Quantum Computing Architectures (BME) / Quantum bits in solids (ELTE)
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Control questions, exercises (v2)

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The exam will consist of a written and an oral part. This file contains a few control questions and exercises, similar to those that will appear in the written part of the exam. Of course, these or similar ones might appear also in the oral part. In the oral part, we will ask about a specific topic that was covered in the lectures; for example, describe electron-phonon interaction and its consequences with respect to spin-qubit dynamics; or, describe how to do single-qubit gates on a transmon qubit, etc.

I. QUANTUM BITS

1. List the three Pauli matrices. Determine their eigenvalues and normalized eigenstates.
2. What is the unitary matrix representing the Hadamard gate? What is the result of the Hadamard gate acting on the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? What is the result of the Hadamard gate acting on the state $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?
3. What is the unitary matrix representing a two-qubit $\sqrt{\text{SWAP}}$ gate in the basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$? What is the result of the $\sqrt{\text{SWAP}}$ gate acting on the state $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$?
4. Determine the polarization vectors associated to the following three states: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$.
5. Consider the single-qubit state $\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$. When we measure the qubit, what is the probability of measuring 1? What is the state of the qubit after the measurement?
6. Consider the two-qubit state $\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|10\rangle + \frac{1}{\sqrt{3}}|01\rangle$. When we measure the first qubit, what is the probability of measuring 0? And that of measuring 1? What is the state of the system after measuring 0? And after measuring 1?

II. CONTROL OF QUANTUM SYSTEMS

1. Let H be an N-dimensional time independent Hamiltonian, with known energy eigenvalues E_n and eigenstates ψ_n fulfilling $H\psi_n = E_n\psi_n$. Assume that the system is initialized in the state ψ_i at $t = 0$. Express the time evolution of this state, $\psi(t)$, using E_n, ψ_n and ψ_i.
2. A quantum system is described by the time-dependent Hamiltonian $H(t)$. Write out the time-dependent Schrödinger equation. Perform a time-dependent unitary transformation $U(t)$ on the time-dependent Schrödinger equation. Express the quantity playing the role of the Hamiltonian in the transformed equation, using $H(t)$ and $U(t)$.
3. Consider the Hamiltonian

$$H = H_0 + H_1, \tag{1}$$

where

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \text{ and } H_1 = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \tag{2}$$

with $0 < t \ll \Delta$. Express the lower-energy eigenstate using first-order perturbation theory, and the lower energy eigenvalue using second-order perturbation theory.

4. In single-electron spin resonance, how does the Rabi time depend on the external homogeneous magnetic field B_0? How does the Rabi time depend on the amplitude B_{ac} of the ac magnetic field?

5. Consider the simplest model of two electrons in a double quantum dot, with two spin states and a single orbital in each dot. What is the dimension of the two-electron Hilbert space? Specify the unitary transformation between the product basis and the singlet-triplet basis. What is the dimension of the three-electron Hilbert space?

6. Write out the Jaynes-Cummings Hamiltonian. Sketch the energy level diagram for the resonant but uncoupled case. Within the same diagram, sketch the coupling matrix elements. How does the coupling matrix element change as we climb up the Jaynes-Cummings ladder?

7. Sketch the energy level diagram of the Jaynes-Cummings model, as well as the coupling matrix elements, in the situation where it is used to perform dispersive readout. What is the dispersive shift of the resonator if the qubit is in its ground state? What is the dispersive shift if the qubit is in its excited state?

8. A harmonic oscillator can mediate interaction between two qubits. Assume that this system is described by the two-qubit Jaynes-Cummings Hamiltonian discussed at the lecture. Set the resonator eigenfrequency to 9.5 GHz, and qubit Larmor frequencies to 10 GHz, and the coupling strength to $g/2\pi = 50$ MHz. Sketch and label the 7 energy levels that play a role in the sqrt-of-iswap gate mediated by virtual photon exchange. Estimate the time scale of that gate.

III. QUBITS BASED ON THE ELECTRON SPIN

1. In GaAs, where the effective mass of the conduction-band electrons is $m \approx 0.063 m_e$, we make a quantum dot with an orbital level spacing of 1 meV. Estimate the spatial extension of the electron occupying the ground-state orbital of this quantum dot. Estimate the charging energy, i.e., the Coulomb repulsion energy between two electrons occupying this dot.

