Chapter 5. Global and Local Metrics

- *Why did Einstein take seven years to go from special relativity to general relativity?*
- *Why are so many different kinds of flat maps used to plot Earth's curved surface?*
- *Why use coordinates at all? Why not just measure distances directly, say with a ruler?*
- *Why does the spacetime metric use differentials?*
- *Are Schwarzschild global coordinates the only way to describe spacetime around a black hole?*

CHAPTER 5

Global and Local Metrics

Edmund Bertschinger & Edwin F. Taylor *

The basic demand of the special theory of relativity (invariance of the laws under Lorentz-transformations) is too narrow, i.e., that an invariance of the laws must be postulated relative to nonlinear *transformations for the co-ordinates in the four-dimensional continuum.*

This happened in 1908. Why were another seven years required for the construction of the general theory of relativity? ***The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning.***

—Albert Einstein [boldface added]

5.1 ■ EINSTEIN'S PERPLEXITY

Why seven years between special relativity and general relativity?

Einstein's seven-year puzzle

It took Albert Einstein seven years to solve the puzzle compressed into the two-paragraph quotation above. The first paragraph complains that special relativity (with its restriction to flat spacetime coordinates) is too narrow. Einstein demands that a *nonlinear* coordinate system—that is, one that is *arbitrarily stretched*—should also be legal. *Nonlinear* means that it can be stretched by different amounts in different locations.

Stretch coordinates arbitrarily.

In the second paragraph, Einstein explains his seven-year problem: He tried to apply to a stretched coordinate system the same rules used in special relativity. Einstein's phrase **immediate metrical meaning** describes something that can be measured directly—for example, the radar-measured distance between the top of the Eiffel Tower and the Paris Opera building. Einstein says that since we can use nonlinear stretched coordinates, these coordinate separations need not be something we can measure directly, for example with a ruler.

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

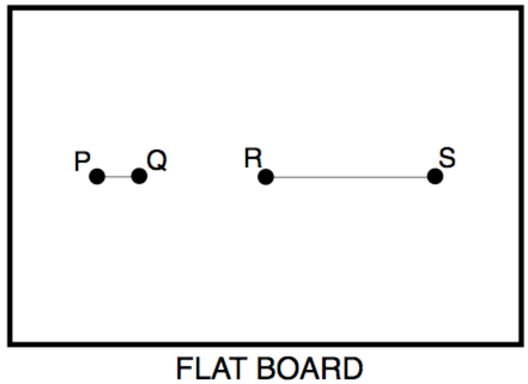

FIGURE 1 Compare distances between two different pairs of points on a flat wooden cutting board. First measure with a ruler the distance between the pair of points P and Q. Then measure the distance between the pair of points R and S. Measured distance PQ is *smaller* than the measured distance RS. We require no coordinate system whatsoever to verify this inequality; we measure distances directly on a flat surface.

Solving Einstein's puzzle leads to the global metric.

What is the relation between the coordinate separations between two points and the directly-measured distance between those two points? How does this distinction affect predictions of special and general relativity? Answering these questions reveals the unmeasurable nature of global coordinate separations, but nevertheless the central role of the *global metric* in connecting different local inertial frames in which we carry out measurements.

5.2 ■ EINSTEIN'S PERPLEXITY ON A WOODEN CUTTING BOARD

Move beyond high school geometry and trigonometry!

Simplify: From curved spacetime to a flat cutting board.

We transfer Einstein's puzzle from spacetime to space and—to simplify further—measure the distance between two points on the flat surface of a wooden cutting board (Figure 1).

Measure distance directly, with a ruler.

A pair of points, P and Q, lie near to one another on the surface. A second pair of points, R and S, are farther apart than points P and Q. How do we know that distance RS is greater than distance PQ? We measure the two distances directly, with a ruler. To ensure accuracy, we borrow a ruler from the local branch of the National Institute of Standards and Technology. Sure enough, with our official centimeter-scale ruler we verify distance RS to be greater than distance PQ. *We do not need any coordinate system whatsoever to measure distance PQ or distance RS or to compare these distances on a flat surface.*

Difference in Cartesian coordinates verifies difference in distances.

Next, apply coordinates to the flat surface. Do not draw coordinate lines directly on the cutting board; instead spread a fishnet over it (Figure 2). When we first lay down the fishnet, its narrow strings look like Cartesian square coordinate lines. Adjacent strings are one centimeter apart. The x-coordinate separation between P and Q is 1 centimeter, and the x-coordinate separation

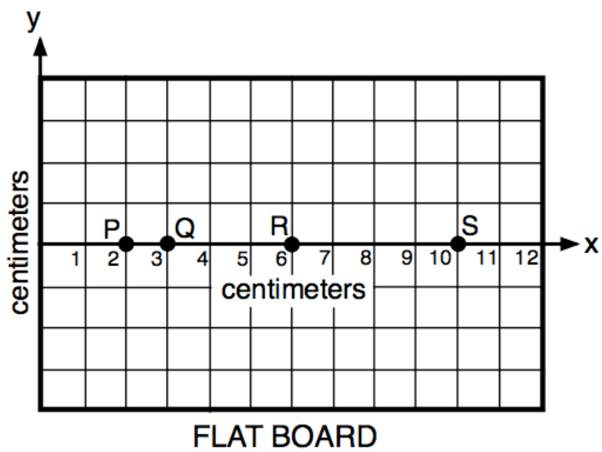

FIGURE 2 A fishnet with one-centimeter separations covers the wooden cutting board. Expressed in these coordinates, the coordinate separation PQ is 1 centimeter, while the coordinate separation RS is 4 centimeters. In this case a coordinate separation *does* have "an immediate metrical meaning" in Einstein's phrase. *Interpretation:* In this case we can derive from coordinate separations the values of directly-measured distances.

between R and S is 4 centimeters, confirming the inequality in our direct distance measurements. In this case each difference (or separation) in Cartesian coordinates, PQ and RS, *does* have "an immediate metrical meaning;" in other words, it corresponds to the *directly-measured distance.*

Stretch fishnet by variable amounts in x-direction.

Moving ahead, suppose that instead of string, we make the fishnet out of rubber bands. As we lay the rubber band fishnet loosely on the cutting board, we do something apparently screwy: As we tack down the fishnet, we stretch it along the x-direction by different amounts at different horizontal positions. Figure 3 shows the resulting "stretch" coordinates along the x-direction.

"Stretch" coordinate separation not equal to measured distance.

Now check the x-coordinate difference between P and Q in Figure 3, a difference that we call Δx_{PQ}. Then $\Delta x_{\text{PQ}} = 5 - 2 = 3$. Compare this with the x-coordinate separation between R and S: $\Delta x_{\text{RS}} = 10 - 9 = 1$. Lo and behold, the coordinate separation Δx_{PQ} is *greater* than the coordinate separation Δx_{RS}, even though our directly-measured distance PQ is *less* than the distance RS. This contradiction is the simplest example we can find of the great truth that Einstein grasped after seven years of struggle: *coordinate separations need not be directly measurable.*

Stretch coordinates form a legal map.

"No fair!" you shout. "You can't just move coordinate lines around arbitrarily like that." Oh yes we can. Who is to prevent us? Any coordinate system constitutes a **map**. What is a map? Applied to our cutting board, a map is simply a rule for assigning numbers that uniquely specify the location of every individual point on the surface. Our coordinate system in Figure 3 does that job nicely; it is a legal and legitimate map. However, the amount of stretching—what we call the **map scale**—varies along the x-direction.

