Chapter 6. Diving

- *Am I comfortable as I fall toward a black hole?*
- *How fast am I going when I reach the event horizon? Who measures my speed?*
- *How long do I live, measured on my wristwatch, as I fall into a black hole?*
- *How much does the mass of a black hole increase when a stone falls into it? when I fall into it?*
- *How close to a black hole can I stand on a spherical shell and still tolerate the "acceleration of gravity"?*

CHAPTER

6 Diving

Edmund Bertschinger & Edwin F. Taylor *

Many historians of science believe that special relativity could have been developed without Einstein; similar ideas were in the air at the time. In contrast, it's difficult to see how general relativity could have been created without Einstein – certainly not at that time, and maybe never.

—David Kaiser

6.1 ■ GO STRAIGHT: THE PRINCIPLE OF MAXIMAL AGING IN GLOBAL COORDINATES

"Go straight!" spacetime shouts at the stone.
The stone's wristwatch verifies that its path is straight.

Section 5.7 described how an observer passes through a sequence of local inertial frames, making each measurement in only one of these local frames. Special relativity describes motion in each such local inertial frame. The observer is just a stone that acts with purpose. Now we ask how a (purposeless!) free stone moves in global coordinates.

Section 1.6 introduced the Principle of Maximal Aging that describes motion in a single inertial frame. To describe global motion, we need to extend this principle to a *sequence* of adjacent local inertial frames. Here, without proof, is the simplest possible extension, to a *single adjacent pair* of local inertial frames.

Definition: **Principle of Maximal Aging** in curved spacetime

DEFINITION 1. Principle of Maximal Aging (curved spacetime)
The *Principle of Maximal Aging* states that a free stone follows a worldline through spacetime such that its wristwatch time (aging) is a maximum when summed across every adjoining pair of local inertial frames along its worldline.

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 Latest drafts at the dropsite exploringblackholes.com.

Box 1. What Then Is Time?

What then is time? If no one asks me, I know what it is. If I wish to explain it to him who asks me, I do not know.

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The world was made, not in time, but simultaneously with time. There was no time before the world.

—St. Augustine (354–430 C.E.)

Time takes all and gives all.

—Giordano Bruno (1548–1600 C.E.)

Everything fears Time, but Time fears the Pyramids.

—Anonymous

Philosophy is perfectly right in saying that life must be understood backward. But then one forgets the other clause—that it must be lived forward.

—Søren Kierkegaard

As if you could kill time without injuring eternity.

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Time is but the stream I go a-fishing in.

—Henry David Thoreau

Although time, space, place, and motion are very familiar to everyone, . . . it is useful to distinguish these quantities into absolute and relative, true and apparent, mathematical and common.

—Isaac Newton

Time is defined so that motion looks simple.

—Misner, Thorne, and Wheeler

Nothing puzzles me more than time and space; and yet nothing troubles me less, as I never think about them.

—Charles Lamb

Either this man is dead or my watch has stopped.

—Groucho Marx

“What time is it, Casey?”

“You mean right now?”

—Casey Stengel

It’s good to reach 100, because very few people die after 100.

—George Burns

The past is not dead. In fact, it’s not even past.

—William Faulkner

Time is Nature’s way to keep everything from happening all at once.

—Graffito, men’s room, Pecan St. Cafe, Austin, Texas

What time does this place get to New York?

—Barbara Stanwyck, during trans-Atlantic crossing on the steamship *Queen Mary*

Objection 1. *Now you have gone off the deep end! In Chapter 1, Speeding, you convinced me that the Principle of Maximal Aging was nothing more than a restatement of Newton’s First Law of Motion, the observation that in flat spacetime the free stone moves at constant speed along a straight line in space. But in curved spacetime the stone’s path will obviously be curved. You have violated your own Principle.*

On the contrary, we have changed the Principle of Maximal Aging as little as possible in order to apply it to curved spacetime. We require the free stone to move along a straight worldline across *each one* of the pair of adjoining local inertial frames, as demanded by the special relativity Principle of Maximal Aging in each frame. We allow the stone only the choice of one map coordinate of the event, at the boundary between these two frames. That single generalization extends the Principle of Maximal Aging from flat to curved spacetime. And the result is a single kink in the worldline. When we shrink all adjoining inertial frames along the worldline to the calculus limit, then the result is what you predict: a curved worldline in global coordinates.

Now we can use the more general Principle of Maximal Aging to discover a constant of motion for a free stone, what we call its *map energy*.

6.2 ■ MAP ENERGY FROM THE PRINCIPLE OF MAXIMAL AGING

The global metric plus the Principle of Maximal Aging leads to map energy as a constant of motion.

Map energy: a constant of motion

This section uses the Principle of Maximal Aging from Section 6.1, plus the Schwarzschild global metric to derive the expression for map energy of a free stone near a nonspinning black hole. For a free stone, map energy is a constant of motion; its value remains the same as the stone moves. Our derivation uses a stone that falls inward along the r-direction, but at the end we show that the resulting expression for map energy also applies to a stone moving in any direction; energy is a *scalar*, which has no direction.

?

Objection 2. *Here is a fundamental objection to the Principle of Maximal Aging: You nowhere derive it, yet you expect us readers to accept this arbitrary Principle. Why should we believe you?*

!

Guilty as charged! Our major tool in this book is the metric, which—along with the topology of a spacetime region—tells us everything we can know about the shape of spacetime in that region. But the shape of spacetime revealed by the metric tells us nothing whatsoever about how a free stone moves in this spacetime. For that we need a second tool, the Principle of Maximal Aging which, like the metric, derives from Einstein's field equations. In this book the metric plus the Principle of Maximal Aging—both down one step from the field equations—are justified by their immense predictive power. Until we derive the metric in Chapter 22, we apply the slogan, "Handsome is as handsome does!"

Find maximal aging: find natural motion.

The Principle of Maximal Aging maximizes the stone's total wristwatch time across *two adjoining* local inertial frames. Figure 1 shows the Above Frame A (of average map coordinate $\bar{r}_{\rm A}$) and adjoining Below Frame B (of average map coordinate $\bar{r}_{\rm B}$). The stone emits initial flash 1 as it enters the top of Frame A, emits middle flash 2 as it transits from Above Frame A to Below Frame B, and emits final flash 3 as it exits the bottom of Below Frame B. We use the three *flash emission events* to find maximal aging.

Outline of the method: Fix the r- and ϕ-coordinates of all three flash emissions and fix the t-coordinates of upper and lower events 1 and 3. Next vary the t-coordinate of the middle flash emission 2 to maximize the total *wristwatch time* (aging) of the stone across both frames.

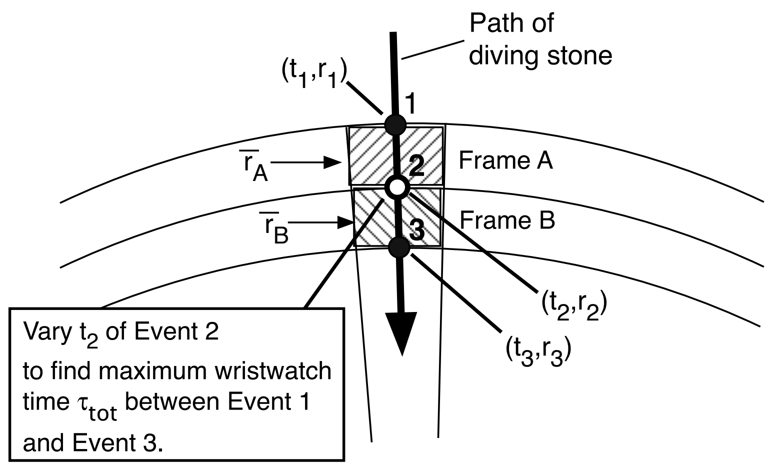

FIGURE 1 Use the Principle of Maximal Aging to derive the expression for Schwarzschild map energy. The diving stone first crosses the Above Frame A, then crosses the Below Frame B, emitting flashes at events 1, 2, and 3. Fix all three coordinates of events 1 and 3; but fix only the r- and ϕ-coordinates of intermediate event 2. Then vary the t-coordinate of event 2 to maximize the *total wristwatch time* (aging) across both frames between fixed end-events 1 and 3. This leads to expression (8) for the stone's map energy, a constant of motion.

So much for t-coordinates. How do we find *wristwatch times* across the two frames? The Schwarzschild metric ties the increment of wristwatch time to changes in r- and t-coordinates for a stone that falls inward along the r-coordinate. Write down the approximate form of the global metric twice, first for Above frame A (at average $\bar{r}_A$) and second for the Below frame B (at average $\bar{r}_B$). Take the square root of both sides:

Approximate the Schwarzschild metric for each frame.

$$\tau_A \approx \left[\left(1-\frac{2M}{\bar{r}_A}\right)(t_2-t_1)^2 + (\text{terms without } t\text{-coordinate})\right]^{1/2} \tag{1}$$

$$\tau_B \approx \left[\left(1-\frac{2M}{\bar{r}_B}\right)(t_3-t_2)^2 + (\text{terms without } t\text{-coordinate})\right]^{1/2} \tag{2}$$

We are interested only in those parts of the metric that contain the map t-coordinate, because we take derivatives with respect to that t-coordinate. To prepare for the derivative that leads to maximal aging, take the derivative of τ_A with respect to t_2 of the intermediate event 2. The denominator in the resulting derivative is just τ_A:

$$\frac{d\tau_A}{dt_2} \approx \left(1-\frac{2M}{\bar{r}_A}\right)\frac{(t_2-t_1)}{\tau_A} \tag{3}$$

The corresponding expression for $d\tau_B/dt_2$ is:

$$\frac{d\tau_B}{dt_2} \approx -\left(1 - \frac{2M}{\bar{r}_B}\right)\frac{(t_3 - t_2)}{\tau_B} \tag{4}$$

Add the two wristwatch times to obtain the summed wristwatch time τ_{tot} between first and last events 1 and 3:

$$\tau_{tot} = \tau_A + \tau_B \tag{5}$$

Maximize aging summed across both frames.

