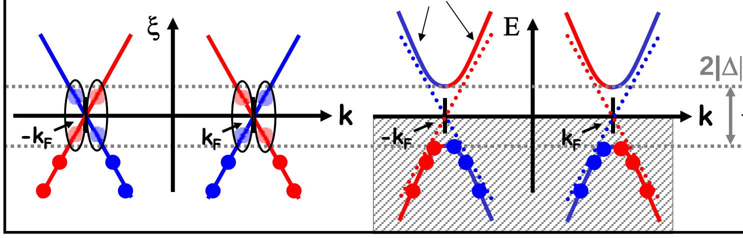
The superconducting ground state: all the negative energy states are occupied

$$|G\rangle_{SC} = \Pi\, \gamma_{k0}^{+}|0\rangle_{eh} = \Pi(-v_k^{*}e_k^{+} + u_k^{*}h_k^{+})|0\rangle_{eh} =$$

$$= \ldots = \Pi(-v_k^{*}c_{k\uparrow}^{+}c_{-k\downarrow}^{+} + u_k^{*})|0\rangle_{e} \leftarrow \text{BCS ground state}$$

The evolution of the quasiparticle charge as a function of ξ

Quasiparticle picture: The red parts are hole-like states in the sense that the sign of k is opposite to the sign of $v \sim dE_k/dk$. However, in all states the electron (e_k) and the hole (h_k) quasiparticles are superimposed, and energy is gained by the pairing. Similarly the blue parts are electron-like states.



The positive energy states are created by the γ^{+}_{k1} operator

Gap in the spectrum

The negative energy states are created by the γ^{+}_{k0} operator

Derivation of the BdG equation from the SC Hamiltonian (for a k-state)

From now on u,v is replaced by u*, v*!

$$|\psi_{k1}\rangle = \gamma_{k1}^{+}|0\rangle_{eh} = (u\, e_k^{+} + v\, h_k^{+})|0\rangle_{eh} \quad \Leftrightarrow \quad \psi_{k1}(r) = e^{ikr}\begin{pmatrix} u \\ v \end{pmatrix} \begin{matrix} \leftarrow \text{elektron} \\ \leftarrow \text{hole} \end{matrix}$$

$$\hat{H}_S = \sum_k \xi_k (e_k^{+}e_k - h_k^{+}h_k) + \Delta e_k^{+}h_k + \Delta^{*}h_k^{+}e_k$$

$$\hat{H}_S\psi_k = E\psi_k \qquad \Rightarrow \begin{pmatrix} \xi_k & \Delta \\ \Delta^{*} & -\xi_k \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = E\begin{pmatrix} u \\ v \end{pmatrix} \qquad \Rightarrow \begin{pmatrix} H_N & \Delta \\ \Delta^{*} & -H_N \end{pmatrix}\psi_{k0}(r) = E_k\psi_{k0}(r)$$

$$\hat{H}_N = \frac{\hat{p}^2}{2m^{*}} - \mu$$

Help: $e_k|0\rangle = h_k|0\rangle = 0, \quad e_k e_k^{+}|0\rangle = (1 - e_k^{+}e_k)|0\rangle = 1\cdot|0\rangle, \quad \{e_k, h_k\} = 0$

After some algebra: $\xi_k u e_k^{+}|0\rangle - \xi_k v h_k^{+}|0\rangle + \Delta v e_k^{+}|0\rangle + \Delta^{*} u h_k^{+}|0\rangle = E u e_k^{+}|0\rangle + E v h_k^{+}|0\rangle$

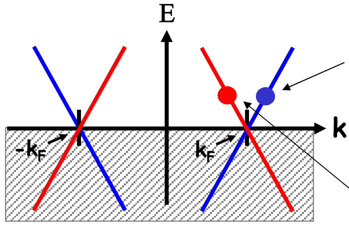
Similarly:

$$|\psi_{k0}\rangle = \gamma_{k0}|0\rangle_{eh} = (-v^{*}e_k^{+} + u^{*}h_k^{+})|0\rangle_{eh} \quad \Leftrightarrow \quad \psi_{k0}(r) = \begin{pmatrix} -v^{*} \\ u^{*} \end{pmatrix} e^{ikr}$$

$$\Rightarrow \begin{pmatrix} \hat{H}_N & \Delta \\ \Delta^{*} & -\hat{H}_N \end{pmatrix}\psi_{k0}(r) = -E_k\psi_{k0}(r)$$

Solution in a normal metal

We consider positive energy solutions, the negative energy states are occupied (Fermi sphere).



Electron eigenfunction:

$$\begin{pmatrix} p^2/2m-\varepsilon_F & 0 \\ 0 & -(p^2/2m-\varepsilon_F) \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}e^{ik_e x}=E\begin{pmatrix}1\\0\end{pmatrix}e^{ik_e x}$$

$$\frac{\hbar^2 k_e{}^2}{2m}-\frac{\hbar^2 k_F^2}{2m}=E \Rightarrow |k_e|>k_F \text{ at } E>0 \qquad k_e=k_F\sqrt{1+E/\varepsilon_F}$$

Hole eigenfunction:

$$\begin{pmatrix} p^2/2m-\varepsilon_F & 0 \\ 0 & -(p^2/2m-\varepsilon_F) \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}e^{ik_h x}=E\begin{pmatrix}0\\1\end{pmatrix}e^{ik_h x}$$

$$\frac{\hbar^2 k_F^2}{2m}-\frac{\hbar^2 k_h{}^2}{2m}=E \Rightarrow |k_h|<k_F \text{ at } E>0 \qquad k_h=k_F\sqrt{1-E/\varepsilon_F}$$

Solution in a superconductor

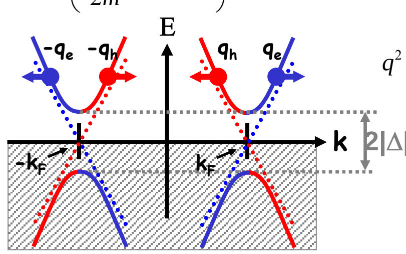
$$\begin{pmatrix} p^2/2m-\varepsilon_F & |\Delta|e^{i\phi} \\ |\Delta|e^{-i\phi} & -(p^2/2m-\varepsilon_F) \end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}e^{iqx}=E\begin{pmatrix}u\\v\end{pmatrix}e^{iqx}$$

$$\left(\frac{\hbar^2q^2}{2m}-\varepsilon_F-E\right)u+|\Delta|e^{i\phi}v=0$$

$$|\Delta|e^{i\phi}u-\left(\frac{\hbar^2q^2}{2m}-\varepsilon_F+E\right)v=0$$

$$v=-\frac{\left(\frac{\hbar^2q^2}{2m}-\varepsilon_F-E\right)}{|\Delta|e^{i\phi}}u$$

$$E^2(q)=\left(\frac{\hbar^2q^2}{2m}-\varepsilon_F\right)^2+|\Delta|^2$$



$$q^2=k_F^2\left(1\pm\frac{\sqrt{E^2-|\Delta|^2}}{\varepsilon_F}\right)$$

"+" $\Rightarrow$ electron-like, $|q_e|>k_F$

"−" $\Rightarrow$ hole-like, $|q_h|<k_F$

(for E<0 it is vice versa)

For $|E|<|\Delta|$:

$$q^2=k_F^2\left(1\pm i\underbrace{\frac{\sqrt{|\Delta|^2-E^2}}{\varepsilon_F}}_{<<1}\right)$$

komplex part of q -> evanescent mode

Eigenvectors: (u,v)=?
(actually it is enough to calculate v/u)

For $|E|>|\Delta|$:

$$\frac{v_e}{u_e}=\left(\frac{E\mp\mathrm{sgn}(E)\sqrt{E^2-|\Delta|^2}}{|\Delta|e^{i\phi}}\right)$$

For $|E|<|\Delta|$:

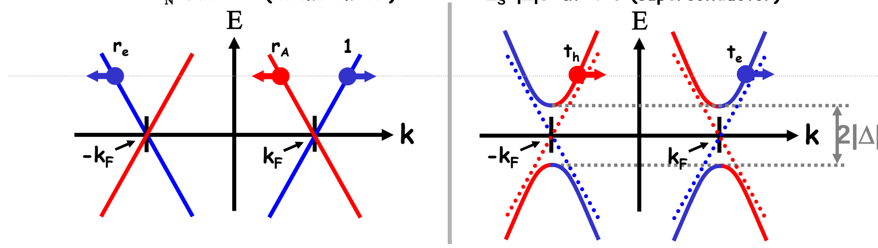
$$\frac{v_e}{u_e}=\left(\frac{E\mp\mathrm{sgn}(E)i\sqrt{|\Delta|^2-E^2}}{|\Delta|e^{i\phi}}\right)$$