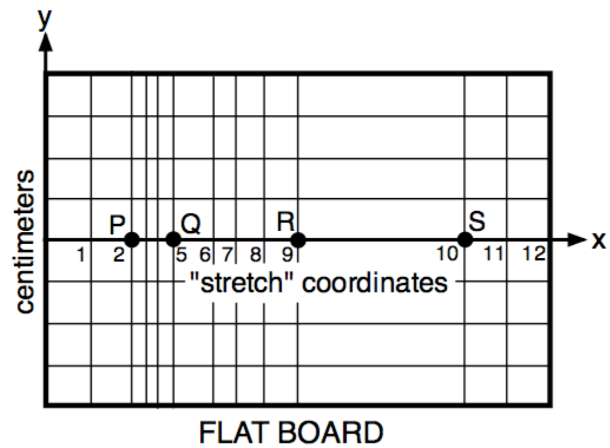

FIGURE 3 Global coordinate system that covers our entire cutting board, but in this case made with a rubber fishnet tacked down so as to stretch the x separation of fishnet cords by different amounts at different locations along the horizontal direction. The coordinate separation $\Delta x_{\text{PQ}} = 3$ between points P and Q is greater than the coordinate separation $\Delta x_{\text{RS}} = 1$ between points R and S, even though the measured *distances* between each of these pairs show the reverse inequality. Einstein was right: In this case coordinate separations do *not* have "an immediate metrical meaning;" in other words, coordinate separations do *not* tell us the values of directly-measured distances.

Of course, for convenience we usually *choose* the map scale to be everywhere uniform, as displayed in Figure 2. This choice is perfectly legal. We call this legality of Cartesian coordinates Assertion 1:

Assertion 1 for a FLAT SURFACE: CAN draw map with everywhere-uniform map scale.

Assertion 1. ON A FLAT SURFACE IN SPACE, we CAN FIND a global coordinate system such that every coordinate separation IS a directly-measured distance.

Standard Cartesian (x, y) coordinates allow us to use the power of the Pythagorean Theorem to predict the directly-measured distance s between two points anywhere on the board in Figure 2:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 \qquad \text{(flat surface: } \textit{Choose} \text{ Cartesian coordinates.)} \quad (1)$$

Cartesian separations: Pythagoras works!

The coordinate separations Δx and Δy and the resulting measured distance Δs can be as small or as large as we want, as long as the map scale is uniform everywhere on the flat cutting board.

In contrast, we *cannot* apply the Pythagorean Theorem using the "stretch" coordinates in Figure 3 to find the distance between a pair of points that are far apart in the x-direction. Why not? Because a large separation between two points can span regions where the map scale varies noticeably, that is, where rubber bands stretch by substantially different amounts. For example in Figure 3, the x-coordinate separation between points Q and S on

Stretch coordinates: Pythagoras fails on a flat surface.

the flat surface is $\Delta x_{\rm QS} = 5$, whereas points P and S have a much greater x-coordinate separation: $\Delta x_{\rm PS} = 8$. This is true even though the directly-measured *distance* between P and S is only slightly greater than the directly-measured *distance* between Q and S.

Stretched-fishnet coordinates of Figure 3, provide a case in which the Pythagorean Theorem (1) gives incorrect answers—coordinate separations are *not* the same as directly-measured distances. This yields Assertion 2, an alternative to Assertion 1:

Assertion 2 for a FLAT SURFACE: We are FREE to choose variable map scale over the surface.

Assertion 2. ON A FLAT SURFACE IN SPACE, we are FREE TO CHOOSE a global coordinate system for which coordinate separations ARE NOT directly-measured distances.

5.3 ■ GLOBAL SPACE METRIC FOR A FLAT SURFACE

Space metric to the rescue.

How can we predict measured distances using arbitrary coordinates? *Answer:* The metric!

Einstein tells us that we are free to stretch or contract conventional (in this case Cartesian) coordinates in any way we want. But if we do, then the resulting coordinate separations lose their "immediate metrical meaning;" that is, a coordinate separation between a pair of points no longer predicts the distance we measure between these points. If the coordinate separation can no longer tell us the distance between two points, what can? Our simple question about space on a flat cutting board is a preview of the far more profound question about spacetime with which Einstein struggled: How can we predict the measured wristwatch time τ or the measured ruler distance σ between a pair of events using the differences in *arbitrary* global coordinates between them? The answer was a breakthrough: "The metric!" Here's the path to that answer, starting with our little cutting board.

Space metric gives differential ds from differentials dx and dy.

Begin by recognizing that very close to any point on the flat surface the coordinate scale is nearly uniform, with a multiplying factor (local map scale) to correct for the local stretching in the x-coordinate. Strictly speaking, the coordinate scale is uniform only vanishingly close to a given point. *Vanishingly close?* That phrase instructs us to use the vanishingly small calculus limit: differential coordinate separations. For the coordinates of Figure 3, we find the differential distance ds from a **global space metric** of the form:

$$ds^2 = F(x_{\rm stretch})dx^2_{\rm stretch} + dy^2_{\rm stretch} \qquad \text{(variable } x\text{-stretch)} \qquad (2)$$

To repeat, we use the word *global* to emphasize that x is a valid coordinate everywhere across our cutting board covered by the stretched fishnet. In (2), $F(x)$—actually the square root of $F(x)$—is the map scale that corrects for the stretch in the horizontal coordinate *differentially close to that value of* x. If $F(x)$ is defined everywhere on the cutting board, however, then equation (2) is also valid at every point on the board.

Metric works well LOCALLY, even with stretched coordinates.

The global space metric is a tremendous achievement. On the right side of metric (2) the function $F(x)$ corrects the squared differential dx^2_{stretch} to give the correct squared differential distance ds^2 on the left side.

We have gained a solution to Einstein's puzzle for the simplified case of differential separations on a flat surface in space. But we seem to have suffered a great loss as well: calculus insists that the differential distance ds predicted by the space metric is vanishingly small. We cannot use our official centimeter-scale ruler to measure a vanishingly small differential distance. How can we possibly predict a measured distance—for example the distance between points P and S on our flat cutting board? We want to predict and then make *real* measurements on *real* flat surfaces!

Differential distance ds is too small to measure. . .

. . . but we can predict measured distance from summed (integrated) ds.

Differential calculus curses us with its stingy differential separations ds, but integral calculus rescues us. We can sum ("integrate") differential distances ds along the curve. The result is a predicted *total distance* along the curved path, a prediction that we can verify with a tape measure. As a special case, let's predict the distance s along the straight horizontal x-axis from point P to point S in Figure 3. Call this distance s_{PS}. "Horizontal" means no vertical, so that $dy = 0$ in equation (2). The distance s_{PS} is then the sum (integral) of $ds = [F(x)]^{1/2}\,dx$ from $x = 2$ to $x = 10$, where the scale function $[F(x)]^{1/2}$ varies with the value of x:

$$s_{\text{PS}} = \int_{x=2}^{x=10} [F(x_{\text{stretch}})]^{1/2}\,dx_{\text{stretch}} \qquad \text{(horizontal distance: P to S)} \qquad (3)$$

When we evaluate this integral, we can once again use our official centimeter-scale ruler to verify by direct measurement that the total distance s_{PS} between points P and S predicted by (3) is correct.