Recall that we keep constant the total t-coordinate separation across both frames. To find the maximum total wristwatch time, take the derivative of both sides of (5) with respect to t_2, substitute from (3) and (4), and set the result equal to zero in order to find the maximum:

$$\frac{d\tau_{tot}}{dt_2} = \frac{d\tau_A}{dt_2} + \frac{d\tau_B}{dt_2} \approx \left(1 - \frac{2M}{\bar{r}_A}\right)\frac{(t_2 - t_1)}{\tau_A} - \left(1 - \frac{2M}{\bar{r}_B}\right)\frac{(t_3 - t_2)}{\tau_B} \approx 0 \tag{6}$$

From the last approximate equality in (6),

$$\left(1 - \frac{2M}{\bar{r}_A}\right)\frac{(t_2 - t_1)}{\tau_A} \approx \left(1 - \frac{2M}{\bar{r}_B}\right)\frac{(t_3 - t_2)}{\tau_B} \tag{7}$$

The expression on the left side of (7) depends only on parameters of the stone's motion across the Above Frame A; the expression on the right side depends only on parameters of the stone's motion across the Below Frame B. Hence the value of either side of this equation must be independent of *which* adjoining pair of frames we choose to look at: this pair can be *anywhere* along the worldline of a stone. Equation (7) displays a quantity that has the same value on *every* local inertial frame along the worldline. We have found the expression for a quantity that is a constant of motion.

Map energy of a stone in Schwarzschild coordinates

Now shrink differences $(t_2 - t_1)$ and $(t_3 - t_2)$ in (7) to their differential limits. In this process the average r-coordinate becomes exact, so $\bar{r} \to r$. Next use the result to *define* the stone's **map energy per unit mass**:

$$\frac{E}{m} \equiv \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \text{(map energy of a stone per unit mass)} \tag{8}$$

Far from the black hole, map energy takes special relativity form.

Why do we call the expression on the right side of (8) *energy* (per unit mass)? Because when the mass M of the center of attraction becomes very small—or when the stone is very far from the center of attraction—the limit $2M/r \to 0$ describes a stone in flat spacetime. That condition reduces (8) to $E/m = dt/d\tau$, which we recognize as equation (28) in Section 1.7 for E/m in flat spacetime. Hence we take the right side of (8) to be the general-relativistic generalization, near a nonspinning black hole, of the special relativity expression for E/m.

Map energy E same unit as m

Note that the right side of (8) has no units; therefore both E and m on the left side must be expressed in the *same* unit, a unit that we may choose for our convenience. *Both* numerator and denominator in E/m may be expressed in kilograms or joules or electron-volts or the mass of the proton, or any other common unit.

Map energy expression valid for *any* motion of the stone.

Our derivation of map energy employs only the t-coordinate in the metric. It makes no difference in the outcome for map energy—expression (8)—whether dr or $d\phi$ is zero or not. This has an immediate consequence: The expression for map energy in Schwarzschild global coordinates is valid for a free stone moving on *any* orbit around a spherically symmetric center of attraction, not just along the inward r-direction. We will use this generality of (8) to predict the general motion of a stone in later chapters.

6.3 ■ UNICORN MAP ENERGY VS. MEASURED SHELL ENERGY

Map energy is like a unicorn: a mythical beast

Map energy E/m is a unicorn: a mythical beast.

The expression on the right side of equation (8) is like a unicorn: a mythical beast. Nobody measures directly the r- or t-coordinates in this expression, which are Schwarzschild global map coordinates: entries in the mapmaker's spreadsheet or accounting form. Nobody measures E/m on the left side of (8) either; the map energy is also a unicorn. If this is so, why do we bother to derive expression (8) in the first place? Because E/m has an important virtue: It is a constant of motion of a free stone in Schwarzschild global coordinates; it has the same value at every event along the global worldline of a stone. The value of E/m helps us to predict its global motion (Chapters 8 and 9). But it does not tell us the value of the energy measured by an observer in a local inertial frame.

What is the stone's energy measured by the shell observer? The special relativity energy expression is valid for the shell observer. Equation (9) in Section 5.7 gives us:

$$\Delta t_{\text{shell}} = \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \tag{9}$$

Then:

$$\frac{E_{\text{shell}}}{m} = \lim_{\Delta\tau \to 0} \frac{\Delta t_{\text{shell}}}{\Delta\tau} = \lim_{\Delta\tau \to 0} \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \frac{\Delta t}{\Delta\tau} \tag{10}$$

As we shrink increments to the differential calculus limit, the average r-coordinate becomes exact: $\bar{r} \to r$. The result is:

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{dt}{d\tau} \quad \text{(shell energy of a stone per unit mass)} \tag{11}$$

Into this equation substitute expression (8) for the stone's map energy to obtain:

$$\frac{E_{\text{shell}}}{m} = \frac{1}{(1 - v_{\text{shell}}^2)^{1/2}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{E}{m} \qquad (12)$$

Shell energy

where we have added the special relativity expression (28) in Section 1.7. Equation (12) tells us how to use the map energy—a unicorn—to predict the frame energy directly measured by the shell observer as the stone streaks past.

Expression (12) for shell energy E_{shell} applies to a stone moving in any direction, not just along the r-coordinate. Why? Energy—including map energy E—is a *scalar*, a property of the stone independent of its direction of motion.

Different shell observers compute same map energy.

The shell observer knows only his local shell frame coordinates, which are restricted in order to yield a local inertial frame. He observes a stone zip through his local frame and disappear from that frame; he has no global view of the stone's path. However, equation (12) is valid for a stone in *every* local shell frame and for *every* direction of motion of the stone in that frame. The shell observer uses this equation and his local r—stamped on every shell—to compute the map energy E/m, then radios his result to every one of his fellow shell observers, for example, "The green-colored free stone has map energy $E/m = 3.7$." A different shell observer, at different map r, measures a different value of shell energy E_{shell}/m of the green stone as it streaks through his own local frame, typically in a different direction. However, armed with (12), every shell observer verifies the constant value of map energy of the green stone, for example $E/m = 3.7$.

In brief, each local shell observer carries out a real measurement of shell energy; from this result plus his knowledge of his r-coordinate he derives the value of the map energy E/m, then uses this map energy—a constant of motion—to predict results of shell energy measurements made by shell observers distant from him. The result is a multi-shell account of the entire orbit of the stone.

The entire scheme of shell observers depends on the existence of local shell frames, which cannot be built inside the event horizon. Now we turn to the experience of the diver who passes inward across the event horizon.

6.4 ■ RAINDROP CROSSES THE EVENT HORIZON

Convert t-coordinate to raindrop wristwatch time.

How to get inside the event horizon?

The Schwarzschild metric satisfies Einstein's field equations everywhere in the vicinity of a nonrotating black hole (except on its singularity at $r = 0$). Map coordinates alone may satisfy Schwarzschild and Einstein, but they do not satisfy us. We want to make every measurement in a local inertial frame. Shell frames serve this purpose nicely outside the event horizon, but we cannot construct stationary shells inside the event horizon. Moreover, the expression $(1 - 2M/r)^{-1/2}$ in energy equation (12) becomes imaginary inside the event horizon, which provides one more indication that shell energy does not apply there.

Yet everyone tells us that an unfortunate astronaut who crosses inward through the event horizon at $r = 2M$ inevitably arrives at the lethal central singularity $r = 0$. In the following chapter we build a local frame around a falling astronaut. To prepare for such a local diving frame, we start here as simply as possible: We ask the stone wearing a wristwatch that began our study of relativity (Section 1.1) to take a daring dive, to drop from initial rest far from the black hole and plunge inward to $r = 0$. We call this diving, wristwatch-wearing stone a **raindrop**, because on Earth raindrops fall from rest at a great height. By definition, the raindrop has no significant spatial extent; it has no frame, it is just a stone wearing a wristwatch.

Raindrop defined: stone dropped from rest far away

DEFINITION 2. Raindrop

A **raindrop** is a stone wearing a wristwatch, that freely falls inward starting from initial rest far from the center of attraction.

Map energy of a raindrop

Examine the map energy (8) of a raindrop. Far from the black hole $r \gg 2M$ so that $(1 - 2M/r) \to 1$. For a stone at rest there, $dr = d\phi = 0$ and the Schwarzschild metric tells us that $d\tau \to dt$. As a result, (8) becomes:

$$\frac{E}{m} \equiv \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \to 1 \quad \text{(raindrop: released from rest at } r \gg 2M) \qquad (13)$$

The raindrop, released from rest far from the black hole, must fall inward along a radial line. In other words, $d\phi = 0$ along the raindrop worldline. Formally we write:

$$\frac{d\phi}{d\tau} = 0 \qquad \text{(raindrop)} \qquad (14)$$

The raindrop-stone, released from rest at a large r map coordinate, begins to move inward, gradually picks up speed, finally plunges toward the center. As the raindrop hurtles inward, the value of $E/m\,(= 1)$ remains constant. Equation (12) then tells us that as r decreases, $2M/r$ increases, and so E_{shell} must also increase, implying an increase in v_{shell}. The local shell observer measures this increased speed directly. Equation (12) with $E/m = 1$ for the raindrop yields:

Shell energy of the raindrop

$$\frac{E_{\text{shell}}}{m} = \left(1 - v_{\text{shell}}^2\right)^{-1/2} = \left(1 - \frac{2M}{r}\right)^{-1/2} \qquad \text{(raindrop)} \qquad (18)$$

It follows immediately that:

$$v_{\text{shell}} = -\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop shell velocity)} \qquad (19)$$

where the negative value of the square root describes the stone's inward motion. Equation (19) shows that the shell-measured speed of the