"−" $\Rightarrow$ electron-like,
"+" $\Rightarrow$ hole-like

Solution for an NS-interface

NS interface at x=0, which has perfect transparency (T=1) if the superconductivity is suppressed

$\Delta_N=0$ at x<0 (normal metal) | $\Delta_S=|\Delta|e^{i\phi}$ at x>0 (superconductor)



$$\Psi_N(x)=1\begin{pmatrix}1\\0\end{pmatrix}e^{ik_e x}+r_e\begin{pmatrix}1\\0\end{pmatrix}e^{-ik_e x}+r_A\begin{pmatrix}0\\1\end{pmatrix}e^{ik_h x}$$

Incoming electron; Normal reflected el.; Andreev reflected hole

$$\Psi_S(x)=t_e\begin{pmatrix}u_e\\v_e\end{pmatrix}e^{iq_e x}+t_h\begin{pmatrix}u_h\\v_h\end{pmatrix}e^{-iq_h x}$$

Transmitted electron and hole-like quasiparticles

Matching the wave functions:

$\Psi_N(0)=\Psi_S(0)$ | $\Psi'_N(0)-\Psi'_S(0)=0$

$1+r_e=t_e u_e+t_h u_h \quad (1)$ | $k_e-r_e k_e=t_e u_e q_e-t_h u_h q_h$

$r_A=t_e v_e+t_h v_h \quad (2)$ | $r_A k_h=t_e v_e q_e-t_h v_h q_h$

Andreev approximation: $\Delta \ll E_F \Rightarrow k_e\approx k_h\approx q_e\approx q_h$ → $1-r_e=t_e u_e-t_h u_h \quad (3)$, $r_A=t_e v_e-t_h v_h \quad (4)$

$(2)-(4)\Rightarrow 2t_h v_h=0\Rightarrow t_h=0$

$(1)-(3)\Rightarrow 2r_e=2t_h u_h=0$

$(1)+(3)\Rightarrow 2=2t_e u_e\Rightarrow t_e=1/u_e$

$(2)+(4)\Rightarrow 2r_A=2t_e v_e=2v_e/u_e$

$$r_e=0,\ r_A=v_e/u_e$$

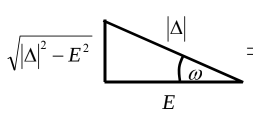
Andreev reflection amplitude for perfectly transmitting barrier

For $|E|>|\Delta|$:

$$r_A(E,\phi)=\frac{v_e}{u_e}=\frac{E-\mathrm{sgn}(E)\sqrt{E^2-|\Delta|^2}}{|\Delta|e^{i\phi}}=\begin{cases}\dfrac{|E|-\sqrt{E^2-|\Delta|^2}}{|\Delta|}e^{-i\phi}, & E>0\\[2ex] \dfrac{|E|-\sqrt{E^2-|\Delta|^2}}{|\Delta|}e^{i(\pi-\phi)}, & E<0\end{cases}$$

For $|E|<|\Delta|$:

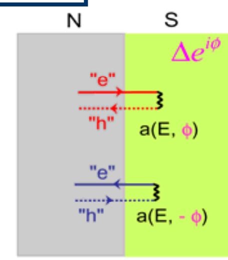
$$r_A(E,\phi)=\frac{v_e}{u_e}=\frac{E-\mathrm{sgn}(E)i\sqrt{|\Delta|^2-E^2}}{|\Delta|}e^{-i\phi}=(\cos\omega-i\sin\omega)e^{-i\phi}=e^{-i(\omega+\phi)}\Rightarrow\begin{cases}|r_A(E,\phi)|=1\\ \arg(r_A(E,\phi))=-\arccos(E/|\Delta|)-\phi\end{cases}$$



$$\Rightarrow \frac{|E|}{|\Delta|}=\cos\omega,\quad \frac{\sqrt{|\Delta|^2-E^2}}{|\Delta|}=\sin\omega$$

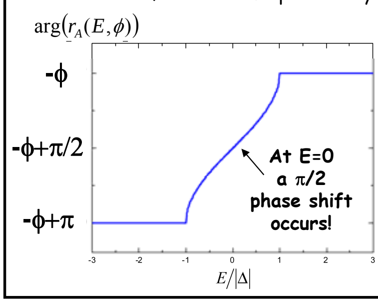
Andreev reflection amplitude for perfectly transmitting barrier -summary

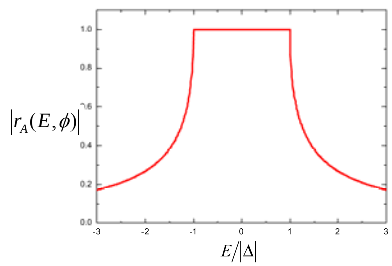
- For a perfectly transmitting channel there is no normal reflection (r_e=0)
- For |E|<|Δ| the Andreev reflection probability is 1.
- At E=0 a phase shift of -ϕ+π/2 occurs
- For an incoming electron the Andreev reflection amp. is $r_A(E,\phi)$, whereas for an incoming hole it is $r_A(E,-\phi)$



In an NS junction with perfect transmission the normal reflection is prohibited, and all the incoming electrons are Andreev reflected with a probability of one. → For each incoming electron a charge of 2e is transmitted

$$G_{NS} = 2G_N$$

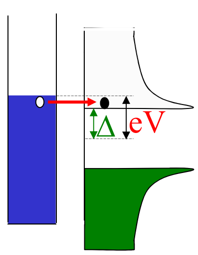


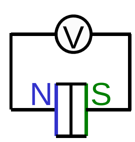


Opposite limit: a weakly transmitting tunnel barrier between the N and S electrode

For a tunnel barrier with low transmission (T<<1) the Andreev reflection is suppressed (the probability for the transmission of 2e charges ~T^2), thus the transport is dominated by the direct quasiparticle tunneling at |eV|>|Δ|.

The dI/dV curve of the NS tunnel junction directly measures the superconducting density of states!



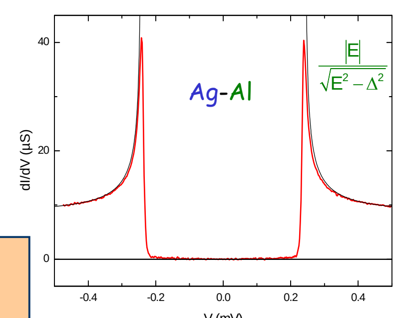


$$I^+ \sim T \cdot \int \mathrm{d}\varepsilon\, g_N(\varepsilon - eV)\, f_N(\varepsilon - eV) \cdot g_S(\varepsilon)(1 - f_S(\varepsilon))$$

$$I^- \sim T \cdot \int \mathrm{d}\varepsilon\, g_S(\varepsilon)\, f_S(\varepsilon) \cdot g_N(\varepsilon - eV)(1 - f_N(\varepsilon - eV))$$

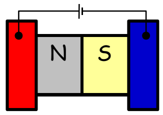
$$I = I^+ - I^- \sim T \cdot g_N(\varepsilon_F) \int \mathrm{d}\varepsilon\, g_S(\varepsilon)\left(f_N(\varepsilon - eV) - f_S(\varepsilon)\right)$$

$$\frac{\mathrm{d}I}{\mathrm{d}V} \sim T \cdot g_N(\varepsilon_F) \int \mathrm{d}\varepsilon\, g_S(\varepsilon)\, f_N'(\varepsilon - eV) \overset{T=0}{\approx} T \cdot g_N(\varepsilon_F) \cdot g_S(eV)$$



BTK theory

(conductance of a ballistic NS junction)

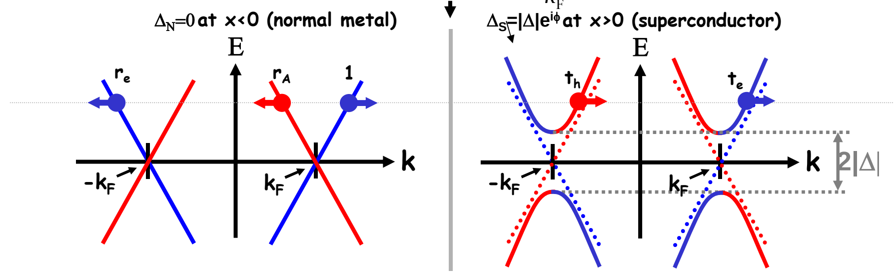


G.E. Blonder, M.Tinkham, T.M.Klapwijk, PRB **25**, 4515 (1982)

In a realistic junction between a normal metal and a superconductor a finite interface scattering has to be considered.