The example of metric (2) leads to our third important assertion:

Assertion 3 for a FLAT SURFACE: Metric gives us ds, whose integral predicts measured distance s.

<u>Assertion 3.</u> ON A <u>FLAT</u> SURFACE IN SPACE when using a global coordinate system for which coordinate separations ARE NOT directly-measured distances, a space metric is REQUIRED to give the differential distance ds whose integrated value predicts the measured distance s between points.

5.4 ■ GLOBAL SPACE METRIC FOR A CURVED SURFACE

Squash a spherical map of Earth's surface onto a flat table? Good luck!

In Sections 5.2 and 5.3, we chose variably-stretched coordinates on a flat surface. Then we corrected the effects of the variable stretching using a metric. This is a cute mathematical trick, but who cares? We are not *forced* to use stretched coordinates on a flat cutting board, so why bother with them at all? To answer these questions, apply our ideas about maps to the curved surface of Earth. Chapter 2 derived a global metric—equation (3), Section 2.3—for the spherical surface of Earth using angular coordinates λ for latitude and ϕ

for longitude, along with Earth's radius R. Here we convert that global metric to coordinates x and y:

$$\begin{aligned} ds^2 &= R^2 \cos^2 \lambda \, d\phi^2 + R^2 d\lambda^2 \qquad (0 \leq \phi < 2\pi \text{ and } -\pi/2 \leq \lambda \leq \pi/2) \qquad (4) \\ &= \cos^2\left(\frac{R\lambda}{R}\right)(Rd\phi)^2 + (Rd\lambda)^2 \qquad (\text{metric: Earth's surface}) \\ &= \cos^2\left(\frac{y}{R}\right) dx^2 + dy^2 \qquad (0 \leq x < 2\pi R \text{ and } -\pi R/2 \leq y \leq \pi R/2) \end{aligned}$$

On a sphere, we define $y \equiv R\lambda$ and $x \equiv R\phi$ (the latter from the definition of radian measure).

Undistorted flat maps of Earth impossible.

Compare the third line of (4) with equation (2). The y-dependent coefficient of dx^2 results from the fact that as you move north or south from the equator, lines of longitude converge toward a single point at each pole. That coefficient of dx^2 makes it impossible to cover Earth's spherical surface with a flat Cartesian map without stretching or compressing the map at some locations.

A curved surface *forces* us to use stretched coordinates.

Throughout history, mapmakers have struggled to create a variety of flat projections of Earth's spherical surface for one purpose or another. But each projection has *some* distortion. *No uniform projection of Earth's surface can be laid on a flat surface without stretching or compression in some locations.* If this is impossible for a spherical Earth with its single radius of curvature, it is certainly impossible for a general curved surface—such as a potato—with different radii of curvature in different locations. In brief, it is impossible to completely cover a curved surface with a single Cartesian coordinate system. (Is a cylindrical surface curved? No; technically it is a flat surface, like a rolled-up newspaper, which Cartesian coordinates can map exactly.) We bypass formal proof and state the conclusion:

Assertion 4 for a CURVED SURFACE: Everywhere-uniform map scale is IMPOSSIBLE.

<u>Assertion 4.</u> ON A <u>CURVED</u> SURFACE IN SPACE, it is IMPOSSIBLE to find a global coordinate system for which coordinate separations EVERYWHERE on the surface are directly-measured distances.

The dy on the third line of equation (4) is still a directly-measured distance: the differential distance northward from the equator. That is true for a sphere, whose constant R-value allows us to define $y \equiv R\lambda$. But Earth is not a perfect sphere; rotation on its axis results in a slightly-bulging equator. Technically the Earth is an **oblate spheroid**, like a squashed balloon. In that case neither x or y coordinate separations are directly-measured distances. And most curved surfaces are more complex than the squashed balloon. Einstein was right: In most cases coordinate separations *cannot* be directly-measurable distances.

Metric required on curved surface.

No possible uniform map scale over the entire surface of Earth? Then there is an inevitable distinction between a coordinate separation and measured distance. The space metric is no longer just an option, but has become the indispensable practical tool for predicting distances between two points from their coordinate separations.

Assertion 5 for a CURVED SURFACE: Metric REQUIRED to calculate distance.

> **Assertion 5. ON A CURVED SURFACE IN SPACE, a global space metric is REQUIRED to calculate the differential distance ds between a pair of adjacent points from their differential coordinate separations.**

As before, integrating the differential ds yields a measured total distance s along a path on the curved surface, whose predicted length we can verify directly with a tape measure.

Space summary

> **SPACE SUMMARY:** *On a flat surface in space we can choose Cartesian coordinates, so that the Pythagorean theorem—with no differentials—correctly predicts the distance s between two points far from one another. On a curved surface we cannot. But on any curved surface we can use a space metric to calculate ds between a pair of adjacent points from values of the differential coordinate separations between them. Then we can integrate these differentials ds along a given path in space to predict the directly-measured length s along that path.*

"Connectedness" = **topology.**

The combination of global coordinates plus the global metric is even more powerful than our summary implies. Taken together, the two describe a curved surface completely. In principle we can use the global coordinates plus the metric to reconstruct the curved surface exactly. (Strictly speaking, the global coordinate system must include information about ranges of its coordinates, ranges that describe its "connectedness"—technical name: its **topology**.)

5.5 ■ GLOBAL SPACETIME METRIC

Visit a neutron star with wristwatch, tape measure—and metric—in your back pocket.

To distorted space add warped t. Result? Trouble for Einstein!

What does all this curved-surface-in-space talk have to do with Einstein's perplexity during his journey from special relativity to general relativity? As usual, we express the answer as an analogy between a curved surface in space and a curved region of spacetime. Spacetime around a black hole multiplies the complications of the curved surface: not only is space distorted compared with its Euclidean description but the fourth dimension, the t-coordinate, is warped as well. All this complicates our new task, which is to predict our measurement of ruler distance σ or wristwatch time τ between a *pair of events in spacetime.*

Here we simply state, for flat and curved regions of spacetime, five assertions similar to those stated earlier for flat and curved surfaces in space.

Assertion A for FLAT SPACETIME: Everywhere-uniform map scale possible.

> **Assertion A. IN A FLAT REGION OF SPACETIME, we CAN FIND a global coordinate system in which every coordinate separation IS a directly-measured quantity.**

In Chapter 1 we introduced a pair of expressions for flat spacetime called the *interval*, similar to the Pythagorean Theorem for a flat surface. One form of the interval predicts the wristwatch time τ between two events with a timelike relation. The second form tells us the ruler distance σ between two events with a spacelike relation:

$$\Delta\tau_{\text{lab}}^2 = \Delta t_{\text{lab}}^2 - \Delta s_{\text{lab}}^2 \qquad \text{(flat spacetime, timelike-related events)} \qquad (5)$$
$$\Delta\sigma^2 = \Delta s_{\text{lab}}^2 - \Delta t_{\text{lab}}^2 \qquad \text{(flat spacetime, spacelike-related events)}$$

In *flat* spacetime, each space coordinate separation Δs_{lab} and time coordinate separation Δt_{lab} measured in the laboratory frame can be as small or as great as we want. On to our second assertion:

Assertion B for FLAT SPACETIME: We are free to choose a variable map scale over the region.