Box 2. Slow speed + weak field $\Longrightarrow$ Mass + Newtonian KE and PE

"If you fall, I'll be there." —Floor

The map energy E/m may be a unicorn in general relativity, but it is a genuine race horse in Newtonian mechanics. We show here that the map energy E/m of a stone moving at non-relativistic speed in a weak gravitational field reduces to the mass of the stone plus the familiar Newtonian energy (kinetic + potential). Rearrange (12) to read:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)^{1/2} \left(1 - v_{\text{shell}}^2\right)^{-1/2} \qquad (15)$$

For $r \gg 2M$ (weak gravitational field) and $v_{\text{shell}}^2 \ll 1$ (non relativistic stone speed) use the approximation inside the front cover twice:

$$\left(1 - \frac{2M}{r}\right)^{1/2} \approx 1 - \frac{M}{r} \qquad (r \gg 2M) \qquad (16)$$

$$\left(1 - v_{\text{shell}}^2\right)^{-1/2} \approx 1 + \frac{1}{2}v_{\text{shell}}^2 \qquad (v_{\text{shell}}^2 \ll 1)$$

Substitute these into (15) and drop the much smaller product $(M/2r)v_{\text{shell}}^2$. The result is an approximation:

$$E \approx m + \frac{1}{2}mv_{\text{shell}}^2 - \frac{Mm}{r} \qquad (17)$$

$$\left(r \gg 2M,\ \ v_{\text{shell}}^2 \ll 1\right)$$

In this equation $-Mm/r$ is the gravitational potential energy of the stone. (In conventional mks units it would read $-GM_{\text{kg}}m_{\text{kg}}/r$.) We recognize in (17) Newtonian's kinetic energy (KE) plus his potential energy (PE) of a stone, with the added stone's mass m.

As a jockey in curved spacetime, you must beware of riding the unicorn map energy E/m; gravitational potential energy is a fuzzy concept in general relativity. Dividing energy into separate kinetic and potential forms works only under special conditions, such as those given in equation (16).

Except for these special conditions, we expect the map constant of motion E to differ from E_{shell}: The local shell frame is inertial and excludes effects of curved spacetime. In contrast, map energy E—necessarily expressed in map coordinates—includes curvature effects, which Newton attributes to a "force of gravity."

The approximation in (17) is quite profound. It reproduces a central result of Newtonian mechanics without using the concept of force. In general relativity, we can always eliminate gravitational force (see inside the back cover).

raindrop—the magnitude of its velocity—increases to the speed of light at the event horizon. This is a limiting case, because we cannot construct a shell—even in principle—at the exact location of the event horizon.

Objection 3. *I am really bothered by the idea of a material particle such as a stone traveling across the event horizon as a particle. The shell observer sees it moving at the speed of light, but it takes light to travel at light speed. Does the stone—the raindrop—become a flash of light at the event horizon?*

No. Be careful about limiting cases. No shell can be built *at* the event horizon, because the initial gravitational acceleration increases without limit there (Section 6.7). An observer on a shell just outside the event horizon clocks the diving stone to move with a speed *slightly less* than the speed of light. Any directly-measured stone speed less than the speed of light is perfectly legal in relativity. So there is no contradiction.

Raindrop dr/dt

Compare the shell velocity (19) of the raindrop with the value of dr/dt at a given r-coordinate. To derive dr/dt, solve the right-hand equation in (13) for $d\tau$ and substitute the result into the Schwarzschild metric with $d\phi = 0$. Result for the raindrop:

Sample Problems 1. The Neutron Star Takes an Aspirin

Neutron Star Gamma has a total mass 1.4 times that of our Sun and a map $r_0 = 10$ kilometers. An aspirin tablet of mass one-half gram falls from rest at a large r coordinate onto the surface of the neutron star. An advanced civilization converts into useful energy the entire kinetic energy of the aspirin tablet, measured in the local surface rest frame. Estimate how long this energy will power a 100-watt bulb. Repeat the analysis to find the useful energy for the case of an aspirin tablet falling from a large r coordinate onto the surface of Earth.

SOLUTION

From the value of the mass of our Sun (inside the front cover), the mass of the neutron star is $M \approx 2 \times 10^3$ meters. Hence $2M/r_0 \approx 2/5$. Far from the neutron star the total map energy of the aspirin tablet equals its rest energy, namely its mass, hence $E/m = 1$. From (18), the shell energy of the aspirin tablet just before it hits the surface of the neutron star rises to the value

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r_0}\right)^{-1/2} \approx 1.3 \qquad \text{(Neutron Star)} \tag{20}$$

The shell *kinetic energy* of the half-gram aspirin tablet is 0.3 of its rest energy. The rest energy is $m = 0.5$ gram = 5 $\times 10^{-4}$ kilogram or $mc^2 = 4.5 \times 10^{13}$ joules. The fraction 0.3 of this is 1.35×10^{13} joules. One watt is one joule/second; a 100-watt bulb consumes 100 joules per second. At that rate, the bulb can burn for 1.35×10^{11} seconds on the kinetic energy of the aspirin tablet. One year is about 3×10^7 seconds. Result: The kinetic energy of the half-gram aspirin tablet falling to the surface of Neutron Star Gamma from a large r coordinate provides energy sufficient to light a 100-watt bulb for approximately 4500 years!

What happens when the aspirin tablet falls from a large r coordinate onto Earth's surface? Set the values of M and r_0 to those for Earth (inside front cover). In this case $2M \ll r_{\text{E}}$, so equation (20) becomes, to a very good approximation:

$$\frac{E_{\text{shell}}}{m} \approx \left(1 + \frac{M}{r_0}\right) \approx 1 + 6.97 \times 10^{-10} \qquad \text{(Earth)} \tag{21}$$

Use the same aspirin tablet rest energy as before. The lower fraction of kinetic energy yields 3.14×10^4 joules. At 100 joules per second the kinetic energy of the aspirin tablet will light the 100-watt bulb for 314 seconds, or a little more than 5 minutes.

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2} \qquad \text{(raindrop)} \tag{22}$$

Raindrop dr/dt: a unicorn!

Equation (22) shows an apparently outrageous result: as the raindrop reaches the event horizon at $r = 2M$, its Schwarzschild dr/dt drops to zero. (This result explains the strange spacing of event-dots along the orbit approaching the event horizon in Figure 3.6.) Does any local observer witness the stone coasting to rest? No! Repeated use of the word "map" reminds us that map velocities are simply spreadsheet entries for the Schwarzschild mapmaker and need not correspond to direct measurements by any local observer. Figure 2 shows plots of both shell speeds and map $dr/d\tau$ of the descending raindrop. Nothing demonstrates more clearly than the diverging lines in Figure 2 the radical difference between (unicorn) map entries and the results of direct measurement.

Does the raindrop cross the event horizon or not? To answer that question we need to track the descent with its directly-measured wristwatch time, not the global t-coordinate. Use equation (13) to convert global coordinate differential dt to wristwatch differential $d\tau$. With this substitution, (22) becomes:

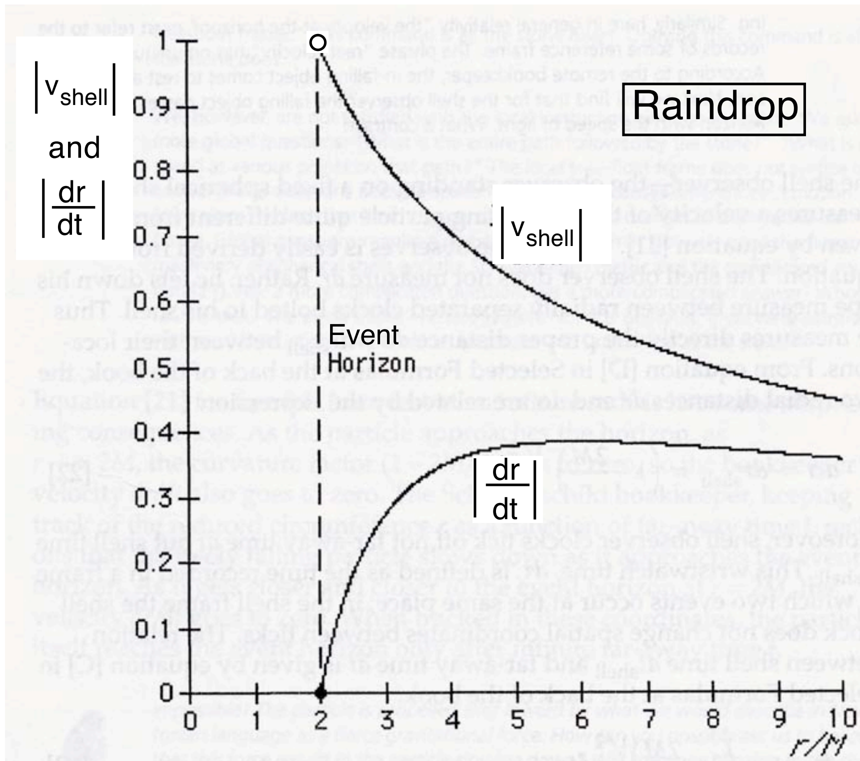

FIGURE 2 Computer plot of the speed $|v_{\rm shell}|$ of a raindrop directly measured by shell observers at different r-values, from (19), and its Schwarzschild map speed $|dr/dt|$ from (22). Far from the black hole the raindrop is at rest, so both speeds are zero, but both speeds increase as the raindrop descends. Map speed $|dr/dt|$ is not measured but computed from spreadsheet records of the Schwarzschild mapmaker. At the event horizon, the measured shell speed rises to the speed of light, while the computed map speed drops to zero. The upper open circle at $r = 2M$ reminds us that this is a limiting case, since no shell can be constructed *at* the event horizon. (Why not? See the Appendix, Section 6.7.)

$$\frac{dr}{d\tau_{\rm raindrop}} = -\left(\frac{2M}{r}\right)^{1/2} \qquad (23)$$

Raindrop crosses the event horizon.