The BTK theory calculates the I-V curve of an NS junction by modeling the interface scattering with a Dirac-delta potential described by a dimensionless scattering strength, Z.

$$V(x) = Z\frac{2E_F}{k_F}\delta(x)$$ dimensionless „barrier strength"



$$\Psi_N(x) = 1\begin{pmatrix}1\\0\end{pmatrix}e^{ik_e x} + r_e\begin{pmatrix}1\\0\end{pmatrix}e^{-ik_e x} + r_A\begin{pmatrix}0\\1\end{pmatrix}e^{ik_h x}$$

$$\Psi_S(x) = t_e\begin{pmatrix}u_e\\v_e\end{pmatrix}e^{iq_e x} + t_h\begin{pmatrix}u_h\\v_h\end{pmatrix}e^{-iq_h x}$$

Andreev reflection normal reflection quasiparticle transmission

Matching the wave functions: $\Psi_N(0) = \Psi_S(0) \equiv \Psi(0)$, $\Psi'_N(0) - \Psi'_S(0) = Z\frac{2E_F}{k_F}\frac{2m}{\hbar^2}\Psi(0)$

Reflection probabilities:

The probability for Andreev reflection: $A = |r_A|^2$

The probability for normal reflection: $B = |r_e|^2$

$E < \Delta$	$E > \Delta$
$A = \frac{\Delta^2}{E^2 + (\Delta^2 - E^2)(1+2Z^2)^2}$	$A = \frac{\varepsilon^2 - 1}{[\varepsilon + (1+2Z^2)]^2}$
$B = 1 - A$	$B = \frac{4Z^2(1+Z^2)}{[\varepsilon + (1+2Z^2)]^2}$

$$\varepsilon = \frac{E}{\sqrt{E^2 - \Delta^2}}$$

-For Z=0 and E<Δ all the incoming electrons are Andreev reflected

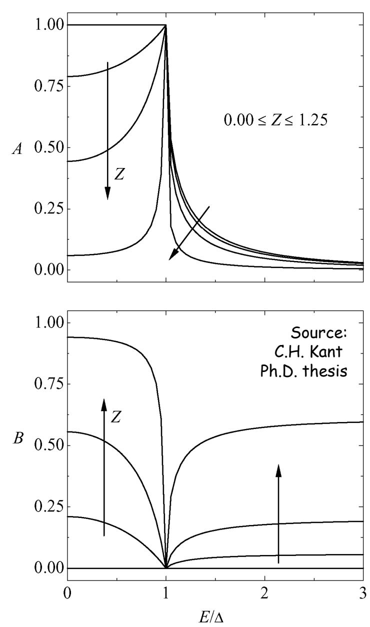
-At E<Δ the probability for quasiparticle transmission is zero, i.e. A+B=1.

-At E>>Δ A=0 and the probability for normal reflection is:

$$B(E >> \Delta) = R_N = 1 - T_N = \frac{Z^2}{1+Z^2}$$

$$\Rightarrow T_N = \frac{1}{1+Z^2}; \quad Z^2 = \frac{1-T_N}{T_N}$$

-The Andreev reflection probability at zero energy: $A(E=0) = R_A = \frac{1}{(1+2Z^2)^2} = \left(\frac{T_N}{2-T_N}\right)^2$



Calculation of the current

Let us calculate the current at the normal side:

$$I = eS\int v(E)\rho(E)\left[1 + A(E) - B(E)\right]\left[f(E - eV) - f(E)\right]dE$$

↑ The area of the contact (S)

The conductance, G_{NS}=dI/dV:

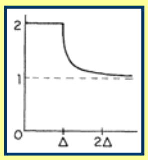
$$G_{NS} = -e^2 S v_F \rho_F \int \left[1 + A(E) - B(E)\right] f'(E - eV)dE$$

The normal state conductance (Δ=0):

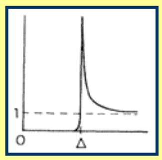
$$G_{NN} = \frac{e^2 S v_F \rho_F}{1 + Z^2}$$

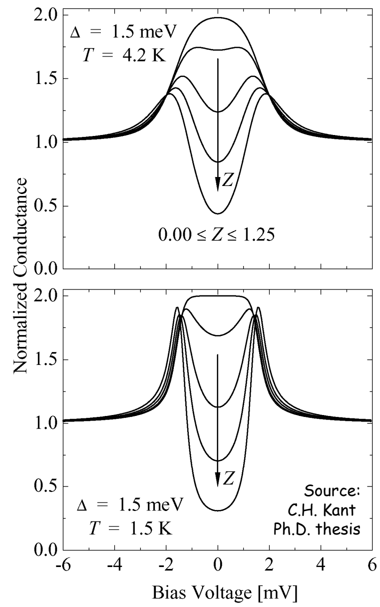
$$\frac{G_{NS}}{G_{NN}} = -(1 - Z^2)\int \left[1 + A(E) - B(E)\right] f'(E - eV)dE$$

Z=0 limit: $G(eV \ll \Delta)=2G_N$, for each incoming electron a hole is reflected, and a charge of 2e is transmitted

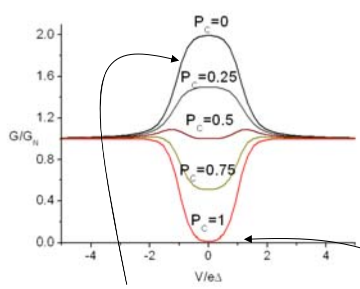


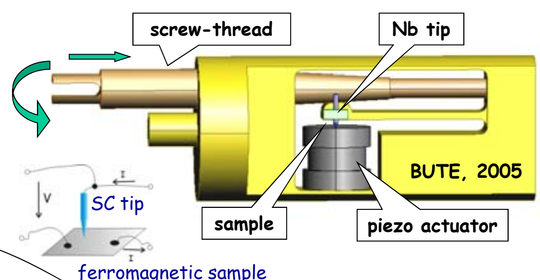
Z>>1 limit: conventional NIS tunneling curve, $G(eV<\Delta)=0$, sharp peak at Δ





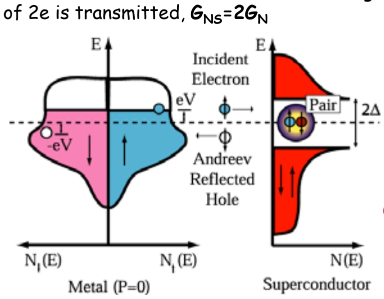
Application: measurement of spin-polarization with SF contact





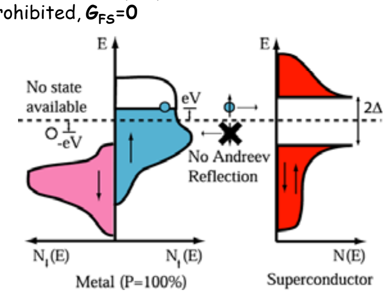
For a normal metal with P=0 an incoming electron is Andreev reflected, thus for each incoming e⁻ a charge of 2e is transmitted, $G_{NS}=2G_N$

In a half-metal (P=1) Andreev reflection is prohibited, $G_{FS}=0$

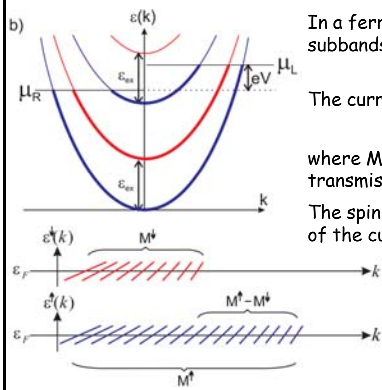


The fit of the I-V curves tells the spin-polarization!