Assertion B. IN A FLAT REGION OF SPACETIME we are FREE TO CHOOSE a global coordinate system in which coordinate separations ARE NOT directly-measured quantities.

In this case we can choose not only stretched space coordinates but also a system of scattered clocks that run at different rates. If we choose such a "stretched" (but perfectly legal) global spacetime coordinate system, the interval equations (5) are no longer valid, because any of these coordinate separations may span regions of varying spacetime map scales. So we again retreat to a differential version of this equation, adding coefficients similar to that of space metric (2). A simple timelike metric might have the general form:

$$d\tau^2 = J(t,y,x)dt^2 - K(t,y,x)dy^2 - L(t,y,x)dx^2 \qquad (6)$$

Spacetime metric delivers $d\tau$ from differentials dt, dy, and dx.

Here each of the coefficient functions J, K, and L may vary with x, y, and t. (The coefficient functions are not entirely arbitrary: the condition of flatness imposes differential relations between them, which we do not state here.) Given such a metric for flat spacetime, we are free to use this metric to convert differentials of global coordinates (right side of the metric) to measured quantities (left side of the metric). This leads to our third assertion:

Assertion C for FLAT SPACETIME: Variable map scale requires metric to calculate $d\tau$ or $d\sigma$.

Assertion C. IN A FLAT REGION OF SPACETIME, when we choose a global coordinate system in which coordinate separations are not directly-measured quantities, then a global spacetime metric is REQUIRED to calculate the differential interval, $d\tau$ or $d\sigma$, between two adjacent events using their differential global coordinate separations.

On the other hand, in a region of curved spacetime—analogous to the situation on a curved surface in space—we *cannot* set up a global coordinate system with the same map scale everywhere in the region.

Assertion D for CURVED SPACETIME: Everywhere-uniform map scale is IMPOSSIBLE.

Assertion D. IN A CURVED REGION OF SPACETIME it is IMPOSSIBLE to find a global coordinate system in which coordinate separations EVERYWHERE in the region are directly-measured quantities.

Assertion E for CURVED SPACETIME: Metric REQUIRED to calculate $d\tau$ or $d\sigma$.

Assertion E. IN A CURVED REGION OF SPACETIME, a global spacetime metric is REQUIRED to calculate the differential interval, $d\tau$ or $d\sigma$, between a pair of adjacent events from their differential global coordinate separations.

Spacetime summary

SPACETIME SUMMARY: *In flat spacetime we can choose coordinates such that the spacetime interval—with no differentials—correctly predicts the wristwatch time (or the ruler distance) between two events far from one another. In curved spacetime we cannot. But in curved spacetime we can use a spacetime metric to calculate $d\tau$ or $d\sigma$ between adjacent events from the values of the differential coordinate separations between them. Then we can integrate $d\tau$ along the worldline of a particle, for example, to predict the directly-measured time lapse τ on a wristwatch that moves along that worldline.*

"Connectedness" = **topology.**

As in the case of the curved surface, a complete description of a spacetime region results from the combination of global spacetime coordinates and global metric—along with the connectedness (topology) of that region. For example, we can in principle use Schwarzschild's global coordinates and his metric to answer all questions about spacetime around the black hole.

5.6 ■ ARE WE SMARTER THAN EINSTEIN?

Did Einstein fumble his seven-year puzzle?

Einstein's struggle

We have now solved the puzzle that troubled Einstein for the seven years it took him to move from special relativity to general relativity. Surely Einstein would understand in a few seconds the central idea behind cutting-board examples in Figures 1 through 3. However, the extension of this idea to the four dimensions of spacetime was not obvious while he was struggling to create a brand new theory of spacetime that is curved, for example, by the presence of Earth, Sun, neutron star, or black hole. Is it any wonder that during this intense creative process Einstein took a while to appreciate the lack of "immediate metrical meaning" of differences in global coordinates?

One co-author didn't get it.

It is embarrassing to admit that one co-author of this book (EFT) required more than two years to wake up to the basic idea behind the present chapter, even though this central result is well known to every practitioner of general relativity. Even now EFT continues to make Einstein's original mistake: He confuses global coordinate separations with measured quantities.

You too will probably find it difficult to avoid Einstein's mistake.

FIRST ADVICE FOR THE ENTIRE BOOK

FIRST STRONG ADVICE FOR THIS ENTIRE BOOK

To be safe, it is best to assume that global coordinate separations do not have any measured meaning. Use global coordinates only with the metric in hand to convert a mapmaker's fantasy into a surveyor's reality.

Global coordinate systems come and go; wristwatch ticks and ruler lengths are forever!

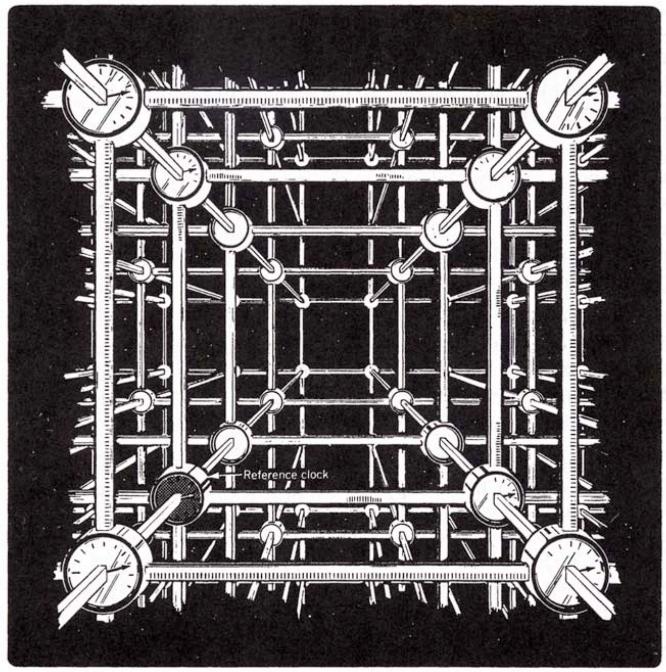

FIGURE 4 On a flat patch we build an inertial Cartesian latticework of meter sticks with synchronized clocks. This is an instrumented room (defined in Section 3.10), on which we impose a local coordinate system—a frame—limited in both space and time. Limited by what? Limited by the sensitivity to curvature of the measurement we want to carry out in that local inertial frame.

5.7 ■ LOCAL MEASUREMENT IN A ROOM USING A LOCAL FRAME

Where we make real measurements

> *Of all theories ever conceived by physicists, general relativity has the simplest, most elegant geometric foundation. Three axioms: (1) there is a global metric; (2) the global metric is governed by the Einstein field equations; (3) all special relativistic laws of physics are valid in every local inertial frame, with its (local) flat-spacetime metric.*
>
> —Misner, Thorne, and Wheeler (edited)

> *No phenomenon is a physical phenomenon until it is an observed phenomenon.*
>
> —John Archibald Wheeler

Special relativity assumes that a measurement can take place throughout an unlimited space and during an unlimited time. Spacetime curvature denies us this scope, but general relativity takes advantage of the fact that almost everywhere on a curved surface, space is locally flat; remember "flat Kansas" in Figure 3, Section 2.2. Wherever spacetime is smooth—namely close to every event except one on a singularity—general relativity permits us to approximate the gently curving stage of spacetime with a local inertial frame. This section sets up the command that we shout loudly everywhere in this book:

Spacetime is locally flat almost everywhere.