Expression (23) combines a map quantity dr with the differential advance of the wristwatch $d\tau_{\rm raindrop}$. It shows that the raindrop's r-coordinate decreases as its wristwatch time advances, so the raindrop passes inward through the event horizon. Indeed, inside the event horizon the magnitude of $dr/d\tau_{\rm raindrop}$ becomes greater than one, and increases without limit as $r \to 0$. But this need not worry us: Both r and dr are map quantities, so $dr/d\tau$ is just an entry on the mapmaker's spreadsheet, not a quantity measured by anyone.

Comment 1. How do we find the value of dr inside the event horizon?
The numerator dr on the left side of (23) has a clear meaning only *outside* the

Box 3. Newton Predicts the Black Hole?

It's remarkable how well much of Newton's mechanics works—sort of—on the stage of general relativity. One example is that Newton appears to predict the r-coordinate of the event horizon $r = 2M$. Yet the meaning of that barrier is strikingly different in the two pictures of gravity, as the following analysis shows.

A stone initially at rest far from a center of attraction drops inward. Or a stone on the surface of Earth or of a neutron star is fired outward along r, coming to rest at a large r coordinate. In either case, Newtonian mechanics assigns the same total energy (kinetic plus potential) to the stone. We choose the gravitational potential energy to be zero at the large r coordinate, and the stone out there does not move. From (17), we then obtain

$$\frac{E}{m} - 1 = \frac{v^2}{2} - \frac{M}{r} = 0 \quad \text{(Newton)} \qquad (24)$$

From (24) we derive the diving (or rising) speed at any r-coordinate:

$$|v| = \left(\frac{2M}{r}\right)^{1/2} \qquad \text{(Newton)} \qquad (25)$$

which is the same as equation (19) for the shell speed of the raindrop. One can predict from (25) the r-value at which the speed reaches one, the speed of light, which yields $r = 2M$, the black hole event horizon. For Newton the speed of light is the **escape velocity** from the event horizon.

Newton assumes a single universal inertial reference frame and universal time, whereas (19) applies only to shell separation divided by shell time. A quite different expression (22) describes dr/dt—map differential dr divided by map differential dt—for raindrops.

Does Newton correctly describe black holes? No. Newton predicts that a stone launched radially outward from the event horizon with a speed less than that of light will rise to higher r, slow, stop without escaping, then fall back. In striking contrast, Einstein predicts that nothing, not even light, can be successfully launched outward from inside the event horizon, and that light launched outward *exactly at* the event horizon hovers there, balanced as on a knife-edge (Box 4).

event horizon, where every shell displays the stamped value of r. Box 7.3 in Section 7.3 describes one practical method by which a descending rain observer can measure map r, both outside and inside the event horizon.

6.5 ■ GRAVITATIONAL MASS

A new way to measure total energy

Mass m of the stone

This book uses the word *mass* in two different ways. Symbol m in equations (8) and (11) represents the inertial mass of a test particle, which we call a *stone*. By definition, the mass of a stone is too small to curve spacetime by a detectable amount. Expression (8) measures the stone's map energy E and mass m in the same units.

Mass M of the center of attraction

The mass M of the center of attraction is quite different: It is the gravitational mass that curves spacetime, as reflected in the global metric expression $(1 - 2M/r)$.

Drop a stone of mass m into a star of mass M.

What happens when a stone of mass m falls into a black hole of mass M? Does the swallowed mass m increase the black hole's mass? Our new understanding of energy helps us to calculate how much the mass of a black hole grows when it swallows matter—and yields a surprising result. To begin, start with a satellite orbiting close to a star. How can we measure the total gravitational mass of the star-plus-satellite system? We make this measurement using the initial acceleration of a distant test particle so remote

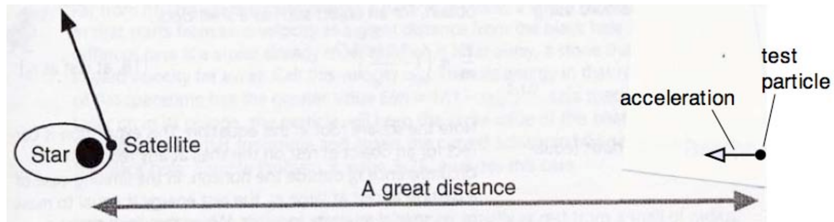

FIGURE 3 Measure the total mass-energy $M_{\rm total}$ of a central star-satellite system using the acceleration of a test particle at a large r coordinate, analyzed using Newtonian mechanics.

that Newtonian mechanics gives a correct result (Figure 3). In units of inverse meters, Newton's expression for this acceleration is:

$$a = -\frac{M_{\rm total}}{r^2} \qquad \text{(Newton)} \qquad (26)$$

Newton says, "Add m to $M_{\rm star}$."

What is $M_{\rm total}$? In Newtonian mechanics total mass equals the mass $M_{\rm star}$ of the original star plus the mass m of the satellite orbiting close to it:

$$M_{\rm total} = M_{\rm star} + m \qquad \text{(Newton)} \qquad (27)$$

Could this also be true in general relativity? The answer is no, but proof requires a sophisticated analysis of Einstein's equations.

Birkhoff's theorem

A mathematical theorem of general relativity due to G. D. Birkhoff in 1923 states that the spacetime outside any spherically symmetric distribution of matter and energy is completely described by the Schwarzschild metric with a *constant* gravitational mass $M_{\rm total}$, no matter whether that spherically symmetric source is at rest or, for example, moving inward or outward along the r-coordinate.

$M_{\rm total}$ includes contracting bubble of dust.

In order to apply Birkhoff's theorem, we approximate the moving satellite of Figure 3 by the inward-falling uniform spherical bubble of Figure 4, a bubble composed of unconnected particles—dust—whose total mass m is the same as that of the satellite in Figure 3. (We use the label "bubble" instead of "shell" to avoid confusion with the stationary concentric shells we construct around a black hole on which we make measurements and observations.) This falling uniform dust bubble satisfies the condition of Birkhoff's theorem, so the Schwarzschild metric applies outside this inward-falling bubble.

How does dust bubble increase $M_{\rm total}$?

Unfortunately, Birkhoff's theorem does not tell us how to calculate the value of $M_{\rm total}$, only that it is a constant for any spherically symmetric configuration of mass/energy. What property of the dust bubble remains constant as it falls inward? Its inertial mass m? Not according to special relativity! Inertial mass is *not* conserved; it can be converted into energy. We had better look for a conserved energy for our infalling dust bubble. Equation (12) is our guide: At a given r-coordinate every particle of dust in the collapsing bubble falls inward at the same rate, so the measure of the total

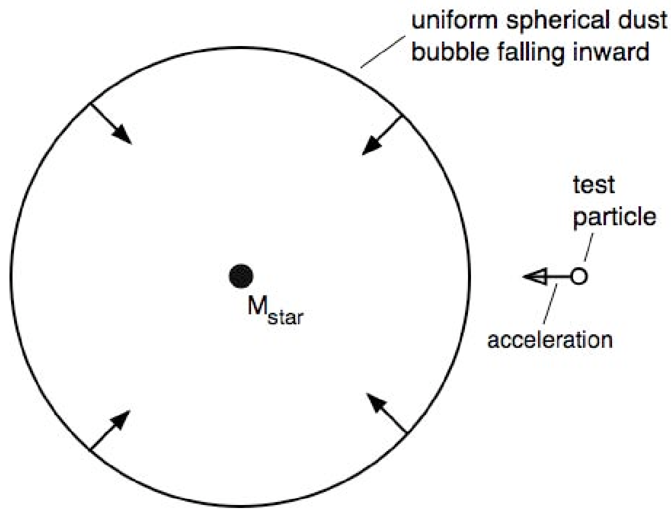

FIGURE 4 Replace the moving satellite of Figure 3 with an inward-falling uniform spherical bubble of dust that satisfies the condition of Birkhoff's theorem, so the Schwarzschild metric applies outside the contracting dust bubble.

shell energy E_{shell} of the bubble at a given r-coordinate is the sum over the individual particles of the dust bubble. Clearly from (12), successive shell observers at successively smaller r-coordinates measure successively higher values of E_{shell} as the collapsing dust bubble falls past them, so we cannot use shell energy in the Birkhoff analysis.

Einstein: "Add dust bubble E to M_{star} to find M_{total}."

However, the Schwarzschild map energy E *does* remain constant during this collapse. So instead of the Newtonian expression (27) we have the trial general relativity replacement:

$$M_{\text{total}} = M_{\text{star}} + E \qquad \text{(Einstein)} \qquad (28)$$

Check validity of (28).

How do we know whether or not the total map energy E of the dust bubble is the correct constant to add to M_{star} in order to yield the total mass M_{total} of the system? One check is that when the satellite/dust bubble is far from the star ($r \gg 2M_{\text{total}}$) but the remote test particle is still exterior to the dust bubble, then $E \to E_{\text{shell}}$ from (12). In addition, for a slow-moving satellite/dust bubble, $E \to E_{\text{shell}} \to m$, and we recover Newton's formula (27), as we should in the limits $r \gg 2M$ and $v^2_{\text{shell}} \ll 1$. And when the satellite/dust bubble falls inward so that our stationary shell observer measures $E_{\text{shell}} > m$, then equation (28) remains valid, because $E(\approx m)$ does not change. Note that Birkhoff's Theorem is satisfied in this approximation.

Result: Convert stone map E into gravitational mass.

If (28) is correct, then general relativity merely replaces Newton's m in (27) with total map energy E, a constant of motion for the satellite/bubble. Thus the mass of a star or black hole grows by the value of the map energy E of a stone or collapsing bubble that falls into it. *The map energy of the stone is converted into gravitational mass.* Earlier we called map energy E "a unicorn, a mythical beast." Now we must admit that this unicorn can add its mass-equivalence to the mass of a star into which it falls.

?