As a first approx.: $G_{FS}(V=0)=2(1-P_C)G_N$



Spin polarization of the current (Landauer picture)



In a ferromagnet an exchange splitting arises between the two spin subbands: $\varepsilon_n^\sigma(k) = \varepsilon(k) + \varepsilon_n - \sigma\varepsilon_{\text{ex}}, \quad \sigma = \pm 1/2$

The current for the two spin subbands: $I^\sigma = \dfrac{e^2}{h} M^\sigma \overline{T}^\sigma V$

where M is the number of open channels, and $\overline{T}$ is the average transmission.

The spin polarization of the current: $P_C = \dfrac{I^\uparrow - I^\downarrow}{I^\uparrow + I^\downarrow} = \dfrac{M^\uparrow \overline{T}^\uparrow - M^\downarrow \overline{T}^\downarrow}{M^\uparrow \overline{T}^\uparrow + M^\downarrow \overline{T}^\downarrow}$

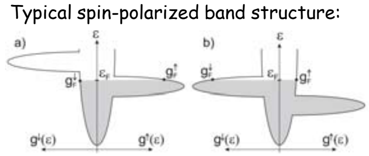
For large M and no spin scattering:

$$\overline{T}^\uparrow \approx \overline{T}^\downarrow \;\Rightarrow\; P_C \approx \frac{M^\uparrow - M^\downarrow}{M^\uparrow + M^\downarrow}$$

Formally:

$$M^\sigma = \frac{2\pi\hbar}{L}\sum_{n=1}^{M^\sigma} v_n^\sigma g_n^\sigma, \quad v_n^\sigma = \frac{\partial \varepsilon_n^\sigma(k)}{\hbar \partial k}, \quad g_n^\sigma = \frac{L}{2\pi}\left(\frac{\partial \varepsilon_n^\sigma(k)}{\partial k}\right)^{-1}$$

Typical spin-polarized band structure:



Average Fermi velocity weighted by the DOS of the channels:

$$\overline{v}_F^\sigma = \frac{\sum_n g_n^\sigma v_n^\sigma}{\sum_n g_n^\sigma} = \frac{\sum_n g_n^\sigma v_n^\sigma}{g_F^\sigma}$$

← total DOS

Note: magnetization accounts for all the spins below the Fermi energy whereas spin polarization is a Fermi surface property. They can even have opposite sign.

With these the current spin polarization: $P_C \approx \dfrac{g_F^\uparrow \overline{v}_F^\uparrow - g_F^\downarrow \overline{v}_F^\downarrow}{g_F^\uparrow \overline{v}_F^\uparrow + g_F^\downarrow \overline{v}_F^\downarrow}$

Inclusion of spin polarization in the BTK theory

Spin polarization on the N side can be considered as a sum of fully polarized and unpolarized currents:

$$I = I_\uparrow + I_\downarrow = \underbrace{2I_\downarrow}_{I_{unpol}} + \underbrace{(I_\uparrow - I_\downarrow)}_{I_{pol}} \qquad P_C = \frac{I^\uparrow - I^\downarrow}{I^\uparrow + I^\downarrow} = \frac{I_{\text{pol}}}{I}$$

$$G(V,T,P_C,Z) = (1-P_C)G_{unpol}(V,T,Z) + P_C G_{pol}(V,T,Z)$$

For the unpolarized current the original BTK result is used. In the polarized current the Andreev reflection is suppressed, $A \to \widetilde{A} = 0$

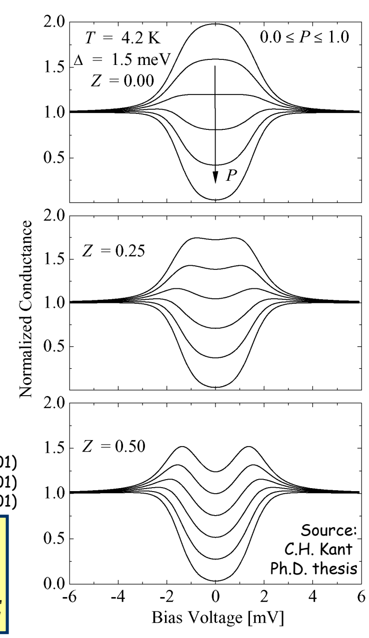
The probability for the normal reflection is rescaled to preserve current conservation: $B \to \widetilde{B}$

Where $\widetilde{B}$ is the reflection probability for those electrons that are not Andreev reflected:

$$\Rightarrow \; \widetilde{B} = \frac{B}{1-A}$$

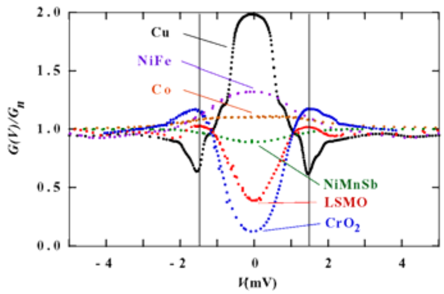
For more details see: G. J. Strijkers et al. Phys. Rev. B **63**, 104510 (2001)
I. I. Mazin et al. J. Appl. Phys. **89**, 7576 (2001)
Y. Ji etal. Phys. Rev. B **64**, 224425 (2001)

$$\frac{G_{\text{NS}}}{G_{\text{NN}}} = -P_{\text{C}}(1-Z^2)\int\left[1-\widetilde{B}(E)\right]f'(E-eV)\mathrm{d}E - (1-P_{\text{C}})(1+Z^2)\int\left[1+A(E)-B(E)\right]f'(E-eV)\mathrm{d}E$$



First measurements: tip-sample approach

R. J. Soulen Jr., J. M. Byers,* M. S. Osofsky, B. Nadgorny, T. Ambrose, S. F. Cheng, P. R. Broussard, C. T. Tanaka, J. Nowak, J. S. Moodera, A. Barry, J. M. D. Coey, Science **282**, 85 (1998)



Material studied	Point	Base	N	P_T (%)	P_C (%)
NiFe	Nb	$Ni_{0.8}Fe_{0.2}$ film	14	25 ± 2	37 ± 5
Co	Nb	Co foil	7	35 ± 3	42 ± 2
Fe	Ta	Fe film	12	40 ± 2	45 ± 2
	Fe	Ta foil	14		46 ± 2
	Nb	Fe film	4		42 ± 2
	Fe	V crystal	10		45 ± 2
Ni	Nb	Ni foil	4	23 ± 3	46.5 ± 1
	Nb	Ni film	5		43 ± 2
	Ta	Ni film	8		44 ± 4
NiMnSb	Nb	NiMnSb film	9	–	58 ± 2.3
LSMO	Nb	$La_{0.7}Sr_{0.3}MnO_3$ film	14	–	78 ± 4.0
CrO_2	Nb	CrO_2 film	9	–	90 ± 3.6

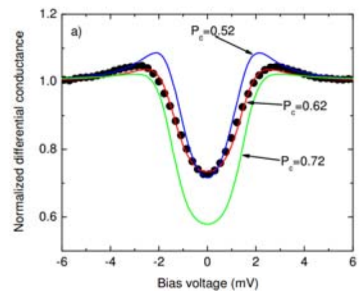
One of the first studies demonstrating the Andreev spectroscopy technique for various ferromagnetic metals.

The spin-polarization is determined by the simple formula:

$$G_{FS}(V=0)=2(1-P_C)G_N$$

Probably rather large diffusive contacts were studied, as the BTK theory would not give good fit to the curves.

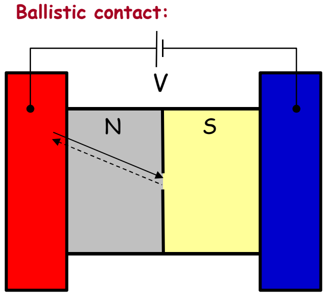
More reliable result: fitting by the modified BTK model:



Proximity effect (why shall we use ballistic contacts?)

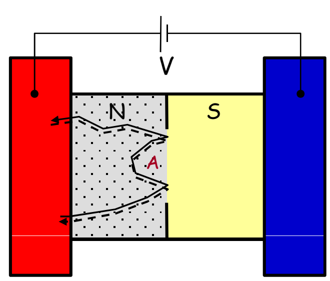
Proximity effect: the Andreev reflection introduces superconducting correlations at the normal side. The Andreev reflected hole is travelling on the time-reversed path of the incoming electron, thus the electron and the hole form phase-conjugated pairs.

Ballistic contact:



In a ballistic contact the reflected hole travels back to the reservoir, where it thermalizes. The incoming states at the NS interface all have the distribution of the left electrode, and no superconducting correlations are present.