SECOND ADVICE FOR THE ENTIRE BOOK

SECOND STRONG ADVICE FOR THIS ENTIRE BOOK

In this book we choose to make every measurement in a local inertial frame, where special relativity rules.

We ride in a *room,* a physical enclosure of fixed spatial dimensions (defined in Section 3.10) in which we make our measurements, each measurement limited in local time. We assume that the room is sufficiently small—and the duration of our measurement sufficiently short—that these measurements can be analyzed using special relativity. This assumption is correct on a *patch.*

Definition: **patch**

DEFINITION 1. Patch

A **patch** is a spacetime region purposely limited in size and duration so that curvature (tidal acceleration) does not noticeably affect a given measurement.

Important: The definition of patch depends on the scope of the measurement we wish to make. Different measurements require patches of different extent in global coordinates. On this patch we lay out a local coordinate system, called a *frame.*

Definition: **frame**

DEFINITION 2. Frame

A **frame** is a local coordinate system of our choice installed onto a spacetime patch.This local coordinate system is limited to that single patch.

Among all possible local frames, we choose one that is inertial:

Definition: **inertial frame**

DEFINITION 3. Inertial frame

An **inertial** or **free-fall frame** is a local coordinate system—typically Cartesian spatial coordinates and readings on synchronized clocks (Figure 4)—for which special relativity is valid. In this book we report every measurement using a local inertial frame.

In general relativity every inertial frame is local, that is limited in spacetime extent. Spacetime curvature precludes a global inertial frame.

Who makes all these measurements? The observer does:

Definition: **observer**

DEFINITION 4. Observer = Inertial Observer

An **observer** is a person or machine that moves through spacetime making measurements, each measurement limited to a local inertial frame. Thus an observer moves through a series of local inertial frames.

Box 1. What moves?

A story—impossible to verify—recounts that at his trial by the Inquisition, after recanting his teaching that the Earth moves around the Sun, Galileo muttered under his breath, “Eppur si muove,” which means “And yet it moves.”

According to special and general relativity, what moves? We quickly eliminate coordinates, events, patches, frames, and spacetime itself:

- Coordinates do not move. Coordinates are number-labels that locate an event; it makes no sense to say that a coordinate number-label moves.
- An event does not move. An event is completely specified by coordinates; it makes no sense to say that an event moves.
- A flat patch does not move. A flat patch is a region of spacetime completely specified by a small, specific range of map coordinates; it makes no sense to say that a range of map coordinates moves.
- A local frame does not move. A frame is just a set of local coordinates—numbers—on a patch; it makes no sense to say that a set of local coordinates move.
- Spacetime does not move. *Spacetime* labels the arena in which events occur; it makes no sense to say that a label moves.

You cannot drop a frame. You cannot release a frame. You cannot accelerate a frame. It makes no sense to say that you can even move a frame. You cannot carry a frame around, any more than you can move a postal zip code region by carrying its number around.

What does move? Stones and light flashes move; observers and rooms move. Whatever moves follows a worldline or worldtube through spacetime.

- A stone moves. Even a stone at rest in a shell frame moves on a worldline that changes global t-coordinate.
- A light flash moves; it follows a *null worldline* along which both r and ϕ can change, but $\Delta\tau = 0$.
- An observer moves. Basically the observer is an instrumented stone that makes measurements as it passes through local frames.
- A room moves. Basically a room is a large, hollow stone.

Why do almost all teachers and special relativity texts—including our own physics text *Spacetime Physics* and Chapter 1 of this book!—talk about “laboratory frame” and “rocket frame”? Because it is a tradition; it leads to no major confusion in special relativity. But when we specify a local rain frame in curved spacetime using (for example) a small range of Schwarzschild global coordinates t, r, and ϕ, then it makes no sense to say that this local rain frame—this range of global coordinates—moves. Stones move; coordinates do not.

The observer, riding in a room (Definition 3, Section 3.10), makes a sequence of measurements as she passes through a series of local inertial frames. As it passes through spacetime, the room drills out a *worldtube* (Definition 4, Section 3.10). Figure 5 shows such a worldtube.

?

Objection 1. *In Definition 4 you say that the observer moves through a series of local inertial frames. But doesn’t a shell observer stay in one local frame?*

!

No! The shell observer is *not* stationary in the global t-coordinate, but moves along a worldline (Figure 5). By definition, a local inertial frame spans a given lapse of frame time $\Delta t_{\rm shell}$, as well as a given frame volume of space. In Figure 5 the first measurement takes place in Frame #1. When the first measurement is over, global t/M has elapsed and the observer leaves Frame #1. A second measurement takes place in Frame #2. The range of r/M and ϕ global coordinates of Frame #2 may be the same as in Frame #1. The shell observer makes a series of measurements, each measurement in a *different* local inertial frame.

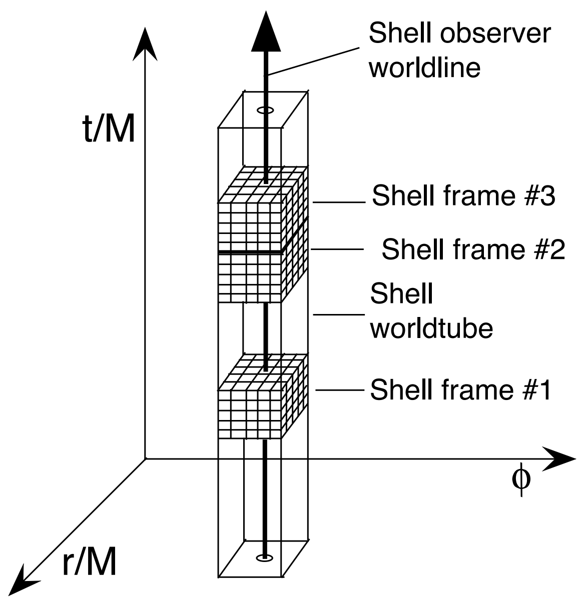

FIGURE 5 A shell worldtube (Section 3.10) that embraces three sample shell frames outside the event horizon. The shell observer carries out an experiment while passing through Frame #1 in the figure. He may then repeat the same experiment or carry out another one in Frames #2 and #3 at greater t coordinates. For simplicity each shell frame is shown as a cube. Each frame is *nailed* to a particular event at map coordinates $(\bar{t}/M, \bar{r}/M, \bar{\phi})$.

Comment 1. Euclid's curved space vs. Einstein's curved spacetime
Figure 5 shows a case in which a shell observer stands at constant r and ϕ coordinates while he passes, with changing map t-coordinate, through a series of local frames, each frame defined over a range of r, ϕ, and t-coordinates. Figure 5 in Section 2.2 showed the Euclidean space analogy in which a traveler passes across a series of local flat maps on her way along the curved surface of Earth from Amsterdam to Vladivostok. Each of these flat maps is essentially a set of numbers: local space coordinates we set up for our own use. Similarly, each local frame of Figure 5 is just a set of numbers, local space and time coordinates we set up for our own use. A frame is not a room; a frame does not fall; a frame does not move; it is just a set of numbers—coordinates—that we use to report results of local measurements (Box 1). Figure 5 shows multiple shell frames, two of them adjacent in t-coordinate. Shell frames can also overlap, analogous to the overlap of adjacent local Euclidean maps in Figure 5, Section 2.2.