Objection 4. *You checked equation (28) only in the Newtonian limit, where the remote dust bubble is at rest or falls inward with small kinetic energy. Is (28) valid for all values of E? Suppose that the dust bubble in Figure 4 is launched inward (or outward) at relativistic speed. In this case does total E still simply add to M_{star} to give total mass M_{total} for the still more distant observer?*

!

Yes it does, but we have not displayed the proof, which requires solution of Einstein's equations. Let a massive star collapse, then explode into a supernova. If this process is spherically symmetric, then a distant observer will detect no change in gravitational attraction in spite of the radical conversions among different forms of energy in the explosion. Indeed, the distant observer has no way to know about these transformations before the outward-blasting bubble of radiation and neutrinos passes her. As they pass, she detects a decline in the gravitational acceleration of the local test particle, because some of the original energy of the central attractor is carried to an r-value greater than hers.

Gravity waves carry off energy.

Is the Birkhoff restriction to spherical symmetry important? It can be: A satellite orbiting around or falling into a star or black hole will emit gravitational waves that carry away some energy, decreasing M_{total}. Chapter 16 notes that a spherically symmetric distribution cannot emit gravitational waves, no matter how that spherical distribution pulses in or out. As a result, equation (28) is okay to use only when the emitted gravitational wave energy is very much less than M_{total}. When that condition is met, the cases shown in Figures 3 and 4 are observationally indistinguishable.

Measuring E from far away.

As long as gravitational wave emission is negligible and we are sufficiently far away, we can, in principle, use (28) to measure the map energy E of *anything* circulating about, diving into, launching itself away from, or otherwise interacting with a center of attraction. Simply use Newtonian mechanics to carry out the measurement depicted in Figure 3, first with the satellite absent, second with the satellite in orbit near the star. Subtract the second value from the first for the acceleration (26) and use (28) to determine the value of $E = M_{\text{total}} - M_{\text{star}}$. As in Box 2, this example shows that E (and not E_{shell}) includes effects of curved spacetime.

Box 4. Event Horizon vs. Particle Horizon

The *event horizon* around any black hole separates events that can affect the future of observers outside the event horizon from events that cannot do so. Barring quantum mechanics, the event horizon never reveals what is hidden behind it. (For a possible exception, see Box 5 on Hawking radiation.)

We can now define a black hole more carefully: *A black hole is a singularity cloaked by an event horizon.*

In Chapter 14 we learn about another kind of horizon, called a **particle horizon**. Some astronomical objects are so far from us that the light they have emitted since they were formed has not yet reached us. In principle more and more such objects swim into our distant field of view every day, as our cosmic particle horizon sweeps past them. In contrast to the event horizon, the particle horizon yields up its hidden information to us—gradually!

In order to avoid confusion among these different kinds of horizons, we try to be consistent in using the full name of the *event horizon* that cloaks a black hole.

6.6 ■ OVER THE EDGE: ENTERING THE BLACK HOLE

No jerk. No jolt. A hidden doom.

Except for the singularity at $r = 0$, no feature of the black hole excites more curiosity than the event horizon at $r = 2M$. It is the point of no return beyond which no traveler can find the way back—or even send a signal—to the outside world. What is it like to fall into a black hole? No one from Earth has yet experienced it. Moreover, we predict that future explorers who do so will not be able to return to report their experiences or to transmit messages about their experience to us outsiders—so we believe! In spite of the impossibility of receiving a final report, there exists a well-developed and increasingly well-verified body of theory that makes clear predictions about our experience as we approach and cross the event horizon of a black hole. Here are some of those predictions.

Predict what no one can verify.

We are not "sucked into" a black hole. Unless we get close to its event horizon, a black hole will no more grab us than our Sun grabs Earth. If our Sun should suddenly collapse into a black hole without expelling any mass, Earth and the other planets would continue on their courses undisturbed (even though, after eight minutes, continuous night would prevail for us on Earth!). The Schwarzschild solution (plus the Principle of Maximal Aging) would still continue to describe Earth's worldline around our Sun, just as it does now. In Section 6.7 you show that for an orbit at r-coordinate greater than about $300M$, Newtonian mechanics predicts gravitational acceleration with an accuracy of about 0.3 percent. We also find (Section 9.5) that no stable circular orbit is possible at r less than $6M$. Even at an r-value between $6M$ and the event horizon at $2M$, we can always escape the grip of the black hole, given sufficient rocket power. Only when we reach or cross the event horizon are we irrevocably swallowed, our fate sealed.

We are not sucked into a black hole.

We detect no special event as we fall inward through the event horizon. Even when we drop across the event horizon at $r = 2M$, we experience no shudder, jolt, or jar. True, tidal forces are ever-increasing as we fall inward, and this increase continues smoothly as we cross the event horizon.

No jolt as we cross the event horizon.

Box 5. Escape from the Black Hole? Hawking Radiation

Einstein's field equations predict that nothing, not even a light signal, escapes from inside the event horizon of a black hole. In 1973, Stephen Hawking demonstrated an exception to this conclusion using quantum mechanics. For years quantum mechanics had been known to predict that particle-antiparticle pairs—such as an electron and a positron—are continually being created and recombined in "empty" space, despite the frigidity of the vacuum. These processes have indirect, but significant and well-tested, observational consequences. Never in cold flat spacetime, however, do such events present themselves to direct observation. For this reason the pairs receive the name "virtual particles." When such a particle-antiparticle pair is produced near, but outside, the event horizon of a black hole, Hawking showed, one member of the pair will occasionally be swallowed by the black hole, while the other one escapes to a large r coordinate—now a *real* particle. Escaped particles form what is called **Hawking radiation.** Before particle emission, we had just the black hole; after particle emission, we have the black hole plus the distant real particle outside the event horizon. Where did the energy of this distant particle come from? In order to conserve mass/energy, the mass of the black hole must decrease in this process. This loss of mass causes the black hole to "evaporate." As the mass of the black hole decreases, the loss rate grows until eventually it becomes explosive, destroying the black hole. For a black hole of several solar masses, however, Hawking's theory predicts that the Earth-time required to achieve this explosive state exceeds the age of the Universe by a fantastic number of powers of ten. For this reason, we ignore Hawking radiation in our description of black holes.

We are not suddenly squashed or torn apart at $r = 2M$, because the event horizon is not a *physical* singularity, as explained in Box 3, Section 3.1. There is no sudden discontinuity in our experience as we pass through the event horizon.

No shell frames inside the event horizon.

Inside the event horizon no shell frames are possible. Outside the event horizon we have erected, in imagination, a set of nested spherical shells concentric to the black hole. We say "in imagination" because no known material is strong enough to withstand the "pull of gravity," which increases without limit as we approach the event horizon from outside (Section 6.7). Locally such a stationary shell can be replaced by a spaceship with rockets blasting in the inward direction to keep it at the same r and ϕ coordinates. Inside the event horizon, however, nothing remains at rest. No shell, no rocket ship can remain at constant r-coordinate there, however ferocious the blast of its engines. The material composing the original star, no matter how strong, was itself unable to resist the collapse that formed the black hole. The same irresistible collapse forbids any stationary structure or object inside the event horizon.

Packages can move inward, not outward.

"Outsiders" can send packages to "insiders." Inside the event horizon, different local frames can still move past one another with measurable relative speeds. Here are some examples. *First local frame:* One traveler may drop from rest just outside the event horizon. *Second local frame:* An unpowered spaceship may fall in from far away. *Third local frame:* Another unpowered spaceship may be hurled inward from outside the event horizon. Light and radio waves can carry messages inward to us. We who have fallen inside the event horizon can still see the stars, though with directions, colors, and intensities that change as we fall (Chapters 11 through 13). Packages and communications sent inward across the event horizon? Yes. How about moving

Box 6. Baked on the Shell?

As you stand on a spherical shell close to the event horizon of a black hole, you are crushed by an unsupportable local gravitational acceleration directed downward toward the center (Section 6.7). If that is not enough, you are also enveloped by an electromagnetic radiation field. William G. Unruh used quantum field theory to show that the temperature *T* of this radiation field (in degrees Kelvin) experienced on the shell is given by the equation

$$T = \frac{h g_{\text{conv}}}{4\pi^2 k_{\text{B}} c} \qquad (29)$$

Here g_{conv} is the local acceleration of gravity expressed in conventional units, meters/second^2; h is Planck's constant; c is the speed of light; and k_{B} is **Boltzmann's constant,** which has the value 1.381×10^{-23} kilogram-meters^2/(second^{2}degree Kelvin). The quantity $k_{\text{B}}T$ has the unit joules and gives the average ambient thermal energy of this radiation field. (The same radiation field surrounds you when you accelerate at the rate g_{conv} in flat spacetime.)

Section 6.7 derives an expression for the local gravitational acceleration on a shell at r. Equation (46) gives the magnitude of this acceleration, expressed in the unit meter^{-1}:

$$g_{\text{shell}} = \frac{g_{\text{conv}}}{c^2} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} \qquad (30)$$

Substitute g_{conv} from (30) into (29) to obtain

$$T = \frac{hc}{4\pi^2 k_{\text{B}}} \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} \qquad (31)$$

with M in meters. This temperature increases without limit as you approach the event horizon. Therefore one would expect the radiation field near the event horizon to shine brighter than any star when viewed by a distant observer. Why doesn't this happen? In a muted way it does happen. Remember that radiation is gravitationally red-shifted as it moves away from any center of attraction. Every frequency is red-shifted by the factor $(1 - 2M/r)^{1/2}$, which cancels the corresponding factor in (31). For radiation coming from near the event horizon, let $r \to 2M$ in the resulting equation. The distant viewer sees the radiation temperature

$$T_H = \frac{hc}{16\pi^2 k_{\text{B}} M} \qquad \text{(distant view of event horizon)} \qquad (32)$$

with M in meters. The temperature T_{H} is called the **Hawking temperature** and characterizes the Hawking radiation from a black hole (Box 5). Notice that this temperature *increases* as the mass M of the black hole *decreases*. Even for a black hole whose mass is only a few times that of our Sun, this temperature is extremely low, so from far away such a black hole really looks *almost* black.