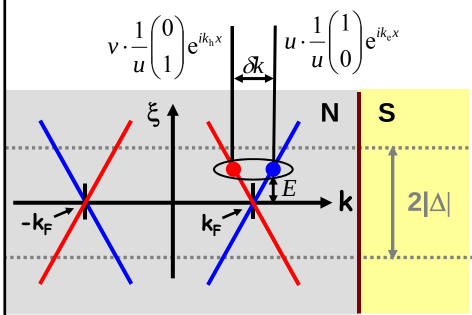
Diffusive contact:



In a diffusive contact an electron and the Andreev reflected hole can bounce back and forth on the same trajectory between different points of the contact, causing a coherent superposition.

Proximity effect - coherence length

The incoming electron and the Andreev reflected hole forms a coherent pair with relative amplitudes u,v. Assuming k_e=k_h their relative phase is just determined by the phase of v/u at any position. Superconducting correlations enter the normal metal -> proximity effect.



Due to the small difference k_e-k_h=δk the phase coherence of the electron-hole pair is lost after a certain distance:

$$\delta k \cdot s \approx 1 \Rightarrow s \approx \frac{1}{\delta k} = \frac{\hbar v_F}{2E} \Rightarrow \tau \approx \frac{\hbar}{2E}$$

Where s is the length along the trajectory required to loose phase coherence, and τ is the corresponding time.

The distance from the barrier, at which phase coherence is lost in the normal metal:

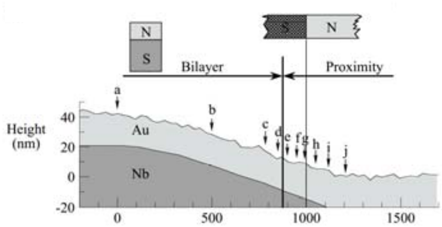
In a ballistic contact: $x_N = v_F \tau = \frac{\hbar v_F}{2E}$

In a diffusive contact: $x_N = \sqrt{D\tau} = \sqrt{\frac{\hbar D}{2E}}$

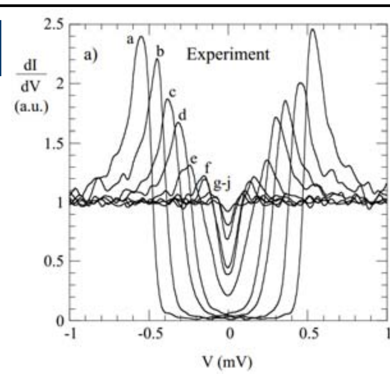
In the worst case, E=Δ the coherence length is:

Pippard coherence length in standard theory of superconductivity → $x_N^{\text{ballistic}} = \frac{\hbar v_F}{2\Delta}, \quad x_N^{\text{diffusive}} = \sqrt{\frac{\hbar D}{2\Delta}}$

Proximity effect - minigap



Forrás: N. Moussy et. al. Europhys. Lett. 55, 861 (2001)



In the normal metal a *minigap* is obeserved by tunneling spectroscopy (with an STM), i.e. a samller gap than that of the bulk superconductor. The larger the distance from the NS barrier the smaller the minigap.

$x_N = \sqrt{\frac{\hbar D}{2E}}$ At smaller E (E<Δ) the coherence is preserved at larger distance - naturally at larger distance smaller energy window will show SC correlations

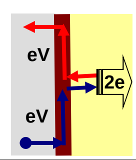
Even at E=0 the coherence length will be finite:

Dephasing due to temperature (2E=kT): $x_N^{kT} = \sqrt{\frac{\hbar D}{kT}}$

Dephasing due to the bias voltage (2E=2eV): $x_N^{eV} = \sqrt{\frac{\hbar D}{2eV}}$ ←

Decoherence due to inelastic scattering: $x_N^{in} = \sqrt{D\tau_{\text{in}}}$

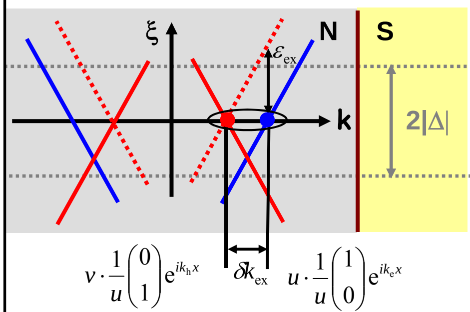
Note: at finite bias both the *e* and the reflected *h* gain an energy eV along the barrier, i.e. a 2eV energy difference arises between the electron-hole pair

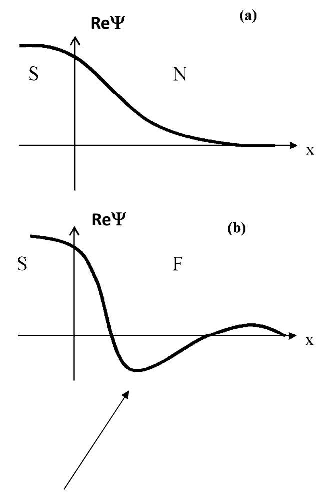


Proximity effects - ferromagnetic electrode

A.I. Buzdin, Rev. Mod. Phys. **77** 935 (2005)

In a ferromagnet the two spin subbands are splitted by an exchange enegy, ε_{ex}. This introduces a wavenumber difference between the incoming electron and the Andreev reflected hole even at E=0.





The relative amplitude of the *e* and *h* wavefunction:

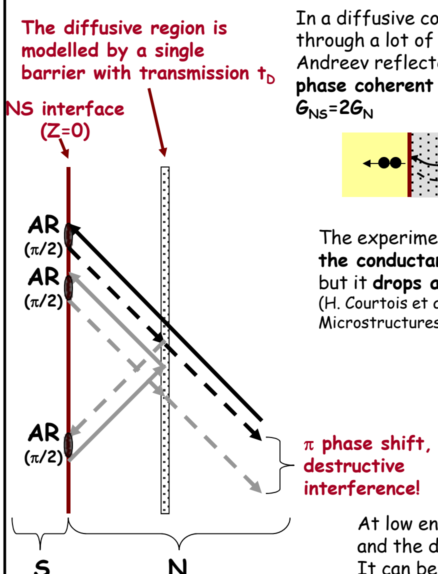
$$\frac{v}{u} e^{i(k_h - k_e)x} = \frac{v}{u} e^{\frac{i\varepsilon_{ex} x}{\hbar v_F}}$$

The order parameter oscillates with the distance from the barrier!

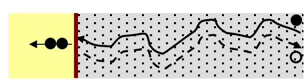
Oscillation of the order parameter (in S Ψ is considered to be a real number)

Proximity effects 1.: reentrance

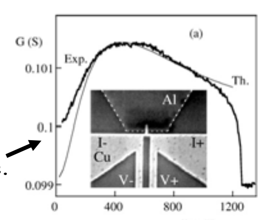
C.W.J. Beenakker, cond-mat/9909293, T.M. Klapwijk, Journal of Superconductivity **17**, 593 (2004), C.W.J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997)



In a diffusive contact, the incoming electron reaches the interface through a lot of scatterings, however in a semiclassical picture the Andreev reflected hole comes back on the time-reversed path, thus **a fully phase coherent NS junction is expected to be completely transparent, $G_{NS}=2G_N$**



The experiments, however show, that **the conductance** increases below the T_c, but it **drops at low enough temperature**.
(H. Courtois et al., Superlattices and Microstructures **25**, 721 (1999))

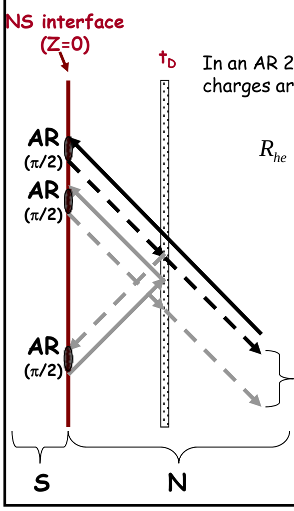


The incoming electron acquires a phase ϕ, whereas the Andeev reflected hole on the time-reversed path acqires a phase –ϕ, but **the Andreev reflection causes a phase shift of π/2**, thus the net phase between the two paths is π!

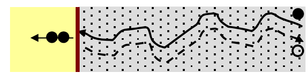
At low enough temperature the coherence length increases, and the destructive interference becomes important.
It can be shown, that at T=0 $G_{NS}=G_N$!