Objection 2. *Whoa! Can a frame exist inside the event horizon?*

Definitely. A frame is a set of coordinates—numbers! Numbers are not things; they can exist anywhere, even inside the event horizon. In contrast, the diver in her unpowered spaceship is a "thing." Even inside the event horizon the she-thing continues to pass through a series of local frames. Inside the event horizon, however, she is doomed to continue to the singularity as her wristwatch ticks inevitably forward.

By definition, we use the flat-spacetime metric to analyze events in a local inertial frame. We write this metric for a local shell frame in a rather strange form which we then explain:

$$\Delta\tau^2 \approx \Delta t_{\rm shell}^2 - \Delta y_{\rm shell}^2 - \Delta x_{\rm shell}^2 \qquad (7)$$

Local flat spacetime → local inertial metric.

Choose the increment $\Delta y_{\rm shell}$ to be vertical (radially outward), and the $\Delta x_{\rm shell}$ increment to be horizontal (tangential along the shell).

Instead of an equal sign, equation (7) has an approximately equal sign. This is because near a black hole or elsewhere in our Universe there is always *some* spacetime curvature, so the equation cannot be exact. The upper case Delta, Δ, also has a different meaning in (7) than in special relativity. In special relativity (Section 1.10) we used Δ to emphasize that in flat spacetime the two events whose separation is described by (7) can be very far apart in space or time and their coordinate separations still satisfy (7) with an equals sign. In equation (7), however, both events must lie in the local frame within which the coordinate separations $\Delta t_{\rm shell}$, $\Delta y_{\rm shell}$, and $\Delta x_{\rm shell}$ are defined.

Connect global and local metrics

How do we connect local metric (7) to the Schwarzschild global metric? We do this by considering a local frame over which global coordinates t, r, and ϕ vary only a little. Small variation allows us to replace r with its average value $\bar{r}$ over the patch and write the Schwarzschild metric in the approximate form:

$$\Delta\tau^2 \approx \left(1 - \frac{2M}{\bar{r}}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1 - \dfrac{2M}{\bar{r}}\right)} - \bar{r}^2\Delta\phi^2 \quad \text{(spacetime patch)} \qquad (8)$$

Equation (8) is no longer global. The value of $\bar{r}$ depends on *where* this patch is located, leading to a local wristwatch time lapse $\Delta\tau$ for a given change Δr. The value of $\bar{r}$ also affects how much $\Delta\tau$ changes for a given change in Δt or $\Delta\phi$. Equation (8) is approximately correct only for limited ranges of Δt, Δr, and $\Delta\phi$. In contrast to the differential global Schwarzschild metric, (8) has become a *local* metric. That is the bad news; now for some good news.

Local shell coordinates

Coefficients in (8) are now constants. So simply equate corresponding terms in the equations (8) and (7):

$$\Delta t_{\rm shell} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2}\Delta t \qquad (9)$$

$$\Delta y_{\rm shell} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2}\Delta r \qquad (10)$$

$$\Delta x_{\rm shell} \equiv \bar{r}\Delta\phi \qquad (11)$$

FIGURE 6 Flat triangular segments on the surface of a Buckminster Fuller geodesic dome. A single flat segment is the geometric analog of a locally flat patch in curved spacetime around a black hole; we add local coordinates to this patch to create a local frame. (Figure 3 in Section 3.3 shows a complete geodesic dome with six-sided segments.)

Substitutions (9), (10), and (11) turn approximate metric (8) into approximate metric (7), which is—approximately!—the metric for flat spacetime. *Payoff:* We can use special relativity analyze local measurements and observations in a shell frame near a black hole.

Objection 3. *What is the meaning of equations (9) through (11)? What do they accomplish? How do I use them?*

These equations are fundamental to our application of general relativity to Nature. On the left are measured quantities: $\Delta t_{\rm shell}$ is the measured shell time between two events, $\Delta y_{\rm shell}$ and $\Delta x_{\rm shell}$ are their measured separations in local space shell coordinates. These equations, plus the local metric (7) unleash special relativity to analyze local measurements in curved spacetime. In this book we choose to report every measurement using a local inertial frame.

Comment 2. Left-handed $(\Delta y_{\rm shell}, \Delta x_{\rm shell})$ local space coordinates
We find it convenient to have the local $\Delta y_{\rm shell}$ point along the outward global Schwarzschild r-coordinate and the local $\Delta x_{\rm shell}$ point along the direction of increasing angle $\Delta\phi$, on the $[r, \phi]$ slice through the center of the black hole. This earns the label **left-handed** for the space part of these local coordinates, which differs from most physics usage.

Figure 6 shows a geometric analogy to a local flat patch: the local flat plane segments on the curved exterior surface of a Buckminster Fuller geodesic dome.

Summary: local notation

We summarize here the new notation introduced in equation (7) and equations (9) through (11):

$\approx$	equality is not exact, due to residual curvature and coordinate conversion (Section 5.8)	(12)
Δ	coordinate separation of two events within the local frame	(13)
$\bar{r}$	average r-coordinate across the patch	(14)

Objection 4. *How large—in $\Delta t_{\mathrm{shell}}$, $\Delta y_{\mathrm{shell}}$, and $\Delta x_{\mathrm{shell}}$—am I allowed to make my local inertial frame? If you cannot tell me that, you have no business talking about local inertial frames at all!*

You are right, but the answer depends on the measurement you want to make. Some measurements are more sensitive than others to tidal accelerations; each measurement sets its own limit on the maximum extent of the local frame in order that it remain inertial for that measurement. If the local frame is too extended in both the $\Delta x_{\mathrm{shell}}$ and $\Delta y_{\mathrm{shell}}$ directions to be inertial, then it may be necessary to restrict the frame time $\Delta t_{\mathrm{shell}}$ during which it is carried out. *Result:* Different measurements prevent us from setting a universal, one-fits-all size for a local inertial frame. Sorry.

Objection 5. *What happens when we choose the size of the local frame too great, so the frame is no longer inertial? How do we know when we exceed this limit?*

There are two answers to these questions. The first is spacetime curvature: Section 1.11 entitled Limits on Local Inertial Frames describes this situation using Newtonian intuition. If two stones initially at rest near Earth are separated radially, the stone nearer the center accelerates downward at a faster rate. If two stones, initially at rest, are separated tangentially, their accelerations do not point in the same directions, Figure 8, Section 1.11. These effects go under the name *tidal accelerations*, because ocean tides on Earth result from differences in gravitational attraction of Moon and Sun at different locations on Earth. If these tidal accelerations exceed the achievable accuracy of an experiment, then the local frame cannot be considered inertial.

The second answer to the question results from the global coordinate system itself and the process by which the local inertial frame is derived from it. This part is treated in Section 5.8.

Box 2. Who cares about local inertial frames?