The radiation field described by equations (29) through (32), although perfectly normal, leads to strange conclusions. Perhaps the strangest is that this radiation goes entirely undetected by a free-fall observer. The free-fall diving traveler observes no such radiation field, while for the shell observer the radiation is a surrounding presence. This paradox cannot be resolved using the classical general relativity theory presentged in this book; see Kip Thorne's *Black Holes and Time Warps: Einstein's Outrageous Legacy*, page 444.

How realistic is the danger of being baked on a shell near the event horizon of a black hole? In answer, compute the local acceleration of gravity for a shell where the radiation field reaches a temperature equal to the freezing point of water, 273 degrees Kelvin. From (29) you can show that $g_{\text{conv}} = 6.7 \times 10^{22}$ meters/second^2, or almost 10^{22} times the acceleration of gravity on Earth's surface. Evidently we will be crushed by gravity long before we are baked by radiation!

outward through the event horizon? No. Box 4 tells us—and Section 7.6 demonstrates—that when a diver fires a light flash radially outward at the instant she passes inward through the event horizon, that light flash hovers at the same r-coordinate at the event horizon. Nothing moves faster than light, so if light cannot move outward through the event horizon, then packages and stones definitely cannot move outward there either.

Surf a collapsing galaxy group.

Inside the event horizon life goes on—for a while. Make a daring dive into an already mature black hole? No. We and our exploration team want to be still more daring, to follow a black hole as it forms. We go to a multiple-galaxy system so crowded that it teeters on the verge of gravitational collapse. Soon after our arrival at the outskirts, it starts the collapse, at first slowly, then more and more rapidly. Soon a mighty avalanche thunders

Box 7. General relativity is a classical (non-quantum) theory.

Newton's laws describe the motion of a stone in flat spacetime at speeds very much less than the speed of light. For higher speeds we need relativity. Newton's laws correctly describe slow-speed motion of a "stone" more massive than, say, a proton. To describe behavior of smaller particles we need quantum physics.

Does this mean that we have no further use for Newton's laws of motion? Not at all! Newton's laws are *classical*, that is non-quantum. In this book we repeatedly use Newton's mechanics as a simple, intuitive first cut at prediction and observation. And with it we check every prediction of relativity in the limit of slow speed and vanishing spacetime curvature. We expect that Newton's laws of motion will be scientifically useful as long as humanity survives.

General relativity is also a *classical*—non-quantum—theory. General relativity does not predict Hawking radiation (Box 5) or the Hawking temperature (Box 6). These are predictions of quantum field theory, predictions that we mention as important asides to our classical analysis.

General relativity does not correctly represent every property of the black hole, any more than Newton's mechanics correctly predicts the motion of fast-moving particles. We still expect that general relativity—along with Newton's mechanics—will be scientifically useful during the long future of humanity.

(silently!) toward the center from all directions, an avalanche of objects and radiation, a cataract of momentum-energy-pressure. The matter of the galaxies and with it our group of enterprising explorers pass smoothly across the event horizon at Schwarzschild $r = 2M$.

"Publish *and* perish."

From that moment onward we lose all possibility of signaling to the outer world. However, radio messages from that outside world, light from familiar stars, and packages fired after us at sufficiently high shell speed continue to reach us. Moreover, communications among us explorers take place now as they did before we crossed the event horizon. We use the familiar categories of space and time to share our findings. With our laptop computers we turn out an exciting journal of observations, measurements, and conclusions. (Our motto: "Publish *and* perish.")

Killer tides.

After crunch there is no "after."

Tides become lethal. Nothing rivets our attention more than the tidal forces that pull heads up and feet down with ever-increasing tension (Sections 1.11 and 10.2). Before much time has passed on our wristwatch, we can predict, this differential pull will reach the point where we can no longer survive. Moreover, we can foretell still further ahead and with certainty the instant of total crunch. That crunch swallows up not only the stars beneath us, not only we explorers, but time itself. All worldlines inside the event horizon terminate on the singularity. For us an instant comes after which there is no "after." Chapters 7 and 21 give more details of life inside the event horizon.

6.7 ■ APPENDIX: INITIAL SHELL GRAVITATIONAL ACCELERATION FROM REST

Unlimited gravitational acceleration on a shell near the event horizon.

Is gravity real or fictitious?

When you stand on a shell near a black hole, you experience gravity—a pull downward—just as you do on Earth. On the shell this gravity can be great: near the event horizon it increases without limit, as we shall see. On the other hand, "In general relativity . . . gravity is *always* a fictitious force which we can eliminate by changing to a local frame that is in free fall . . ." (inside the back cover). So is this "gravity" real? Falls kill and injure many people every year. Anything that can kill you is definitely real, not fictitious! Here we avoid philosophical issues by asking a practical question: "When the shell observer drops a stone from rest, what *initial* acceleration does he measure?"

Practical experiment to define gravity

To begin, we behave like an engineer: Use a thought experiment to define what we mean by the initial gravitational acceleration of a stone dropped from rest on a shell at r_0. Use the heavy machinery of general relativity to find the magnitude of the newly-defined acceleration experienced by a shell observer.

Figure 5 presents our method to measure quantities used to define initial gravitational acceleration on a shell. The shell is at map r_0. At a shell distance $|\Delta y_{\rm shell}|$ below the shell lies a stationary platform onto which the shell observer drops a stone. The time lapse $\Delta t_{\rm shell}$ for the drop is measured as follows:

Specific instructions for experiment to define gravity

1. The shell observer starts his clock at the instant he drops the stone.
2. When the stone strikes the platform, it fires a laser flash upward to the shell clock.
3. The shell observer determines the shell time lapse between drop and impact, $\Delta t_{\rm shell}$, by deducting flash transit shell time from the time elapsed on his clock when he receives the laser flash.

To calculate the "flash transit shell time" in Step 3, the shell observer divides the shell distance $|\Delta y_{\rm shell}|$ by the shell speed of light. (In an exercise of Chapter 3, you verified that the shell observer measures light to move at its conventional speed—value one—in an inertial frame.)

Define $g_{\rm shell}$

The shell observer substitutes $\Delta y_{\rm shell}$ and $\Delta t_{\rm shell}$ into the expression that defines uniform acceleration $g_{\rm shell}$:

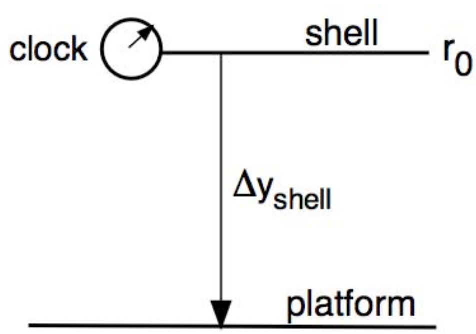

FIGURE 5 Notation for thought experiment to define initial gravitational acceleration from rest in a shell frame. The shell observer at r_0 releases a stone from rest and measures its shell time of fall $\Delta t_{\rm shell}$ onto a lower stationary platform that he measures to be a distance $|\Delta y_{\rm shell}|$ below the shell. From these observations he defines and calculates the value of the stone's initial acceleration $g_{\rm shell}$, equation (33).

$$\Delta y_{\rm shell} = -\frac{1}{2} g_{\rm shell} \Delta t^2_{\rm shell} \qquad \text{(uniform } g_{\rm shell}\text{)} \tag{33}$$

Thus far our engineering definition of $g_{\rm shell}$ has little to do with general relativity. The fussy procedure of this thought experiment reflects the care required when general relativity is added to the analysis, which we do now.

Mapmaker demands constant map energy for falling stone.

What does the Schwarzschild mapmaker say about the acceleration of a dropped stone? She insists that, whatever motion the free stone executes, its map energy E/m must remain a constant of motion. So start with the map energy of a stone bolted to the shell at r_0. From map energy equation (15) with $v_{\rm shell} = 0$ and $r = r_0$, we have:

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} \qquad \text{(stone released from rest at } r_0\text{)} \tag{34}$$

Now release the stone from rest. The mapmaker insists that as the stone falls its map energy remains constant, so equate the right sides of (34) and (8), square the result, and solve for $d\tau^2$:

$$d\tau^2 = \left(1 - \frac{2M}{r_0}\right)^{-1} \left(1 - \frac{2M}{r}\right)^2 dt^2 \tag{35}$$

Substitute this expression for $d\tau^2$ into the Schwarzschild metric for radial motion ($d\phi = 0$), namely

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{36}$$

Divide corresponding sides of equations (36) and (35), then solve the resulting equation for $(dr/dt)^2$:

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r_0}\right)^{-1} \left(1 - \frac{2M}{r}\right)^2 \left(\frac{2M}{r} - \frac{2M}{r_0}\right) \qquad \text{(from rest at } r_0\text{)} \tag{37}$$

We want the acceleration of the stone in Schwarzschild map coordinates. Take the derivative of both sides with respect to the t-coordinate and cancel the common factor $2(dr/dt)$ from both sides of the result to obtain:

$$\frac{d^2r}{dt^2} = -\left(\frac{M}{r^2}\right)\left(1 - \frac{2M}{r}\right)\left(1 - \frac{2M}{r_0}\right)^{-1}\left(\frac{4M}{r_0} + 1 - \frac{6M}{r}\right) \tag{38}$$

This equation gives the map acceleration at r of a stone released from rest at r_0. This acceleration depends on r, so is clearly *not* uniform as the stone falls, but *decreases* as r gets smaller, going to zero as r reaches the event horizon. We know that map acceleration is a unicorn, a result of Schwarzschild map coordinates, not measured by any inertial observer. We are interested in the *initial* acceleration at the instant of release from rest. Set $r = r_0$ in equation (38), which then reduces to the relatively simple form:

$$\left(\frac{d^2r}{dt^2}\right)_{r_0} = -\frac{M}{r_0^2}\left(1 - \frac{2M}{r_0}\right) \qquad \text{(initial, from rest at } r_0\text{)} \tag{39}$$