Proximity effects 1.: reentrance

C.W.J. Beenakker, cond-mat/9909293, T.M. Klapwijk, Journal of Superconductivity **17**, 593 (2004), C.W.J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997)



The zero-bias conductance of an NS junction:



$$G_{NS} = 2\cdot\frac{2e^2}{h} N R_{he}$$

In an AR 2 electron charges are transmitted — Number of channels — The probability that an incoming e^- is Andreev reflected as a hole

$$R_{he} \approx \left| t_D^* i t_D + t_D^* i r_D i r_D^* i t_D + \ldots \right|^2 = \left| \frac{i t_D^* t_D}{1 - i i r_D^* r_D} \right|^2 = \frac{T_D^2}{\left(1+R_D\right)^2} = \frac{T_D^2}{\left(2-T_D\right)^2}$$

$$G_{NS} = 2\cdot\frac{2e^2}{h}\sum_n \frac{T_n^2}{\left(2-T_n\right)^2} \overset{!}{<} 2G_N = 2\cdot\frac{2e^2}{h}\sum_n T_n$$

Averaging with random matrix theory:

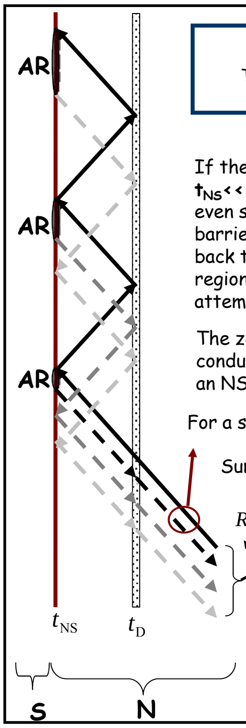
π phase shift, destructive interference!

$$G_{NS} = 2\cdot\frac{2e^2}{h}\left\langle \sum_n \frac{T_n^2}{\left(2-T_n\right)^2} \right\rangle = G_N$$

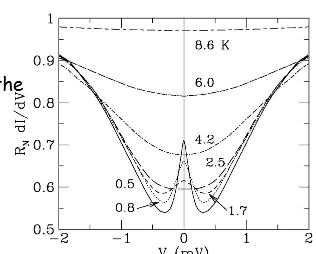
Note: without the phase shift of „i" $G_{NS}=2G_N$ would come!

Proximity eff. 2.: reflectionless tunneling

T.M. Klapwijk, Journal of Superconductivity **17**, 593 (2004), C.W.J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997)



If the NS interface has small transmission, t_{NS}**<<1**, the amplitude of Andreev reflection is even smaller, $r_A \sim t^2_{NS}$ (2 electron charges cross the barrier). However, the electron can be reflected back to the NS interface by the disordered region several times, thus it can repeatedly attempt the Andreev reflection.



A. Kastalsky et al. Phys. Rev. Lett. **67**, 3026 (1991)

The zero-bias conductance of an NS junction: $G_{NS} = 2\cdot\frac{2e^2}{h} N R_{he}$

For a single process: $R_{he} = \left| t_D^* r_A t_D \right|^2 = R_A T_D^2 \approx T_{NS}^2 T_D^2$

The conductance is considerably larger!

Summing up the multiple attempts: (the phase is same for all!)

$$R_{he} \approx \left| t_D^* r_A t_D + t_D^* r_{NS}^* r_D^* r_A r_D r_{NS} t_D + \ldots \right|^2 = \left| \frac{t_D^* r_A t_D}{1 - r_{NS}^* r_{NS} r_D^* r_D} \right|^2 \approx \frac{T_{NS}^2 T_D^2}{\left(T_{NS} + T_D - T_{NS}T_D\right)^2} \overset{T_{NS} >> T_D}{\approx} T_{NS} T_D^2$$

General statement: (Beenakker, Rev. Mod. Phys. **69**, 731 (1997))

$$R_{NS}^{\text{class}} = \frac{h}{2e^2 N}\left(g_N^{-1} + 2T_{NS}^{-2}\right)$$

$$R_N^{\text{class}} = \frac{h}{2e^2 N}\left(g_N^{-1} + T_{NS}^{-1}\right)$$

$$T_{NS} >> g_N$$

$$R_{NS}(B=0, V=0) \approx R_N^{\text{class}}$$

$$R_{NS}(B>0, V=0) \approx R_{NS}^{\text{class}}$$

S_1 I S_2

Josephson effect (traditional approach)

$\psi_1 = \sqrt{\rho_1} e^{i\phi_1}$ $\quad \psi_2 = \sqrt{\rho_2} e^{i\phi_2}$ Macroscopic wave functions. $|\psi|^2 \sim$ particle density (ρ) + phase difference ($\delta = \phi_2 - \phi_1$)

We apply a voltage of eV on the junction!

$$i\hbar \frac{d\psi_1}{dt} = \frac{2eV}{2}\psi_1 + T\psi_2 \;\Rightarrow\; i\hbar\left(\frac{1}{2\sqrt{\rho_1}}\dot{\rho}_1 e^{i\phi_1} + \sqrt{\rho_1} e^{i\phi_1} i\dot{\phi}_1\right) = \frac{2eV}{2}\sqrt{\rho_1} e^{i\phi_1} + T\sqrt{\rho_2} e^{i\phi_2}$$

$$i\hbar \frac{d\psi_2}{dt} = -\frac{2eV}{2}\psi_2 + T\psi_1 \;\Rightarrow\; \ldots$$

Dividing by $e^{i\phi 1}$ (or $e^{i\phi 2}$) and writing the equations separately for the real and imaginary part:

$$\dot{\rho}_1 = \frac{2T}{\hbar}\sqrt{\rho_1\rho_2}\sin\delta, \quad \dot{\rho}_2 = -\frac{2T}{\hbar}\sqrt{\rho_1\rho_2}\sin\delta$$

$$\dot{\phi}_1 = -\frac{T}{\hbar}\sqrt{\frac{\rho_2}{\rho_1}}\cos\delta - \frac{2eV}{2\hbar}, \quad \dot{\phi}_2 = -\frac{T}{\hbar}\sqrt{\frac{\rho_1}{\rho_2}}\cos\delta + \frac{2eV}{2\hbar}$$

The current is proportional to $d\rho_1/dt = -d\rho_2/dt$: $\quad I = I_0 \sin\delta$

Subtracting the equations for the phase: $\quad \dot{\delta} = \frac{2eV}{\hbar} \;\Rightarrow\; \delta(t) = \delta_0 + \frac{2e}{\hbar}\int V(t)\,dt$

Josephson equations

S_1 I S_2

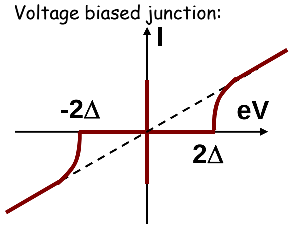
Josephson effect (traditional approach)

$$\delta = \phi_2 - \phi_1$$

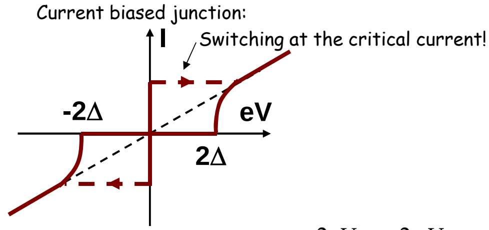
Applying a constant bias voltage: $\quad I = I_0 \sin\left(\delta_0 + \frac{2eVt}{\hbar}\right)$ An AC current with $\omega = 2eV/\hbar$ is flowing. The DC current averages to zero.

At zero bias voltage a maximal suppercurrent of I_0 can flow between the two sides!

Voltage biased junction:



Current biased junction:



Let us superimpose an AC (microwave) voltage on a DC voltage!

$$V(t) = V_0 + V_1\cos\omega t \;\Rightarrow\; \delta(t) = \delta_0 + \frac{2eV_0}{\hbar}t + \frac{2e}{\hbar}\frac{V_1}{\omega}\sin\omega t$$

$$I = I_0\sin\left(\delta_0 + \frac{2eV_0}{\hbar}t + \frac{2e}{\hbar}\frac{V_1}{\omega}\sin\omega t\right) \approx I = I_0\sin\left(\delta_0 + \frac{2eV_0}{\hbar}t\right) + I_0\frac{2e}{\hbar}\frac{V_1}{\omega}\sin(\omega t)\cos\left(\delta_0 + \frac{2eV_0}{\hbar}t\right) =$$

$$= I_0\sin\left(\delta_0 + \frac{2eV_0}{\hbar}t\right) + I_0\frac{2e}{\hbar}\frac{V_1}{2\omega}\left(\sin\left(\omega t - \delta_0 - \frac{2eV_0}{\hbar}t\right) + \sin\left(\omega t + \delta_0 + \frac{2eV_0}{\hbar}t\right)\right)$$

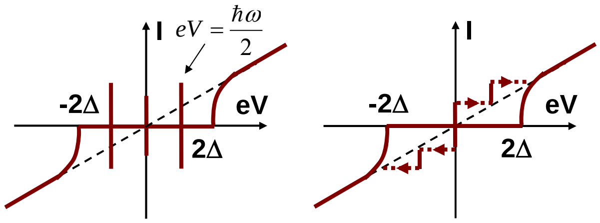
At $V_0 = \pm\hbar\omega/2e$ a DC current will flow!