Sections 5.1 through 5.6 make no reference to local inertial frames. Nor are they necessary. The left side of the global metric predicts differentials $d\tau$ or $d\sigma$ (or $d\tau = d\sigma = 0$) between adjacent events. Of course we cannot measure differentials directly, because they are, by definition, vanishingly small. We need to integrate them; for example we integrate wristwatch time along the worldline of a stone. The resulting predictions are sufficient to analyze results of any experiment or observation. No local inertial frames are required, and most general relativity texts do not use them.

Our approach in this book is different; we *choose* to predict, carry out, and report all measurements with respect to a local inertial frame. *Payoff:* In each local inertial frame we can unleash all the concepts and tools of special relativity, such as directly-measured space and time coordinate separations, measurable energy and momentum of a stone; Lorentz transformations between local inertial frames.

We may report local-frame measurements in the calculus limit, as we often do on Earth. For example, we record the motion of a light flash in our local inertial frame. Rewrite (7) as

$$\Delta\tau^2 \approx \Delta t_{\text{shell}}^2 - \Delta s_{\text{shell}}^2 \qquad (15)$$

where Δs_{shell} is the distance between two events measured in the shell frame. Now let a light flash travel directly between the two events in our local frame. For light $\Delta\tau = 0$ and we write its speed (a positive quantity) as:

$$\left|\frac{\Delta s_{\text{shell}}}{\Delta t_{\text{shell}}}\right| \approx 1 \qquad \text{(speed of light flash)} \qquad (16)$$

Can take calculus limit in local frame.

We may want to know the instantaneous speed, which requires the calculus limit. In this process all increments shrink to differentials and $\bar{r} \rightarrow r$. For the light flash the result is:

$$v_{\text{shell}} \equiv \lim_{\Delta t_{\text{shell}}\rightarrow 0}\left|\frac{\Delta s_{\text{shell}}}{\Delta t_{\text{shell}}}\right| = 1 \qquad \text{(instantaneous light flash speed)} \qquad (17)$$

Equation (17) reassures us that the speed of light is exactly one when measured in a local shell frame at any r (outside the event horizon, where shells can be constructed). The measured speed of a stone is always less than unity:

$$v_{\text{shell}} \equiv \lim_{\Delta t_{\text{shell}}\rightarrow 0}\left|\frac{\Delta s_{\text{shell}}}{\Delta t_{\text{shell}}}\right| < 1 \qquad \text{(instantaneous stone speed)} \qquad (18)$$

5.8 ■ THE TROUBLE WITH COORDINATES

Coordinates, as well as spacetime curvature, limit accuracy.

Can use global metric exclusively.

We need global coordinates and cannot apply general relativity without them. Only global coordinates can connect widely separated local inertial frames in

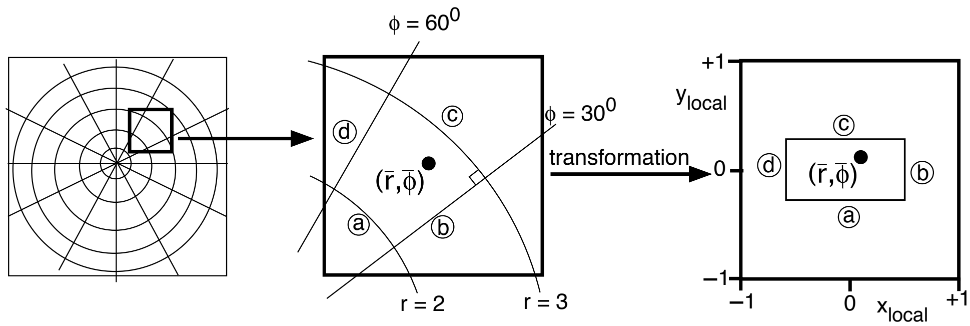

FIGURE 7 Inaccuracies due to polar coordinates on a flat sheet of paper. Coordinates in the middle frame are curved.

which we make measurements. Indeed, we can choose to use only global coordinates to apply general relativity (Box 2). Instead, in this book we *choose* to design and carry out measurements in a local inertial frame in order to unleash the power and simplicity of special relativity. In this process we fix average values of global coordinates to make constant the coefficients in the global metric. This allows us to write down the relation between global and local coordinates, equations (9) through (11), in order to generate a local flat spacetime metric (7).

We *choose* to use local coordinates.

But our choice has a cost that has nothing to do with spacetime curvature, illustrated by analogy to a flat geometric surface in Figure 7. The left frame shows polar coordinates laid out on the entire flat sheet. Choose a small area of the sheet (expanded in the second frame). That small area is, a *patch* (Definition 1) with a small section of *global* coordinates superimposed. This is a *frame* (Definition 2) whose local coordinate system is derived from global coordinates. The third frame shows Cartesian coordinates that cover the same patch, converting it to a local Cartesian frame, analogous to an inertial frame (Definition 3). What is the relation between the second frame and the third frame?

Approximation due to coordinate conversion

The exact differential separation between adjacent points is

$$ds^2 = dr^2 + r^2 d\phi^2 \qquad (19)$$

In order to provide some "elbow room" to carry out local measurements on our small patch, we expand from differentials to small increments with the approximations:

$$\begin{aligned} \Delta s^2 &\approx \Delta r^2 + \bar{r}^2 \Delta\phi^2 \\ &\approx \Delta x^2 + \Delta y^2 \end{aligned} \qquad (20)$$

Because of the average $\bar{r}$ due to curved coordinates, equation (20) is not exact. The approximation of this result has nothing to do with curvature, since the surface in the left panel is flat. A similar inexactness haunts the relation

Approximate due to (1) residual curvature plus (2) coordinate conversion.

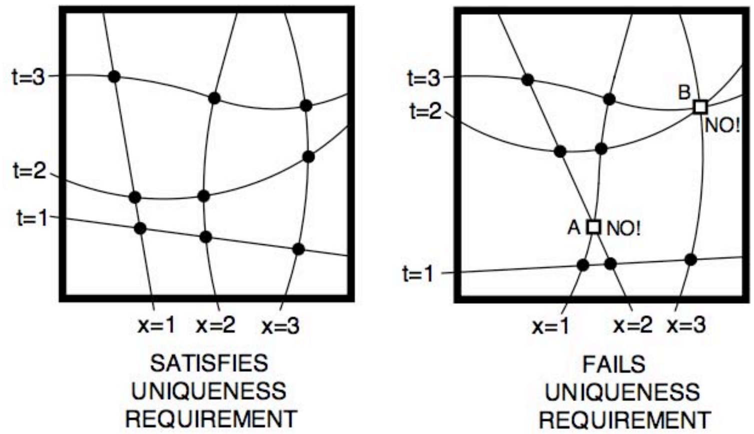

FIGURE 8 Left panel. Example of global coordinates that satisfy the uniqueness requirement: every event shown (filled circles) has a unique value of x and t. Right panel: Example of a global coordinate system that fails to satisfy the uniqueness requirement; Event A has two x-coordinates: $x = 1$ and $x = 2$; Event B has two t-coordinates: $t = 2$ and $t = 3$.

between global and local coordinates in equations (9) through (11). These equations are approximate for two reasons: (1) the residual curvature of spacetime across the local frame and (2) the conversion between global and local coordinates. In this book we emphasize the first of these, but the second is ever-present.

5.9 ■ REQUIREMENTS OF GLOBAL COORDINATE SYSTEMS

Which coordinate systems can we use in a global metric?