Acceleration in map coordinates

What is the meaning of this acceleration in Schwarzschild map coordinates? It is only a spreadsheet entry, an accounting analysis by the mapmaker, not the result of a direct observation by anyone. Observation requires an experiment on the shell, which we have already designed, leading to the expression (33). What is the relation between our engineering definition of acceleration and acceleration (39) in Schwarzschild coordinates? To compare the two expressions, expand the Schwarzschild r-coordinate of the dropped stone close to the radial position r_0 using a Taylor series for a short lapse Δt:

$$r = r_0 + \left(\frac{dr}{dt}\right)_{r_0}\Delta t + \frac{1}{2}\left(\frac{d^2r}{dt^2}\right)_{r_0}(\Delta t)^2 + \frac{1}{6}\left(\frac{d^3r}{dt^3}\right)_{r_0}(\Delta t)^3 + ... \tag{40}$$

Because Δt is small, we disregard terms higher than quadratic in Δt. This allows us to approximate uniform gravity (constant acceleration) and to compare mapmaker accounting entries with observed shell acceleration. Since the stone drops from rest at r_0, the initial map speed is zero: $(dr/dt)_{r_0} = 0$. With these considerations, insert (39) into (40) and obtain:

$$r - r_0 = \Delta r \approx -\frac{1}{2}\left[\left(1 - \frac{2M}{r_0}\right)\frac{M}{r_0^2}\right](\Delta t)^2 \tag{41}$$

This equation has a form similar to that of our experimental definition (33) of shell gravitational acceleration, except the earlier equation employs vertical shell separation Δy_{shell} and shell time lapse Δt_{shell}. Convert these to Schwarzschild quantities using standard transformations—equations (5.8) and (5.9):

$$\Delta y_{\text{shell}} = \left(1 - \frac{2M}{r_0}\right)^{-1/2}\Delta r \quad \text{and} \quad \Delta t_{\text{shell}}^2 = \left(1 - \frac{2M}{r_0}\right)(\Delta t)^2 \tag{44}$$

With these substitutions, and after rearranging terms, equation (33) becomes:

$$\Delta r = -\frac{1}{2}\left[\left(1 - \frac{2M}{r_0}\right)^{3/2} g_{\text{shell}}\right](\Delta t)^2 \tag{45}$$

As we go to the limit $\Delta t \to 0$, the extra terms in (40) become increasingly negligible, so (41) approaches an equality and we can equate square-bracket expressions in (41) and (45). Replacing the notation r_0 with r yields the magnitude of the initial acceleration of a stone dropped from rest on a shell at any r-coordinate:

Initial shell acceleration

Sample Problems 2. Initial Gravitational Acceleration on a Shell

1. On a shell at $r/M = 4$ near a black hole, the initial gravitational acceleration from rest is how many times that predicted by Newton?
2. On a shell at $r/M = 2.1$ near a black hole, the initial gravitational acceleration is how many times that predicted by Newton?
3. What is the minimum value of r/M so that, at or outside of that r-coordinate, Newton's formula for gravitational acceleration yields values that differ from Einstein's by less than ten percent? by less than one percent?
4. Compute the weight in pounds of a 100-kilogram astronaut on the surface of a neutron star with mass equal to $1.4M_{\text{Sun}}$ and $M/r_0 = 2/5$.

SOLUTIONS

1. At $r/M = 4$ the factor $(1 - 2M/r)^{-1/2}$ in (46) predicts a gravitational acceleration $2^{1/2} = 1.41$ times that predicted by Newton.
2. Even at $r/M = 2.1$ the gravitational acceleration is still the relatively mild multiple of 4.6 times the Newtonian prediction.
3. Setting $(1 - 2M/r)^{-1/2} = 1.1$ yields $r/M = 11.5$. At or outside this r-coordinate, Newton's prediction will be in error (it will be too low) by less than ten percent. At or outside $r/M = 100$ Newton's prediction will be too low by less than one percent.
4. The Newtonian acceleration in conventional units is:

$$g_{\text{Newton conv}} = \left(\frac{GM_{\text{kg}}}{c^2 r_0^2}\right) c^2 = \left(\frac{M}{r_0^2}\right) c^2 \qquad (42)$$
$$= \left(\frac{M}{r_0}\right)^2 \frac{c^2}{M} = \left(\frac{2}{5}\right)^2 \frac{c^2}{1.4 \times M_{\text{Sun}}}$$

Insert values of c^2 and M_{Sun} (in meters) to yield $g_{\text{Newton conv}} \approx 7.0 \times 10^{12}$ meters/second^2. From (46),

$$\text{weight} = m g_{\text{shell}} = \left(1 - \frac{4}{5}\right)^{-1/2} m g_{\text{Newton}} \qquad (43)$$
$$\approx 16 \times 10^{14} \text{ Newtons}$$

One Newton = 0.225 pounds, so our astronaut weighs approximately 3.5×10^{14} pounds, or 350 trillion pounds (USA measure of weight). It is surprising that, even at the surface of this neutron star, the general relativity result in (43) is greater than Newton's by the rather small factor $5^{1/2} = 2.24$.

$$g_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{M}{r^2} \qquad \text{(initial, drop from rest)} \qquad (46)$$

Sample Problems 2 explore initial shell accelerations under different conditions. It is surprising how accurate Newton's expression $g_{\text{Newton}} = M/r^2$ is even quite close to the event horizon of a black hole—an intellectual victory for Newton that we could hardly have anticipated.

QUERY 1. Gravitational acceleration on Earth's surface

Use values for the constants M_{E} and r_{E} for the Earth listed inside the front cover to show that equation (46) correctly predicts the value of the gravitational acceleration g_{E} at Earth's surface. Check your calculated values against those also listed inside the front cover.

A. Show that in units of length this acceleration has the value $g_{\text{E}} = 1.09 \times 10^{-16}$ meter^{-1}.

B. Show that in conventional units this acceleration has the value $g_{\text{E,conv}} = 9.81$ meters/second^2.

A GRAVITYLESS DAY

I am sitting here 93 million miles from the sun on a rounded rock which is spinning at the rate of 1,000 miles an hour, and roaring through space to nobody-knows-where, to keep a rendezvous with nobody-knows-what . . . and my head pointing down into space with nothing between me and infinity but something called gravity which I can't even understand, and which you can't even buy anyplace so as to have some stored away for a gravityless day . . .

—Russell Baker

6.8 ■ EXERCISES

1. Diving from Rest Far Away

Black Hole Alpha has a mass $M = 10$ kilometers. A stone starting from rest far away falls radially into this black hole. In the following, express all speeds as a decimal fraction of the speed of light.

A. What is the speed of the stone measured by the shell observer at $r = 50$ kilometers?

B. Write down an expression for $|dr/dt|$ of the stone as it passes $r = 50$ kilometers?

C. What is the speed of the stone measured by the shell observer at $r = 25$ kilometers?

D. Write down an expression for $|dr/dt|$ of the stone as it passes $r = 25$ kilometers?

E. In two or three sentences, explain why the change in the speed between Parts A and C is qualitatively different from the change in $|dr/dt|$ between Parts B and D.

2. Maximum Raindrop $|dr/dt|$

A stone is released from rest far from a black hole of mass M. The stone drops radially inward. Mapmaker records show that the the value of $|dr/dt|$ of the stone initially increases but declines toward zero as the stone approaches the event horizon. The value of $|dr/dt|$ must therefore reach a maximum at some intermediate r. Find this r-value for this maximum. Find the numerical value of $|dr/dt|$ at that r-value. Who measures this value?

3. Hitting a Neutron Star

A particular nonrotating neutron star has a mass $M = 1.4$ times the mass of our Sun and $r = 10$ kilometers. A stone starting from rest far away falls onto the surface of this neutron star.

A. If this neutron star were a black hole, what would be the map r-value of its event horizon? What fraction is this of the r-value of the neutron star?

B. With what speed does the stone hit the surface of the neutron star as measured by someone standing (!) on the surface?

C. With what value of $|dr/dt|$ does the stone hit the surface?

D. With what kinetic energy per unit mass does the stone hit the surface according to the surface observer?

Earlier it was thought that astronomical gamma-ray bursts might be caused by stones (asteroids) impacting neutron stars. Carry out a preliminary analysis of this hypothesis by assuming that the stone is made of iron. The impact kinetic energy is very much greater than the binding energy of iron atoms in the stone, greater than the energy needed to completely remove all 26 electrons from each iron atom, and greater even than the energy needed to shatter the iron nucleus into its component 26 protons and 30 neutrons. So we neglect all these binding energies in our estimate. The result is a vaporized gas of 26 electrons and 56 nucleons (protons and neutrons) per incident iron atom. We want to find the average energy of photons (gamma rays) emitted by this gas.

E. Explain briefly why, just after impact, the electrons have very much less kinetic energy than the nucleons. So in what follows we neglect the initial kinetic energy of the electron gas just after impact.

F. The hot gas emits thermal radiation with characteristic photon energy approximately equal to the temperature. What is the characteristic energy of photons reaching a distant observer, in MeV?

NOTE: It is now understood that astronomical gamma-ray bursts release much more energy than an asteroid falling onto a neutron star. Gamma ray bursts are now thought to arise from the birth of new black holes in distant galaxies.

4. A Stone Glued to the Shell Breaks Loose

A stone of mass m glued to a shell at r_0 has map energy given by equation (34). Later the glue fails so that the stone works loose and drops to the center of the black hole of mass M.

A. By what amount ΔM does the mass of the black hole increase?

B. A distant observer measures the mass of black hole plus stone at rest at r_0 using the method of Figure 3. How will the value of this total mass change after the stone has fallen into the black hole?