If we would expand in higher order, we would get that DC current can flow at: $\quad V_0 = \pm\frac{\hbar\omega\cdot n}{2e}$, where n is an integer

S₁ I S₂

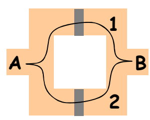
Shapiro steps, SQUID

Shapiro resonances at voltage and current bias:



Superconducting quantum interferometer device (SQUID):

Two Josephson junctions in parallel in a "loop" geometry. The loop encloses a magnetic flux of Φ



The superconductor has a well-defined phase at every position. -> The pase difference between A and B is constant for all trajectories.

$$(\phi_B - \phi_A)_1 = \delta_1 + \frac{2e}{\hbar}\int_1 \mathbf{A}d\mathbf{s} \quad = \quad (\phi_B - \phi_A)_2 = \delta_2 + \frac{2e}{\hbar}\int_2 \mathbf{A}d\mathbf{s}$$

$$\Rightarrow \delta_2 - \delta_1 = \frac{2e}{\hbar}\oint \mathbf{A}d\mathbf{s} = \frac{2e}{\hbar}\Phi = 2\cdot 2\pi\frac{\Phi}{\Phi_0}$$

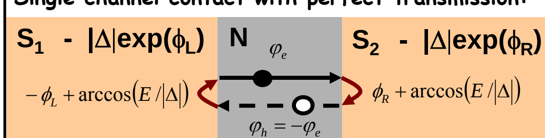
Let us take: $\delta_1 = \delta_0 + \frac{e}{\hbar}\Phi,\ \delta_2 = \delta_0 - \frac{e}{\hbar}\Phi$

$$I = I_1 + I_2 = I_0[\sin(\delta_0 + e\Phi/\hbar) + \sin(\delta_0 - e\Phi/\hbar)] = 2I_0 \sin\delta_0 \cos(e\Phi/\hbar)$$

The maximal value of the critical current is tuned by the magnetic flux: $I_{\max} = 2I_0|\cos(e\Phi/\hbar)|$

Andreev bound states in a short SNS junction with phase bias

Single channel contact with perfect transmission:



The electron-hole par can bounce back and forth in the normal region.

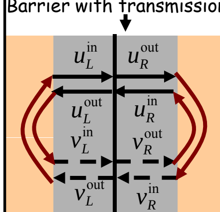
If the aquired phase is $n2\pi$ (constructive interference) a bound state is formed.

The condition for the bound state: $\phi_R - \phi_L + 2\arccos(E/|\Delta|) = n2\pi \Rightarrow$ $E_\pm = \pm|\Delta|\cos(\delta/2)$

$\delta = \phi_R - \phi_L$

Single channel contact with arbitrary transmission:

Barrier with transmission τ



Normal scattering for the electrons:

$$\begin{pmatrix} u_L^{out} \\ u_R^{out} \end{pmatrix} = \underbrace{\begin{pmatrix} -ir & t \\ t & -ir \end{pmatrix}}_{\underline{\underline{S}}} \begin{pmatrix} u_L^{in} \\ u_R^{in} \end{pmatrix}, \quad t, r \text{ are real}, \quad t^2 = \tau,\ r^2 = 1-\tau$$

Normal scattering for holes:

$$\begin{pmatrix} v_L^{out} \\ v_R^{out} \end{pmatrix} = \underline{\underline{S}}^* \begin{pmatrix} v_L^{in} \\ v_R^{in} \end{pmatrix}$$

Andreev reflection at the left:

$$\begin{pmatrix} v_L^{in} \\ u_L^{in} \end{pmatrix} = \begin{pmatrix} r_A(E,\phi_L) & 0 \\ 0 & r_A(E,-\phi_L) \end{pmatrix} \begin{pmatrix} u_L^{out} \\ v_L^{out} \end{pmatrix},$$

Andreev reflection at the right:

$$\begin{pmatrix} v_R^{in} \\ u_R^{in} \end{pmatrix} = \begin{pmatrix} r_A(E,\phi_R) & 0 \\ 0 & r_A(E,-\phi_R) \end{pmatrix} \begin{pmatrix} u_R^{out} \\ v_R^{out} \end{pmatrix}$$

These can be combined to: $\begin{pmatrix} u_L^{in} \\ u_R^{in} \end{pmatrix} = \underline{\underline{M}} \begin{pmatrix} u_L^{in} \\ u_R^{in} \end{pmatrix}, \quad \underline{\underline{M}} = \begin{pmatrix} r_A(E,-\phi_L) & 0 \\ 0 & r_A(E,-\phi_R) \end{pmatrix} \underline{\underline{S}}^* \begin{pmatrix} r_A(E,\phi_L) & 0 \\ 0 & r_A(E,\phi_R) \end{pmatrix} \underline{\underline{S}}$

The condition to get nonzero solution: $\det(\underline{\underline{M}} - \underline{\underline{1}}) = 0 \Rightarrow$ $E_\pm = \pm|\Delta|\sqrt{1 - \tau\sin^2(\delta/2)}$

Current - phase relation (V=0, phase biased junction)

Reminder: calculation of the current in a 1D conductor: $I = \frac{e}{L}\sum_{k,\sigma} v_{k,\sigma} f(\varepsilon_{k,\sigma})$

In an Andreev bound state a singly occupied energy state is considered, which carries 2e charge: $\Rightarrow I = \frac{2e}{L} v$

The velocity can be obtained as: $v = \partial E / \partial p$ As $E = f(p - 2eA) \Rightarrow \partial E/\partial p = -(\partial E/\partial A)/2e$

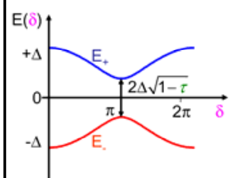
$\Rightarrow I = -\frac{1}{L}\frac{\partial E}{\partial A}$ However, A and the superconducting phase, ϕ are not invariant for a gauge transformation. The energy must be a function of a gauge invariant function of A and ϕ.

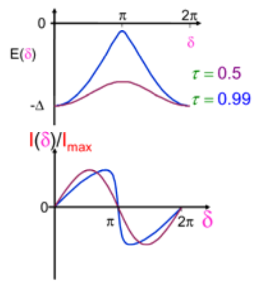
$A' = A + \nabla\chi \Rightarrow \phi' = \phi + 2e\chi/\hbar$ $\Rightarrow \nabla\phi - 2eA/\hbar$ is the proper gauge invariant quantity

$$E = g(\nabla\phi - 2eA/\hbar) \Rightarrow \partial E/\partial A = -\frac{2e}{\hbar}\frac{\partial E}{\partial \underbrace{\nabla\phi}_{(\phi_R-\phi_L)/L}}$$

$$\Rightarrow I(\delta) = \frac{2e}{\hbar}\frac{\partial E}{\partial \delta}$$

$$E_{\pm} = \pm|\Delta|\sqrt{1-\tau\sin^2(\delta/2)}$$





Current phase relation for the occupied (negative energy) Andreev bound state:

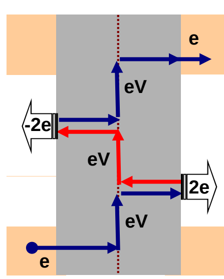
$$I(\delta) = \frac{e|\Delta|}{2\hbar}\frac{\tau\sin\delta}{\sqrt{1-\tau\sin^2(\delta/2)}}$$

For a tunnel junction (τ<<1):

$$I(\delta) = \frac{e|\Delta|}{2\hbar}\tau\sin\delta$$

We get back the Josephson relation!