2. At the lecture, we have discussed how to use spin-to-charge conversion and charge sensing to measure the spin relaxation time T_{spin}. There are four characteristic energy scales in this experiment, one related to the spin relaxation time ($\hbar/T_{\mathrm{spin}}$), one related to the tunnel rate between the quantum dot and the electron reservoir ($\hbar\Gamma$), the Zeeman splitting ($\hbar\omega_L \equiv g^* \mu_B B$), and the thermal energy scale ($k_B T$). What condition(s) among these energy scales should be satisfied to make the experiment work?

3. Consider the gate-voltage-dependent conductance of the QPC appearing on slide 11 of Lecture 3. The QPC is tuned the the working point denoted by the cross. Imagine that a single electron appears at a distance of 200 nm from the QPC. How much does the conductance change due to the appearance of the electron?

4. Estimate the tunnel rate Γ from the measurement data shown in slide 12 of Lecture 3.

5. Estimate the temperature of the experiment from the data set shown in slide 15 of Lecture 3. It is allowed to use the material parameters of GaAs for the estimate.

6. Write out the 6×6 Hamiltonian matrix of the two-electron sector of the two-site Hubbard model, whose energy spectrum is shown in slide 17 of Lecture 3. Do include the magnetic field in the Hamiltonian. To obtain the energy spectrum shown there, we used $t_H = 0.05\,U$ and $g\mu_B B = 0.1\,U$. Assuming $U = 1$ meV, what is the S-T_0 energy splitting at zero detuning $\epsilon = 0$? What is the Larmor frequency of this singlet-triplet qubit at the zero-detuning point? What is the magnetic field value (in Teslas) corresponding to this parameter set?

IV. COHERENT CONTROL OF ELECTRON SPINS

1. An infinitely thin wire carries a current of 1 mA. Calculate the magnetic field induced by this current at a distance of 50 nm from the wire.

2. Consider the measurement cycle on slide 7 of Lecture 4. Estimate the characteristic scale of the maximum dc current achievable by periodically repeating this cycle.

3. By numerical diagonalization of the two-site Hubbard Hamiltonian in a magnetic field, reproduce the two spectra shown in slide 8 of Lecture 4. On that slide, $\beta = g^* \mu_B B$. What is the value of the B-field on the top right figure on slide 8 of Lecture 4, if $U = 1$ meV and $g^* = 0.4$?

4. Consider a single phosphorus atom (P). It has a spin-1/2 nuclear spin. The gyromagnetic ratio of this nuclear spin is $\gamma_n \approx 17\,\mathrm{MHz/T}$. Calculate the thermal population of the ground and excited states of this nuclear spin at magnetic field $B = 1$ T and temperature $T = 100$ mK.

5. Consider the transverse magnetic-field profile $B_\perp(x, y)$ shown in slide 10 of Lecture 4. Assume that we do a μ-EDSR experiment in such a profile, in a GaAs 2DEG, by driving the system along the x axis with a homogeneous ac electric field, in a circularly symmetric quantum dot with orbital level spacing $\hbar\omega_0 = 1$ meV. Estimate the amplitude of the ac electric-field drive required to produce spin Rabi oscillations with Rabi time $T = 2\pi/\Omega = 1\,\mu$s.

6. Consider an electron in a two-dimensional electron gas, subject to Rashba spin-orbit interaction:

$$H = \frac{p_x^2 + p_y^2}{2m} + \alpha(\sigma_y p_x - \sigma_x p_y). \tag{3}$$

   Assume that the electron is occupying a plane-wave state moving along x, with momentum vector $\boldsymbol{k} = (k_x, 0)$. How do the energy eigenvalues depend on k_x? How do the energy eigenstates depend on k_x?

7. Derive the SOI-EDSR Rabi frequency, following the first-order perturbative calculation outlined in slides 17 and 18 of Lecture 4. Note that in the last step leading to $H_{E,q}$, a first-order series expansion in the small parameter $\omega_L/\omega_0 \ll 1$ was made. How many terms are there in the first-order perturbative expressions of the dressed qubit basis states $|\overline{0_x 0_y \uparrow}\rangle$ and $|\overline{0_x 0_y \downarrow}\rangle$?

8. An unreadable formula on the blackboard encodes the spatial dependence of an inhomogeneous magnetic field in the vicinity of a quantum dot:

$$\boldsymbol{B}(x, y, z) = (\beta ?, 0, B_0), \tag{4}$$

   where ? is either x or y. Which one is it? Assume that the lecturer did no mistake.