C. Apply your result of Part A to find the numerical value of the constant K in the equation $\Delta M = Km$ for the three cases: (a) $r_0 \gg 2M$, (b) $r_0 = 8M$ and (c) r_0 is just outside the event horizon. In all cases the observer in Figure 3 is much farther away than r_0.

5. Wristwatch Time to the Center

An astronaut drops from rest off a shell at r_0. How long a time elapses, as measured on her wristwatch, between letting go and arriving at the center of the black hole? If she drops off the shell just outside the event horizon, what is her event-horizon-to-crunch wristwatch time?

Several hints: The first goal is to find $dr/d\tau$, the rate of change of r-coordinate with wristwatch time τ, in terms of r and r_0. Then form an integral whose variable of integration is r/r_0. The limits of integration are from $r/r_0 = 1$ (the release point) to $r/r_0 = 0$ (the center of the black hole). The integral is

$$\frac{\tau}{M} = -\frac{1}{2^{1/2}}\left(\frac{r_0}{M}\right)^{3/2}\int_1^0 \frac{(r/r_0)^{1/2}\, d\,(r/r_0)}{(1-r/r_0)^{1/2}} \tag{47}$$

Solve this integral using tricks, nothing but tricks: Simplify by making the substitution $r/r_0 = \cos^2\psi$ (The “angle” ψ is not measured anywhere; it is simply a variable of integration.) Then $(1 - r/r_0)^{1/2} = \sin\psi$ and $d(r/r_0) = -2\cos\psi\sin\psi\, d\psi$ The limits of integration are from $\psi = 0$ to $\psi = \pi/2$. With these substitutions, the integral for wristwatch time becomes

$$\begin{aligned}\frac{\tau}{M} &= 2^{1/2}\left(\frac{r_0}{M}\right)^{3/2}\int_0^{\pi/2}\cos^2\psi d\psi \\ &= 2^{1/2}\left(\frac{r_0}{M}\right)^{3/2}\left[\frac{\psi}{2}+\frac{\sin 2\psi}{4}\right]\Bigg|_0^{\pi/2}\end{aligned} \tag{48}$$

Both sides of (48) are unitless. Complete the formal solution. For a black hole 20 times the mass of our Sun, how many seconds of wristwatch time elapse between the drop from rest just outside the event horizon to the singularity?

6. Release a stone from rest

You release a stone from rest on a shell of map coordinate r_0.

A. Derive an expression for $|dr/dt|$ of the stone as a function of r. Show that when the stone drops from rest far away, $|dr/dt|$ reduces to the expression (22) for a raindrop. Find the r-value at which map speed is *maximum* and the expression for that maximum map speed. Verify that in the limit in which the stone is dropped from rest far away, these expressions reduce to those found in Exercise 6.2 for the raindrop.

B. Derive an expression for the *shell velocity* of the stone as a function of r. Show that in the limit in which the stone drops from rest far away, the shell velocity reduces to the expression (19) for a raindrop.

C. Sketch graphs of shell speed *vs.* r similar to Figure 2 for the following values of r_0:
   (a) $r_0/M = 10$
   (b) $r_0/M = 6$
   (c) $r_0/M = 3$

7. Hurl a stone inward from far away

You hurl a stone radially inward with speed v_{far} from a remote location. (At a remote r where spacetime is flat, $|dr/dt|$ equals shell speed.)

A. Derive an expression for dr/dt of the stone as a function of r. Show that when you launch the stone from rest, dr/dt reduces to the expression (22) for a raindrop. Find the value of r at which $|dr/dt|$ is *maximum* and the expression for $|dr/dt|$. Verify that in the limit in which the stone is dropped from rest far away, these expressions reduce to those found in Exercise 6.2 for the raindrop.

B. Derive an expression for the *shell velocity* of the stone as a function of r. Show that in the limit in which the stone drops from rest far away, the shell velocity reduces to the expression (19) for a raindrop.

C. Sketch graphs of shell speed *vs.* r similar to Figure 2 for the following values of v_{far}:
   (a) $v_{\text{far}} = 0.20$
   (b) $v_{\text{far}} = 0.60$
   (c) $v_{\text{far}} = 0.90$

8. All Possible Shell Speeds

Think of a shell observer at any $r > 2M$. Consider the following three launch methods for a stone that passes him moving radially inward:(a) released at rest from a shell at $r_0 \geq r$, (b) released from rest far away, and (c) hurled radially inward from far away with initial speed $0 < |v_{\text{far}}| < 1$. Show that, taken together, these three methods can result in all possible speeds $0 \leq |v_{\text{shell}}| < 1$ measured by this shell observer at $r > 2M$.

9. Only One Shell Speed—with the Value One—at the Event Horizon

Show that the three kinds of radial launch of a stone described in Exercise 8 yield the *same* shell speed, namely $|v_{\text{shell}}| = 1$, as a limiting case when the stone moves inward across the event horizon. Your result shows that at the event horizon (as a limiting case): (a) You cannot make the shell-observed speed of a stone *greater* than that of light, no matter how fast you hurl it inward from far away. (b) You cannot make the shell-observed speed of the stone *less* than that of light, no matter how close to the event horizon you release it from rest.

10. Energy from garbage using a black hole

Define an **advanced civilization** as one that can carry out any engineering task not forbidden by the laws of physics. An advanced civilization wants to use a black hole as an energy source. Most useful is a "live" black hole, one that spins (Chapters 17 through 21), with rotation energy available for use. Unfortunately the nonrotating black hole that we study in this chapter is "dead:" no energy can be extracted from it (except for entirely negligible Hawking radiation, Box 5). Instead, our advanced civilization uses the dead (nonspinning) black hole to convert garbage to useful energy, as you analyze in this exercise.

A bag of garbage of mass m drops from rest at a power station located at r_0, onto a shell at r; a machine at the lower r brings the garbage to rest and converts all of the *shell kinetic energy* into a light flash. Express all energies requested below as fractions of the mass m of the garbage.

A. What is the energy of the light flash measured on the shell where it is emitted?

B. The machine now directs the resulting flash of light radially outward. What is the energy of this flash as it arrives back at the power station?

C. Now the conversion machine at r releases the garbage so that it falls into the black hole. What is the increase ΔM in the mass of the black hole? What is its increase in mass if the conversion machine is located—as a limiting case—exactly at the event horizon?

D. Find an expression for the efficiency of the resulting energy conversion, that is (output energy at the power station)/(input garbage mass m) as a function of the converter r and the r_0 of the power station. What is the efficiency when the power station is far from the black hole, $r_0 \rightarrow \infty$, and the conversion machine is on the shell at $r = 3M$? (Except for matter-antimatter collisions, the efficiency of mass-to-energy conversions in nuclear reactions on Earth is never greater than a fraction of one percent.)

E. *Optional:* Check the conservation of *map* energy in all of the processes analyzed in this exercise.

Comment 2. Decrease disorder with a black hole vacuum cleaner?
Suppose that the neighborhood of a black hole is strewn with garbage. We tidy up the vicinity by dumping the garbage into the black hole. This cleanup reduces disorder in the surroundings of the black hole. But wait! Powerful principles of thermodynamics and statistical mechanics demand that the disorder—technical name: **entropy**—of an isolated system (in this case, garbage plus black hole) cannot decrease. Therefore the disorder of the black hole itself must increase when we dump disordered garbage into it. Jacob Bekenstein and Stephen Hawking quantified this argument to define a measure of the entropy of a black hole, which turns out to be proportional to the Euclidean-calculated spherical "area" of the event horizon. See Kip S. Thorne, *Black Holes and Time Warps*, pages 422–448.

11. Temperature of a Black Hole

A Use equation (32) to find the temperature, when viewed from far away, of a black hole of mass five times the mass of our Sun.

B. What is the mass of a black hole whose temperature, viewed from far away, is 1800 degrees Kelvin (the melting temperature of iron)? Express your answer as a fraction or multiple of the mass of Earth. (Equation (32) tells us that "smaller is hotter," which leads to increased emission by a smaller black hole and therefore shorter life. If this analysis is correct, small black holes created in the Big Bang must have evaporated by now.)

6.9 ■ REFERENCES

Initial quote David Kaiser, personal communication.

This chapter owes a large intellectual debt in ideas, figures, and text to *Gravitation* by Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, W. H. Freeman and Company, San Francisco (now New York), 1973. Other quotations from *A Journey into Gravity and Spacetime* by John Archibald Wheeler, W. H. Freeman and Company, New York, 1990. In addition, our treatment was helped by reference to "Nonrotating and Slowly Rotating Holes" by Douglas A. Macdonald, Richard H. Price, Wai–Mo Suen, and Kip S. Thorne in the book *Black Holes: The Membrane Paradigm*, edited by Kip S. Thorne, Richard H. Price, and Douglas A. Macdonald, Yale University Press, New Haven, 1986.

More advanced treatment of the Principle of Maximal Aging in Misner, Thorne, and Wheeler, page 315 and following.

Newton quotation in Box 1: I. Bernard Cohen and Anne Whitman, *Isaac Newton: The Principia*, 1999, University of California Press, page 408

References for Box 6, "Baked on the Shell?" Historical background in Kip S. Thorne, Richard H. Price, and Douglas A. MacDonald *Black Holes: The Membrane Paradigm*, Yale University Press, 1986, pages 280–285 and 35–36; W. G. Unruh, "Notes on Black-Hole Evaporation", Physical Review D, Volume 14, Number 4, 15 August 1976, pages 870–892; William G. Unruh and Robert M. Wald, "What Happens When an Accelerating Observer Detects a Rindler Particle," Physical Review D, Volume 29, Number 6, 15 March 1984, pages 1047–1056.

Russell Baker quotation excerpted from *New York Times,* May 18, 1975, *New York Times Magazine*, page 6. Reprinted in *A Geological Miscellany* by E. J. Jones and G. Y. Craig, editors, 1985, Princeton University Press, Princeton, pages 191-192.