Some restrictions on global coordinates

Thus far we have put no restrictions on global coordinate systems for global metrics in general relativity. The basic requirements are a global coordinate system that (a) uniquely specifies the spacetime location of every event, and (b) when submitted to Einstein's equations results in a global metric. Here are some technical requirements, quoted from advanced theory without proof.

Unique set of coordinates for each event

UNIQUENESS REQUIREMENT

The global coordinate system must provide a unique set of coordinates for each separate event in the spacetime region under consideration.

The uniqueness requirement seems reasonable. A set of global coordinates, for example t, r, ϕ, must allow us to distinguish any given event from every other event. That is, no two distinct events can have every global coordinate the same; nor can any given event be labelled by more than one set of coordinates. The left panel in Figure 8 shows an example of global coordinates that satisfy the uniqueness requirement; the right panel shows an example of global coordinates that fails this requirement.

Box 3. Find a particular local inertial frame.

How can we locate and label a particular local inertial frame on a shell around a black hole?

Ask a simpler question: How do we label and find one particular flat triangular surface on a Buckminster Fuller geodesic dome (Figure 6)? One way is simply to number each flat surface: triangle #523 next to triangle #524 next to triangle #525. Carry out this procedure for every flat triangle on the geodesic dome. The result is a huge catalog that we must consult to locate a given local flat segment on these nested Buckminster Fuller geodesic domes.

We could use a similar sequential numbering scheme to label and find a local inertial shell frame around a black hole, sequential in both space and time. But we already have a simpler way to index a single local inertial frame:

Equations (9) through (11) provide a much simpler indexing scheme: the average values $\bar{t}$, $\bar{r}$, and $\bar{\phi}$. Average $\bar{r}$ gives us the shell, average $\bar{\phi}$ locates the position of the local frame along the shell, and average $\bar{t}$ tells us the global t-coordinate of the frame at that location—local in time as well as space. Three numbers, for example $\bar{t}$, $\bar{r}$, and $\bar{\phi}$, specify precisely the local inertial shell frame in spacetime surrounding a black hole.

In addition to the uniqueness requirement, we must be able to set up a local inertial frame everywhere around the black hole (except on its singularit. To allow this possibility, we add the second, smoothness requirement:

Smooth coordinates

SMOOTHNESS REQUIREMENT

The coordinates must vary smoothly from event to neighboring event. In practice, this means there must be a differentiable coordinate transformation that takes the global metric to a local inertial metric (except on a physical singularity).

Comment 3. The (almost) complete freedom of general relativity

There are an unlimited number of valid global coordinate systems that describe spacetime around a stable object such as a star, white dwarf, neutron star, or black hole (Box 3 in Section 7.5). Who chooses which global coordinate system to use? We do!

Near every event (except on a singularity) there are an unlimited number of possible local inertial frames in an unlimited number of relative motions. Who chooses the single local frame in which to carry out our next measurement? We do!

Nature has no interest whatsoever in which global coordinates we choose or how we derive from them the local inertial frames we employ to report our measurements and to check our predictions. Choices of global coordinates and local frames are (almost) completely free human decisions. Welcome to the wild permissiveness of general relativity!

5.10 ■ EXERCISES

5.1. Rotation of vertical

The inertial metric (7) assumes that the Δx_{shell} and Δy_{shell} are both straight-line separations that are perpendicular to one another. How many kilometers along a great circle must you walk before both the horizontal and vertical directions "turn" by one degree

A. on Earth.
B. on the Moon (radius 1 737 kilometers).
C. on the shell at map coordinate $r = 3M$ of a black hole of mass five times that of our Sun.

5.2. Time warping

In a given global coordinate system, two identical clocks stand at rest on different shells directly under one another, the lower clock at map coordinate $r_{\rm L}$, the higher clock at map coordinate $r_{\rm H}$. By *identical clocks* we mean that when the clocks are side by side the measured shell time between sequential ticks is the same for both. When placed on shells of different map radii, the measured time lapses between adjacent ticks are $\Delta t_{\rm shell\,H}$ and $\Delta t_{\rm shell\,L}$, respectively.

A. Find an expression for the fractional measured time difference f between the shell clocks, defined as:

$$f \equiv \frac{\Delta t_{\rm shell\,H} - \Delta t_{\rm shell\,L}}{\Delta t_{\rm shell\,L}} \qquad (21)$$

This expression should depend on only the map r-values of the two clocks and on the mass M of the center of attraction.

B. Fix $r_{\rm L}$ of the lower shell clock. For what higher $r_{\rm H}$-value does the fraction f have the greatest magnitude? Derive the expression $f_{\rm max}$ for this maximum fractional magnitude.

C. Evaluate the numerical value of the greatest magnitude $f_{\rm max}$ from Item B when $r_{\rm L}$ corresponds to the following cases:

(a) Earth's surface (numerical parameters inside front cover)
(b) Moon's surface (radius 1 737 kilometers, mass 5.45×10^{-5} meters)
(c) on the shell at $r_{\rm L} = 3M$ of a black hole of mass $M = 5M_{\rm Sun}$ (Find the value of $M_{\rm Sun}$ inside front cover)

D. Find the higher map coordinate $r_{\rm H}$ at which the fractional difference in clock rates is 10^{-10} for the cases in Item C.

E. For case (c) in item C, what is the directly-measured distance between the shell clocks?

F. What is the value of $f_{\rm max}$ in the limit $r_{\rm L} \rightarrow 2M$? What is the value of f in the limit $r_{\rm L} \rightarrow 2M$ and $r_{\rm H} = 2M(1+\epsilon)$, where $0 < \epsilon \ll 1$. What does this result say about the ability of a light flash to move outward from the event horizon?

G. Which items in this exercise have different answers when the upper clock and the lower clock do *not* lie on the same radial line, that is when the upper clock is *not* directly above the lower clock?

5.3. Diving inertial frame

Think of a local inertial frame constructed in a free capsule that dives past a local shell frame with local radial velocity v_{rel} measured by the shell observer. Use Lorentz transformations from Chapter 1 to find expressions similar to equations (9) through (11) that give coordinate increments Δt_{dive}, Δy_{dive}, and Δx_{dive} between a pair of events in the diving frame in terms of $\bar{r}$, v_{rel}, and global coordinate increments Δt, Δr, and $\Delta\phi$.

5.4. Tangentially moving inertial frame

Think of a local inertial frame constructed in a capsule that moves instantaneously in a tangential direction with tangential speed v_{rel} measured by the shell observer. Use Lorentz transformations from Chapter 1 to find expressions similar to equations (9) through (11) that give coordinate increments Δt_{tang}, Δy_{tang}, and Δx_{tang} between a pair of events in the tangentially-moving frame in terms of $\bar{r}$, v_{rel}, and global coordinate increments Δt, Δr, and $\Delta\phi$.

5.11 ■ REFERENCES

Albert Einstein quotes from "Autobiographical Notes," in *Albert Einstein: Philosopher-Scientist*, edited by Paul Arthur Schilpp, Volume VII of The Library of Living Philosophers, MJF Books, New York 1970, page 67.

Misner, Thorne, and Wheeler quote from Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, *GRAVITATION*, W. H. Freeman Company, San Francisco [now New York], 1971, pages 302-303.

Wheeler on a phenomenon: Quoted in Robert J. Scully, *The Demon and the Quantum* (2007), page 191.