A more complete descrition at finite bias : Multiple Andreev Reflections



At finite bias the electrons and holes gain or loose an energy of eV when they cross the barrier. We start a quasiparticle with E<–Δ from the left side. It gains an energy eV, but if E+eV<Δ it can only be Andreev reflected as a hole. Going backward the hole gains an energy of eV, and so on. Finally the energy will increase above +Δ and a quasiparticle can leave to the electrodes.

The amplitude of the incoming quasiparticle:

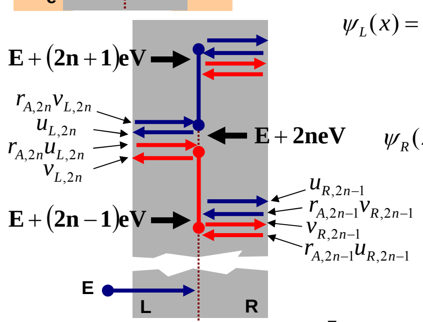
$$J(E) = \sqrt{1-|r_{A(E,0)}|^2}$$

The Andreev reflection amplitude: (ϕ=0, it is included in the timedependence!)

$$r_{A,2n} = r_A(\varepsilon = E + 2neV, \phi = 0)$$

Wavefunctions at the left and right side of the barrier:

$$\psi_L(x) = \begin{pmatrix} \sum_n \left((r_{A,2n}v_{L,2n} + J(E)\delta_{n,0})e^{ik_e x} + u_{L,2n}e^{-ik_e x}\right)e^{-i(E+2neV)t/\hbar} \\ \sum_n \left(r_{A,2n}u_{L,2n}e^{-ik_h x} + v_{L,2n}e^{ik_h x}\right)e^{-i(E+2neV)t/\hbar} \end{pmatrix}$$



$$\psi_R(x) = \begin{pmatrix} \sum_n \left(r_{A,2n+1}v_{R,2n+1}e^{-ik_e x} + u_{R,2n+1}e^{ik_e x}\right)e^{-i(E+(2n+1)eV)t/\hbar} \\ \sum_n \left(r_{A,2n+1}u_{R,2n+1}e^{ik_h x} + v_{R,2n+1}e^{-ik_h x}\right)e^{-i(E+(2n+1)eV)t/\hbar} \end{pmatrix}$$

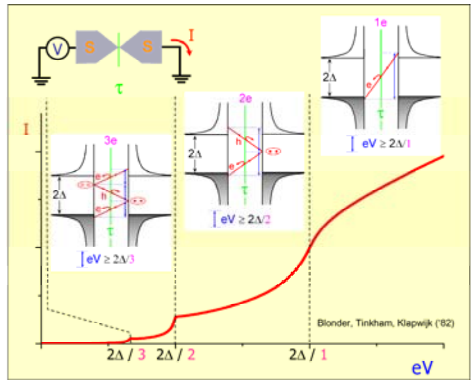
Scattering on the barrier:

$$\begin{pmatrix} u_{L,2n} \\ u_{R,2n+1} \end{pmatrix} = \underline{\underline{S}} \begin{pmatrix} r_{A,2n}v_{L,2n} \\ r_{A,2n+1}v_{R,2n+1} \end{pmatrix}; \quad \begin{pmatrix} v_{L,2n} \\ v_{R,2n-1} \end{pmatrix} = \underline{\underline{S}}^* \begin{pmatrix} r_{A,2n}u_{L,2n} \\ r_{A,2n-1}u_{R,2n-1} \end{pmatrix}$$

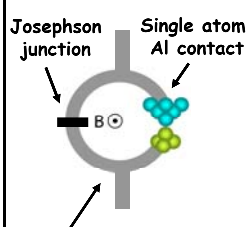
The DC current: $$I_{DC}(V) = \frac{2e^2}{h}\left[V - \frac{1}{e}\int dE\left(J(E)r_{A,0}(v_{L,0}^* + v_{L,0}) + \sum_n \left(1+|r_{A,2n}|^2\right)\left(|v_{L,2n}|^2 - |u_{L,2n}|^2\right)\right)\right]$$

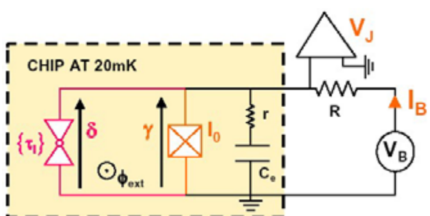
Multiple Andreev Reflections - qualitative picture

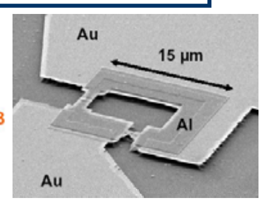
- If $eV>2\Delta$ the quasiparticle started from the left side gains enough energy to reach an empty qausiparticle state on the right side. In this process a single electron charge is transmitted with probability τ.
- If $eV>\Delta$ single qausiparticle transmission is prohibited, but with a single Andreev reflection the charge transfer is already possible. In this process 2e charges are transmitted with probability of τ^2 (the carriers cross twice the barrier)
- At $eV>2\Delta/n$ an n^{th} process with a charge transfer of ne and a probability τ^n becomes available
- Accordingly at the I-V curve shows singularities at $eV=2\Delta/n$.
- If we have a junction with a few conductance channels all the transmission probabilities can be determined by placing the junction between superconducting electrodes and fitting the "subgap" structures in the I-V curve.



Measurement of the current-phase relation



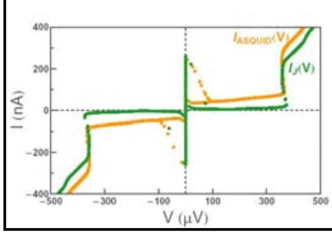


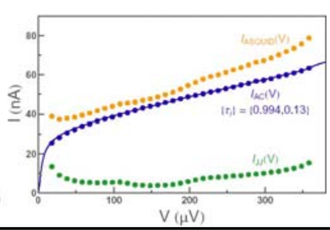


The device is an "atomic SQUID" (ASQUID), i.e. an SC atomic contact (AC) and a Josephson junction (JJ) in parallel. The critical current is much larger for the JJ than for the AC.

To know the transmission probabilities of the AC the subgap structures must be measured in the I-V curve. This is done with a voltage biased measurement. In principle the DC current of the JJ should be zero at finite DC bias, thus the I-V curve would come purely from the AC. In reality the JJ also has structures in the I-V curve due to interferences with the environment. The I-V curve of the JJ can be separately measured by completely breaking the junction.

$$I_{AC}(V) = I_{ASQUID}(V) - I_{JJ}(V)$$

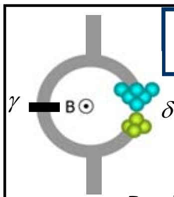




After subtracting the I-V curve of the JJ from that of the ASQUID the transmission probabilities of the AC can be determined.

Measurement of the current-phase relation

M. Chauvin, The Josephson Effect in Atomic Contacts, Ph.D. thesis (2005)



The phase difference on the atomic contact is: $\delta = \gamma + \underbrace{4\pi \cdot \Phi / \Phi_0}_{\varphi}$

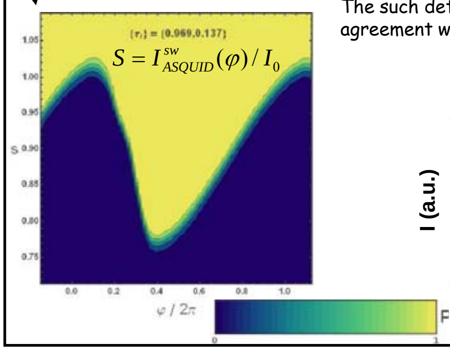
At zero temperature the JJ switches out of its zero bias state at $\gamma=\pi/2$ (at a critical current I_0), therefore the critical current of the ASQUID is: $I^{0}_{ASQUID}(\varphi) = I_0 + I_{AC}(\varphi + \pi/2)$

I.e. the critical current of the JJ is modulated by the current phase relation of the AC.

I.e. the critical current of the JJ is modulated by the current phase relation of the AC.
In experiment we measure the mean switching current instead of the critical current, which is a thermally activated stochastic variable. Short current pulses are applied on the sample and the distribution of the switching current is measured.

$$I^{sw}_{ASQUID}(\varphi) = I^{sw}_{JJ} + I_{AC}(\varphi + \pi/2)$$

The average value of the JJ switching current can be independently measured with an open AC.



The such determined current-phase relation shows remarkable agreement with the theory: