

Chapter 8. Circular Orbits

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- 12 • *How do orbits around a black hole differ from planetary orbits around*
- 13 *our Sun?*
- 14 • *How close to a black hole can a free stone move in a circular orbit?*
- 15 • *Can a stone reach the speed of light in a circular orbit around a black*
- 16 *hole?*
- 17 • *Can I use a black hole circular orbit to travel forward in time? backward*
- 18 *in time?*
- 19 • *What is the source of the energy that the so-called QUASAR radiates*
- 20 *outward in such prodigious quantity?*

CHAPTER

8

21

Circular Orbits

Edmund Bertschinger & Edwin F. Taylor *

22 *How happy is the little Stone*
 23 *That orbits a Black Hole alone**
 24 *And doesn't care about Careers*
 25 *And Exigencies never fears –*
 26 *Whose Coat of elemental Brown*
 27 *A passing Universe put on*
 28 *And independent as the Sun*
 29 *Associates or glows alone*
 30 *Fulfilling absolute Decree*
 31 *In casual simplicity –*

32 —Emily Dickinson

33 *Line two in the original reads:
 34 *That rambles in the Road alone*

8.1.5 ■ STEP OR ORBIT?

36 *“Go straight!” shouts spacetime. The Principle of Maximal Aging interprets*
 37 *that command*

Nature shouts at the
stone “Go straight!”

38 A stone in orbit streaks around a black hole—or around Earth. What tells the
 39 stone how to move? Spacetime grips the stone, giving it the simplest possible
 40 command: “Go straight!” or in the more legalistic language of the Principle of
 41 Maximal Aging, “Follow the worldline of maximal aging across the next two
 42 adjoining local inertial frames.” From instant to instant this directive is
 43 enough to tell the stone what to do next, the next step to take in its motion.

This chapter:
circular orbits

44 This command for its next step is sufficient for the stone, but we want
 45 more: We seek a description of the entire orbit of the stone through
 46 spacetime—its worldline in global coordinates. The present chapter uses the
 47 global metric and the Principle of Maximal Aging to predict circular orbits of
 48 a stone around any spherically symmetric center of attraction. This prediction

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Constants of motion:
map energy and
map angular
momentum

49 uses two map quantities that do not change as the motion progresses: map
50 energy and map angular momentum. In Query 7, Section 7.5, you derived the
51 map energy of a stone in global rain coordinates. Section 8.2 in the present
52 chapter derives an expression for map angular momentum in global rain
53 coordinates. Sections 8.4 shows how to use map angular momentum—together
54 with map energy—to forecast circular orbits. We find that a *free* stone can
55 move (a) in a *stable* circular orbit only at an r -coordinates greater than
56 $r = 6M$, or (b) in an *unstable* circular orbit from $r = 6M$ down to $r = 3M$. No
57 circular orbit for a free stone exists for $r < 3M$.

58 **Comment 1. Global quantities are unicorns**

59 Expressions for global quantities such as map energy and map angular
60 momentum are specific to the global coordinates in which they are expressed.
61 They are unicorns—mythical beasts—unmeasured by a local inertial observer,
62 except by some quirk of the global coordinates (Section 6.3).

63 The circular orbit is a special case of an *orbit*. We have not yet carefully
64 defined an orbit. Here is that definition.

65 **DEFINITION 1. Orbit**

66 An **orbit** is the path of a free stone through spacetime described by a
67 given set of global coordinates. The path of a radially-plunging stone,
68 with $d\phi = 0$ is a special case of the orbit.

69 **Comment 2. Orbit vs. worldline**

70 The *orbit* of a stone is different from its *worldline*. The worldline of a stone
71 (Definition 9, Section 1.5) is its (free or driven) path through spacetime described
72 by its wristwatch time. The description of a worldline does not require either
73 coordinates or the metric.

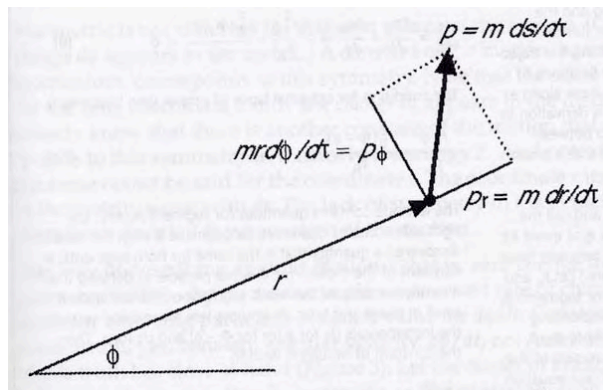


FIGURE 1 In flat spacetime, angular momentum L is the product of r and the ϕ -component of linear momentum $p_\phi = m r d\phi/dt$, which yields $L = m r^2 d\phi/dt$. Here dt is the differential advance of wristwatch time of the stone. Box 1 shows that the same expression, written in global (either Schwarzschild or rain) coordinates, is a constant of motion around a non-spinning black hole.

Section 8.2 Map Angular Momentum of a Stone from Maximal Aging **8-3**

8.2. MAP ANGULAR MOMENTUM OF A STONE FROM MAXIMAL AGING

75 Vary the map angle of an intermediate event on a worldline to find map
76 angular momentum.

77 Here we derive the expression for map angular momentum using global rain
78 coordinates with its T -coordinate. The resulting expression for map angular
79 momentum is also valid in Schwarzschild coordinates. Why? Because both
80 global coordinate systems have the same r and ϕ coordinates, and the global t -
81 or T -coordinate—different in the two global coordinate systems—does not
82 appear in the expression for map angular momentum.

83 Start with the global rain metric, equation (15) in Section 7.4. Write down
84 its approximation at the average r -coordinate \bar{r} :

$$\Delta\tau^2 \approx \left(1 - \frac{2M}{\bar{r}}\right) \Delta T^2 - 2\left(\frac{2M}{\bar{r}}\right)^{1/2} \Delta T \Delta r - \Delta r^2 - \bar{r}^2 \Delta\phi^2 \quad (9)$$

85 Box 1 uses the now-familiar Principle of Maximal Aging to derive the
86 expression for map angular momentum in global rain coordinates. Box 1 tells
87 us that $r^2 d\phi/d\tau$ is a constant of motion for a free stone moving around the
88 non-spinning black hole. Can we recognize this constant as something
89 familiar? Figure 1 shows that in flat spacetime the angular momentum of the
90 stone (symbol L) has the form $L = mr^2 d\phi/d\tau$. So we identify our new
91 constant of motion as the **map angular momentum per unit mass** of the
92 stone: $L/m = r^2 d\phi/d\tau$.

Map angular
momentum

$$\frac{L}{m} \equiv r^2 \frac{d\phi}{d\tau} \quad (\text{map angular momentum}) \quad (10)$$

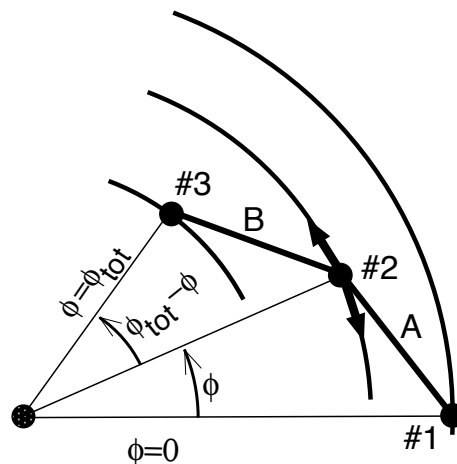


FIGURE 2 [Figure for Box 1.] Derivation of map angular momentum. Find the intermediate map angle ϕ that maximizes the stone's wristwatch time between Events #1 and #3.

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Box 1. Derive the Expression for Map Angular Momentum

Strategy: Apply the Principle of Maximal Aging to maximize the wristwatch time of a free stone that moves along two adjoining worldline segments labeled A and B—for Above and Below—in Figure 2. The stone emits flashes at Events #1, #2, and #3 that mark off the segments. Fix the global rain T - and r -coordinates of all three flashes and the ϕ -coordinates of flashes #1 and #3. Vary the ϕ -coordinate of Event #2 by sliding it along a circle (double-headed arrow in Figure 2) to maximize the total wristwatch time between flashes #1 and #3. Then identify the resulting constant of motion as the map angular momentum per unit mass of the stone. Now the details.

Set the fixed ϕ -coordinate of Event #1 equal to zero and call ϕ_{tot} the fixed final ϕ -coordinate for Event #3. To change the angle ϕ of Event #2, move it in either direction along its circle (double-headed arrow in the figure). Let \bar{r}_A and \bar{r}_B be appropriate average values of the r -coordinate for segments A and B, respectively, and let τ_A and τ_B be the corresponding lapses of wristwatch time of the stone moving along these segments. With these substitutions, and for a small value of τ_A , the approximate global rain metric (9) for higher Segment A becomes:

$$\tau_A \approx [-\bar{r}_A^2 \phi^2 + (\text{terms without } \phi)]^{1/2} \quad (1)$$

To prepare for the derivative that leads to maximal aging, take the derivative of this expression with respect to ϕ :

$$\frac{d\tau_A}{d\phi} \approx -\frac{\bar{r}_A^2 \phi}{\tau_A} \quad (2)$$

Similarly for lower Segment B,

$$\tau_B \approx [-\bar{r}_B^2 (\phi_{\text{tot}} - \phi)^2 + (\text{terms without } \phi)]^{1/2} \quad (3)$$

$$\frac{d\tau_B}{d\phi} \approx \frac{\bar{r}_B^2 (\phi_{\text{tot}} - \phi)}{\tau_B} \quad (4)$$

The total wristwatch time for both segments is $\tau = \tau_A + \tau_B$. Take the derivative of this expression with respect to ϕ , substitute from (2) and (4), and set the resulting derivative equal to zero in order to apply the Principle of Maximal Aging:

$$\frac{d\tau}{d\phi} = \frac{d\tau_A}{d\phi} + \frac{d\tau_B}{d\phi} \approx -\frac{\bar{r}_A^2 \phi}{\tau_A} + \frac{\bar{r}_B^2 (\phi_{\text{tot}} - \phi)}{\tau_B} = 0 \quad (5)$$

The condition for maximal lapse of wristwatch time becomes

$$\frac{\bar{r}_A^2 \phi}{\tau_A} \approx \frac{\bar{r}_B^2 (\phi_{\text{tot}} - \phi)}{\tau_B} \quad (6)$$

or in our original Δ notation:

$$\frac{\bar{r}_A^2 \Delta\phi_A}{\Delta\tau_A} \approx \frac{\bar{r}_B^2 \Delta\phi_B}{\Delta\tau_B} \quad (7)$$

The left side contains quantities for Segment A only; the right side quantities for Segment B only. We have discovered a quantity that has the same value for both segments, a *global constant of motion* for the free stone across every pair of adjoining segments along the worldline of the free stone. In deriving this quantity, we assumed that each segment of the worldline is small. To yield an equality in (7), go to the calculus limit in (7), for which $\bar{r} \rightarrow r$; the constant of motion becomes

$$\lim_{\Delta\tau \rightarrow 0} \left(\bar{r}^2 \frac{\Delta\phi}{\Delta\tau} \right) = r^2 \frac{d\phi}{d\tau} = \text{a constant of motion} \quad (8)$$

where r and τ are in units of meters. The text identifies this constant of motion as L/m , the map angular momentum of the stone per unit mass.

94 Since r and τ are in units of meters, therefore L/m is also in units of meters.

8.3. EQUATIONS OF MOTION FOR A STONE IN GLOBAL RAIN COORDINATES

96 *The stone's wristwatch ticks off $d\tau$. From $d\tau$ find the resulting changes $d\phi$, dr , and dT .*

98 We now have in hand the tools needed to calculate the step-by-step advance of
 99 the free stone in global rain coordinates. Map energy and map angular
 100 momentum—global constants of motion—plus the global metric give us three
 101 equations in the three global rain unknowns dT , dr , and $d\phi$, expressed as
 102 functions of the advance $d\tau$ of the stone's wristwatch. Starting from an
 103 arbitrary initial event, the computer advances wristwatch time and calculates

Equations
of motion

Section 8.3 Equations of Motion for a Stone in Global Rain Coordinates **8-5**

104 the consequent advance of all three map coordinates, then sums the results of
 105 these steps to plot the stone's worldline in global coordinates. We now spell
 106 out this process.

First equation
 of motion

107 The first equation of motion comes from (10):

$$\frac{d\phi}{d\tau} = \frac{L}{mr^2} \quad (11)$$

108
 109 The second equation of motion comes from the expression for E/m ,
 110 equation (35) in Section 7.5:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau} \quad (\text{global rain coordinates}) \quad (12)$$

111 Solve (12) for $dT/d\tau$:

$$\frac{dT}{d\tau} = \left[\frac{E}{m} + \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau} \right] \left(1 - \frac{2M}{r}\right)^{-1} \quad (13)$$

112 Take the differential limit of (9), divide through by $d\tau^2$, and substitute into it
 113 from (11) and (13):

$$1 = \left[\frac{E}{m} + \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau} \right]^2 \left(1 - \frac{2M}{r}\right)^{-1} \quad (14)$$

$$- 2 \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau} \left[\frac{E}{m} + \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau} \right] \left(1 - \frac{2M}{r}\right)^{-1} - \left(\frac{dr}{d\tau}\right)^2 - \left(\frac{L}{mr}\right)^2$$

Second equation
 of motion

114 Multiply out and collect terms. Solve the resulting quadratic equation in
 115 $dr/d\tau$ to yield our second equation of motion for the stone:

$$\frac{dr}{d\tau} = \pm \left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{m^2 r^2}\right) \right]^{1/2} \quad (\text{stone}) \quad (15)$$

Third equation
 of motion

117 The third equation of motion shows how dT varies with stone wristwatch
 118 time lapse $d\tau$. Substitute for $dr/d\tau$ from (15) into (13) and solve for $dT/d\tau$:

$$\frac{dT}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \left\{ \frac{E}{m} \pm \left(\frac{2M}{r}\right)^{1/2} \left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{m^2 r^2}\right) \right]^{1/2} \right\} \quad (16)$$

120 **Comment 3. Plotting the orbit**

121 To plot any orbit of the stone—not just a circular orbit—you (or your computer)

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122 can integrate the derivative $d\phi/dr = (d\phi/d\tau)(d\tau/dr)$ using equations (11) and
 123 (15).

Equations of motion
 in global rain
 coordinates
 124 Taken together, equations (11), (15), and (16) are the *equations of motion*
 125 *of the stone* in global rain coordinates. Their integration yields the worldline of
 126 the stone in global rain coordinates T , r , and ϕ . Interactive software GRorbits
 127 carries out this process, plots the orbit in r and ϕ , and outputs a spreadsheet
 128 of events along the worldline of the stone.

QUERY 1. Crossing the event horizon in global rain coordinates.

A first glance at equation (16) might lead to the conclusion that $dT/d\tau$ blows up at the event horizon, so that a stone requires an unlimited lapse in the T -coordinate to cross there. Set $r = 2M(1 + \epsilon)$ in this equation to show that as $\epsilon \rightarrow 0$ the right side does *not* blow up.

8.4 ■ EFFECTIVE POTENTIAL

136 *Grasp orbit features at a single glance!*

Effective potential:
 the r -component
 of motion

137 The orbit computation in Section 8.3 puts into our hands powerful tools to
 138 describe any motion of the free stone in the equatorial plane of a spherically
 139 symmetric center of attraction. Indeed, the wealth of possible orbits is so great
 140 that we need some classification scheme with which to sort orbits at a glance.
 141 One classification scheme uses the so-called **effective potential** that focuses
 142 on the r -component of motion. Clearer even than our computed orbits, the
 143 effective potential plot instantly shows many central features of our stone's
 144 motion.

Pit in the potential

145 Vicious gravitational effects close to a black hole dominate the effective
 146 potential there. In addition to the attractive potential of gravity at large
 147 r -coordinates and the effective repulsion due to map angular momentum at
 148 intermediate r -values, at still smaller r -coordinates Einstein adds a pit in the
 149 potential, shown at the left of Figures 3 and 4.

150 The potential? A pit in this potential? Can we get this potential from
 151 principles that are simple, clear, and solid? Yes, starting from map energy and
 152 map angular momentum, both of them global constants of motion.

153 To begin this process, square both sides of (15).

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{m^2 r^2}\right) \quad (17)$$

Effective potential
 for a stone

154 Define a function $(V_L(r)/m)^2$ to replace the second term on the right side of
 155 (17). Call this function the square of the **effective potential**.

$$\left(\frac{V_L(r)}{m}\right)^2 \equiv \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{m^2 r^2}\right) \quad (\text{squared effective potential}) \quad (18)$$

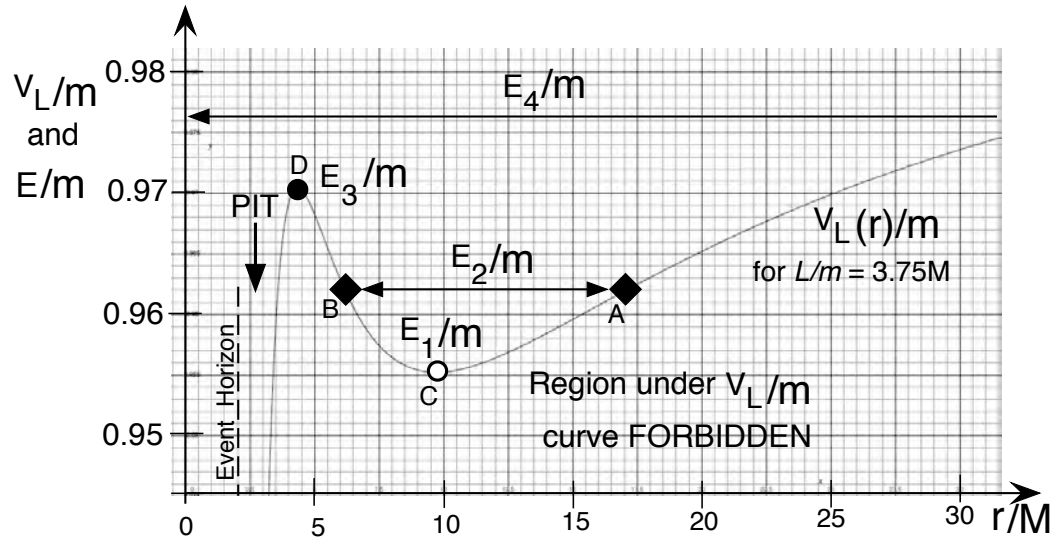


FIGURE 3 Effective potential for a stone that orbits the black hole with map angular momentum $L/m = 3.75M$. When the stone's map energy equals the minimum of the effective potential energy (little open circle at C), the stone is in a stable circular orbit. A stone with somewhat greater map energy, E_2/m , (line with double arrow) oscillates back and forth in r between turning points (little black rotated squares) labeled A and B. When the stone's map energy equals the maximum of the effective potential energy (little filled circle at D), the stone is in an unstable circular orbit. When the map energy E_4/m of an inward-moving stone is greater than the peak of the effective potential (upper horizontal line), the approaching stone crosses the event horizon and plunges to the singularity at $r \rightarrow 0$.

156 Subscript L on $V_L(r)$ reminds us that this effective potential is different for
 157 different values of the map angular momentum L . Substitute (18) into (17)
 158 and take the square root of both sides:

$$\frac{dr}{d\tau} = \pm \left[\left(\frac{E}{m} \right)^2 - \left(\frac{V_L(r)}{m} \right)^2 \right]^{1/2} \quad (19)$$

159 The squared effective potential $(V_L(r)/m)^2$ is what we subtract from the
 160 squared map energy term $(E/m)^2$ to obtain $(dr/d\tau)^2$. The plus sign in (19)
 161 describes increase in r -coordinate, the minus sign describes decreasing r .

162 Figure 3 plots effective potential $V_L(r)/m$ from (18) and shows the r -range
 163 for motion of stones with three different map energies.

164 Note that $dr/d\tau$ in equation (19) is real only where $(E/m)^2$ has a value
 165 greater than $(V_L(r)/m)^2$. This has important consequences: The stone cannot
 166 exist with a map energy in the region under the effective potential curve: that
 167 is the **forbidden map energy region**. As a result, the horizontal map energy
 168 line labeled E_2/m in Figure 3 terminates wherever it meets the $V_L(r)/m$
 169 curve. At these points, called **turning points** in r , the map energy and the
 170 effective potential are equal: $E/m = V_L/m$, so that $dr/d\tau = 0$ in (19). At a
 171 turning point the r -component of map motion goes to zero (while the stone

Forbidden map energy
 region; turning points

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continues to sweep around in the ϕ -direction). In Figure 3 the stone's r map position oscillates back and forth between turning points in r labeled A and B. Earth and each solar planet oscillates back and forth with an r -component of motion similar to that labeled E_2/m in Figure 3, each around a minimum of its own solar effective potential that depends on its map angular momentum.

DEFINITION 2. Forbidden map energy region

Definition:
Forbidden map energy region

The **forbidden map energy region** is a region in a $V_L(r)/m$ vs. r/M plot in which equations of motion of the stone (Section 8.3) become imaginary or complex. Hence the stone cannot move—or even exist—with map energy in the forbidden map energy region.

QUERY 2. Demonstrate forbidden map energy regions

Verify the statement in Definition 2 that “In the forbidden map energy region, the equations of motion of a stone (Section 8.3) become imaginary or complex.” for *each* equation of motion in Section 8.3.

DEFINITION 3. Turning point, circle point, and bounce point

Figures 3 and 4 show little filled circles, little open circles, and little rotated filled squares, each one located on the effective potential curve. These points are called *turning points*. (Section 8.5 defines the meaning of the “half-black” circle numbered one in Figure 4.)

Definition:
Turning point

A **turning point** is a value of r for which $E = V_L(r)$. At a turning point, $dr/d\tau = 0$. Examples of turning points: points A through D in Figure 3 and points 1 through 5 in Figure 4. We distinguish two kinds of turning points: circle point and bounce point:

Definition:
Circle point

A **circle point** is a turning point at a maximum or minimum of the effective potential. At a circle point $dr/d\tau$ equals zero and remains zero, at least temporarily, so a stone at a circle point is in either an unstable or a stable circular orbit. We plot a circle point as either a little filled circle (at an unstable circular orbit r -value) or a little open circle (at a stable circular orbit r -value). Examples of bounce points: C and D in Figure 3 and points labeled 1 through 5 in Figure 4.

Definition:
Bounce point

A **bounce point** is a turning point that is *not* at a maximum or minimum of the effective potential. At a bounce point, $dr/d\tau$ for a free stone reverses sign. We plot a bounce point as a little filled rotated square. Examples of bounce points: A, and B in Figure 3. A stone that moves between bounce points—such as the stone with map energy E_2/m in Figure 3, is in a bound orbit that is *not* circular (Chapter 9).

Three payoffs of effective potential

Here are four important payoffs of the effective potential. First, it gives $dr/d\tau$ in terms of E , L , and r . Second, at every r it shows us the map energy region that is forbidden to the stone. Third, it fixes r -values of the turning points for given E and L . Fourth, and most important, it helps us to categorize—at a glance—different kinds of orbits, including circular orbits.

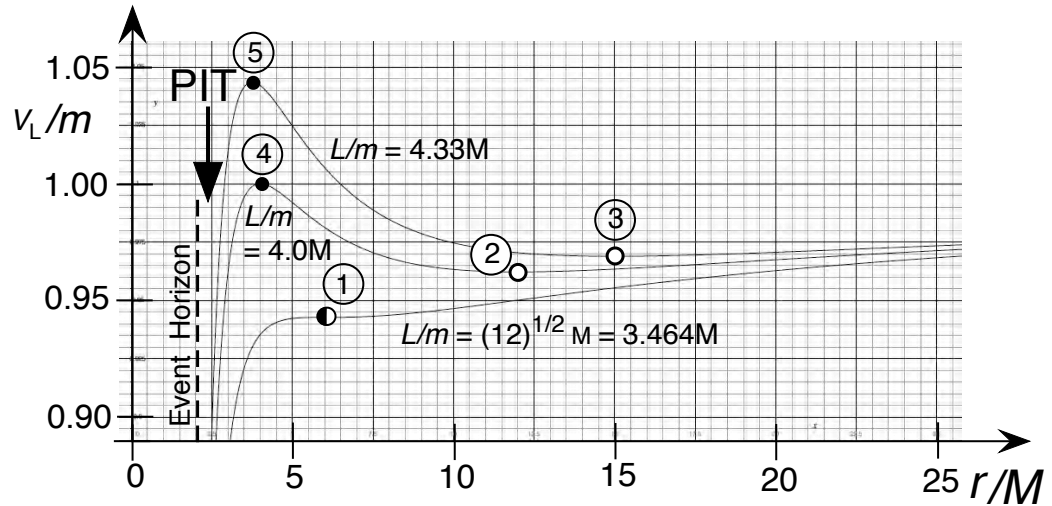


FIGURE 4 The r -coordinates of stable and unstable (knife-edge) circular orbits at points of zero slope of the effective potentials for three values of L/m . Unstable circular orbits (little filled circles numbered 4 and 5) lie between $r = 3M$ and $r = 6M$. Stable circular orbits, little open circles numbered 2 and 3, lie at r greater than $r = 6M$. Orbit numbered 1 (little half-black circle) is the limiting case, stable for increase in r ; unstable for decrease in r . Section 8.5 discusses this “half-stable orbit.” A forbidden map energy region (Definition 2) lies under the curve for each value of L/m .

214

QUERY 3. Compare Newtonian and general-relativistic orbital motion (optional)

The right side of (17)₁₆ tells us a great deal about the difference between the stone’s global motion described in global rain coordinates and its motion described by Newton.

- A. Multiply out the right side of (17) and divide through by 2 to yield

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 = \frac{1}{2} \left[\left(\frac{E}{m} \right)^2 - 1 \right] - \left(-\frac{M}{r} + \frac{L^2}{2m^2 r^2} - \frac{ML^2}{m^2 r^3} \right) \quad (\text{global rain coordinates}) \quad (20)$$

- B. Newton’s expression for angular momentum, with Newton’s “universal time t ” is:

$$\frac{L}{m} \equiv r^2 \frac{d\phi}{dt} \quad (\text{Newton, universal time } t) \quad (21)$$

Show that Newton’s expression for the square of the velocity of the stone is:

$$v^2 = \left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 = \left(\frac{dr}{dt} \right)^2 + \frac{L^2}{m^2 r^2} \quad (\text{Newton}) \quad (22)$$

- C. Now, Newton’s expression for gravitational potential energy per unit mass (chosen to go to zero far from the center of attraction) is $U(r) = -M/r$. Write down Newton’s conservation of energy equation, solve it for the radial velocity, and show the result:

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$$\frac{1}{2} \left(\frac{dr}{dt} \right)^2 = \frac{E}{m} - \left(-\frac{M}{r} + \frac{L^2}{2m^2 r^2} \right) = \frac{E}{m} - \frac{V_{\text{NewtL}}(r)}{m} \quad (\text{Newton}) \quad (23)$$

where **Newton’s effective potential** is $V_{\text{NewtL}}(r)/m$.

D. Sketch for the Newtonian case a diagram like that of Figure 3: a plot of $V_{\text{NewtL}}(r)$ with horizontal lines for different values of E . Describe the resulting orbits and contrast them to those for motion in curved spacetime.

Of course the general relativity expression (20) is not just another version of Newton’s equation (23). But look at the basic similarity of the right sides of these two equations: a constant term from which we subtract a function of the r -coordinate—the “effective potential”—that varies with the value of map angular momentum L .

Conclusion of this analysis: It is the negative third term in the effective potential on the right side of (20), with r^3 in its denominator, that drives the effective potential downward as r becomes smaller as it approaches the event horizon—thereby creating the PIT in the potential labeled in Figures 3 and 4. This third term is the child of spacetime curvature.

237 In a stable circular orbit the stone’s map energy rests at the minimum of
 238 the effective potential; the stone rides round and round the black hole without
 239 changing r -coordinate.

240 **DEFINITION 4. Stable circular orbit**

Stable orbit at effective
 potential minimum

241 A stone in a stable circular orbit has map energy E/m equal to the
 242 *minimum* of the effective potential $V_L(r)/m$, for example the map energy
 243 labeled 1 in Figure 3 and energies labeled 2 and 3 in Figure 4. Any
 244 incremental change in the r -coordinate at constant E/m puts the stone
 245 into the forbidden map energy region under the effective potential curve,
 246 where a stone cannot go.

247 We use a little open circle to locate a stable circular orbit on an effective
 248 potential energy curve. The point labeled 1 in Figure 4 is the stable circular
 249 orbit of minimum r -value analyzed in Section 8.5.

250 Einstein opens up a second set of r -coordinates where the effective
 251 potential also has zero slope, illustrated by point D in Figure 3 and points 4
 252 and 5 in Figure 4. Each of these is a *maximum* of the effective potential curve;
 253 at this r -coordinate the stone experiences no tendency to move either to larger
 254 or smaller r -coordinate, so will stay at the same r -coordinate, riding round
 255 and round the black hole at constant r -coordinate. We call these **unstable** or
 256 **knife-edge** circular orbits, because slight departure from the knife-edge
 257 r -coordinate leads to decisive motion either to larger r , or else—horrors!—to
 258 smaller r that leads to the event horizon.

259 **DEFINITION 5. Unstable (or knife-edge) circular orbit**

260 A stone in an unstable (or knife-edge) circular orbit has map energy

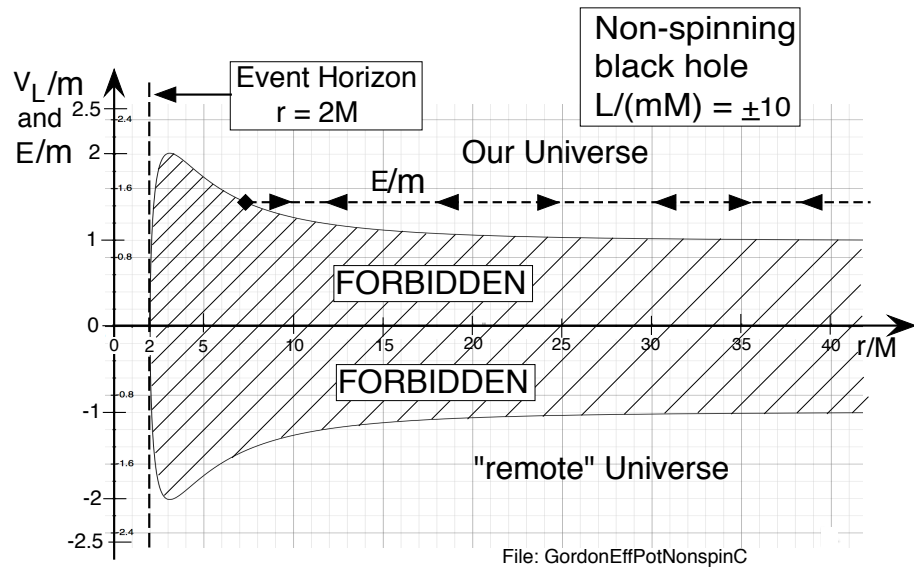


FIGURE 5 Plot of equation (17) for stone map angular momentum $L/(mM) = \pm 10$. In our Universe map energy is positive; it is negative in a “remote” Universe below the forbidden region. We cannot travel between our Universe and the “remote Universe” because the worldlines that connect them must pass inward through the event horizon, then back out again. (Diagonal lines emphasize impenetrability.) So where is this “remote” Universe? See Chapter 21.

Unstable (or knife-edge) orbit at effective potential maximum

261
262
263
264

E/m equal to the *maximum* of the effective potential V_L/m , so that any incremental r -displacement in either direction puts the stone into a region with a gap between E/m and V_L/m such that this displacement increases.

265 We use a little filled circle to locate an unstable circular orbit on an effective
266 potential energy curve.

267 **Comment 4. How long on a knife edge?**

268 Suppose that our spaceship is in a knife-edge orbit, technically an *unstable orbit*.
269 Slight cosmic wind, firing of a projectile, or ejection of the day’s trash may give
270 our spaceship a tiny r -motion. Once displacement from the effective potential
271 peak occurs, the slope of the effective potential urges the spaceship farther away
272 from the point of zero slope, either outward toward larger r -coordinate or inward
273 toward the event horizon. Sooner or later—who knows when?—a stone
274 inevitably falls off the effective potential maximum of an unstable circular orbit.

275 “Why, oh why,” our captain cries, “didn’t I carry along a booster rocket? A
276 tiny rocket boost to push us outward could have reversed our initially
277 slow inward motion and allowed us to escape. But now it’s too late!”

278 Strange results follow from equation (19), which requires that
279 $(E/m)^2 \geq (V_L/m)^2$ in order that $dr/d\tau$ be real. A consequence of this
280 condition is that either $E/m \geq +V_L/m$ or $E/m \leq -V_L/m$. Figure 5 shows this

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281 condition. A stone cannot move, or even exist, with E/m in the region
 282 $+V_L/m > E/m > -V_L/m$. This is a forbidden map energy region, because
 283 $dr/d\tau$ would be imaginary there. *Result:* The forbidden map energy region
 284 divides spacetime outside the event horizon into two isolated regions: one for
 285 positive map energy and the other for negative map energy. The stone cannot
 286 travel directly between them. This definition of a forbidden map energy region
 287 is consistent with that given in Definition 2.

288 Figure 3 shows only positive values of map E/m . This is the region we live
 289 in, where we carry out our measurements and observations, the upper region
 290 of positive map energy in Figure 5. What is the meaning of negative E/m in
 291 the lower region of Figure 5? Can we carry out measurements and observations
 292 there? Remember that map energy is a global map quantity, not a quantity
 293 that we can measure; its negative value tells us nothing about permitted
 294 measurements. In the exercises you show that we can construct local inertial
 295 frames in the negative map energy region, so we can carry out measurements
 296 and observations there, just as we can in the region above the forbidden map
 297 energy region.

298 Can we travel from the upper (positive map energy) region in Figure 5 to
 299 the lower (negative map energy) region? Our own worldline, just like the
 300 worldline of a stone, cannot pass directly through that forbidden map energy
 301 region. Figure 5 shows that the forbidden map energy region ends at the event
 302 horizon, $r = 2M$. Can we make an end run around the forbidden map energy
 303 region by moving in through the event horizon and back out again? No, sorry:
 304 Once inside the event horizon, we cannot come out again; instead we move
 305 relentlessly inward to the singularity. See exercise 11 in Section 8.7.

?

306 **Objection 1.** *Can light move between the upper and lower regions?*

!

307 Nope. Figure 11 in Section 11.8 shows that a corresponding forbidden
 308 region for light separates upper and lower regions. Both for stones and for
 309 light, the two regions are physically isolated.

?

310 **Objection 2.** *Wait! Where is this lower region? It has the same r -values as*
 311 *the upper region but you tell me that it lies "somewhere else," in a negative*
 312 *map energy region we cannot reach. Where is it?*

!

313 The answer is subtle and deep. Later we will understand that global rain
 314 coordinates do not include all of spacetime. We must find other global
 315 coordinates that include such regions. Chapter 21 treats these matters.
 316 Keep on reading!

317 Chapters 17 through 21 examine the spinning black hole. We will find that
 318 for the spinning black hole we may be able to travel between the

Section 8.5 Properties of circular orbits **8-13**

319 corresponding upper and lower regions by dropping through the event horizon
 320 from the upper region, using rocket thrusts while inside the event horizon,
 321 then emerging outward through the event horizon into the lower region. Luc
 322 Longtin summarizes: “The non-spinning black hole is like the spinning black
 323 hole, but with its gate to other universes closed. For the spinning black hole,
 324 the gate is ajar.” (initial quote, Chapter 21)

8.5 ■ PROPERTIES OF CIRCULAR ORBITS

326 *Details! Details!*

327 A series of Queries helps you to explore some properties of circular orbits in
 328 the everyday positive map energy region around the non-spinning black hole.

QUERY 4. Map r -values of circular orbits

- A. A circular orbit is possible at every r -coordinate where the effective potential has zero slope. Take the r -derivative of both sides of (18) for a fixed L/m , set this derivative equal to zero, and show the following result:

$$r^2 - \frac{L^2}{Mm^2}r + 3\frac{L^2}{m^2} = 0 \quad (\text{circular orbit}) \quad (24)$$

- B. Equation (24) is linear in $(L/m)^2$. Solve it to find:

$$\left(\frac{L}{m}\right)^2 = \frac{Mr^2}{r - 3M} \quad (\text{circular orbit, } r > 3M) \quad (25)$$

Note that this expression is valid for both stable and unstable circular orbits and is invalid for $r < 3M$, where L/m would be imaginary. Circular orbits cannot exist for $r < 3M$, and for $r = 3M$ the circular orbit is a limiting case (Item D in Query 8)

- B. Equation (24) is quadratic in r . Solve it to find:

$$r = \frac{L^2}{2m^2M} \left[1 \pm \left(1 - \frac{12M^2m^2}{L^2} \right)^{1/2} \right] \quad (\text{circular orbit, } r > 3M) \quad (26)$$

Refer to Figure 4. Make the argument that the $+$ sign in (26) corresponds to the minimum of the effective potential, that is to a stable circular orbit; and that the $-$ sign corresponds to the maximum of the effective potential, that is to the unstable (knife-edge) circular orbit.

- C. *Optional:* Take the second derivative of (18) and verify that the \pm signs in (26) correspond, respectively, to a minimum and maximum of the effective potential.

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347 Look more closely at equation (26) and the effective potential curve in
 348 Figure 4 with the “half-black” little circle labeled number 1. In order for the
 349 r -coordinate to be real, the square root expression in (26) must be real. This
 350 occurs only when $|L/m| \geq (12)^{1/2}M = 3.4641M$. You can show that for the
 351 minimum map angular momentum, the global r -coordinate of the circular
 352 orbit is $r = 6M$. This is called the **innermost stable circular orbit** and is
 353 located at $r_{\text{ISCO}} = 6M$.

354 **DEFINITION 6. Innermost stable circular orbit (ISCO)**

355 Definition
 356 ISCO

355 The **innermost stable circular orbit (ISCO)**, located at $r_{\text{ISCO}} = 6M$,
 356 divides r -values for unstable circular orbit in the region $3M < r < 6M$
 357 from r -values for stable circular orbits in the region $r > 6M$. We can call
 358 the ISCO “half stable.” An increase in r at the same map energy puts the
 359 stone into a forbidden map energy region (like a stable circular orbit); a
 360 decrease in r at the same map energy puts the stone into a legal map
 361 energy region (like an unstable circular orbit).

362 Section 8.6 describes a so-called *toy model* of a quasar, the brightest
 363 steady source of light in the heavens. This emission comes from the loss of
 364 map energy of a stone that enters a circular orbit at large r and tumbles down
 365 through a series of “stable” circular orbits of smaller and smaller r . When the
 366 stone reaches the innermost stable circular orbit and continues to lose map
 367 energy, it spirals inward across the event horizon, after which we can no longer
 368 detect its radiation.

369 **QUERY 5. Shell speed of a stone in a circular orbit**

Compute the speed of the stone in a circular orbit measured by a shell observer, as follows.

- A. Consider two ticks of the orbiting stone’s clock, separated by wristwatch time $\Delta\tau$ and by zero distance measured in the stone’s local frame, but separated by shell time Δt_{shell} and by shell distance $\Delta x_{\text{shell}} = \bar{r}\Delta\phi$. The relation between Δt_{shell} and $\Delta\tau$ is just the special relativity expression

$$\Delta t_{\text{shell}} = \gamma_{\text{shell}}\Delta\tau = (1 - v_{\text{shell}}^2)^{-1/2}\Delta\tau \tag{27}$$

where γ_{shell} has an obvious definition. From the value of map angular momentum, we can use (27) to calculate shell speed:

$$\begin{aligned} v_{\text{shell}} &= \lim_{\Delta t_{\text{shell}} \rightarrow 0} \left(\frac{\bar{r}\Delta\phi}{\Delta t_{\text{shell}}} \right) = (1 - v_{\text{shell}}^2)^{1/2} \frac{r^2 d\phi}{rd\tau} \\ &= (1 - v_{\text{shell}}^2)^{1/2} \frac{L}{mr} \quad (\phi - \text{motion}) \end{aligned} \tag{28}$$

From this equation, show that

$$v_{\text{shell}}^2 = \left[1 + \left(\frac{mr}{L} \right)^2 \right]^{-1} \quad (\phi - \text{motion}) \tag{29}$$

Section 8.5 Properties of circular orbits **8-15**

From equation (25) show that

$$\left(\frac{mr}{L}\right)^2 = \frac{r}{M} - 3 \quad (\text{circular orbit}) \quad (30)$$

Substitute this into (29) to find

$$v_{\text{shell}}^2 = \frac{M}{r - 2M} = \left(\frac{M}{r}\right) \left(1 - \frac{2M}{r}\right)^{-1} \quad (\text{circular orbit, } r > 3M) \quad (31)$$

Equation (31) is valid for both stable and unstable (knife-edge) circular orbits.

- B. What is the value of the shell speed v_{shell} in the ISCO, the innermost stable circular orbit at $r = 6M$?
- C. Verify that the minimum map r -coordinate for a circular orbit is $r = 3M$. (*Hint:* What is the upper limit of the shell speed of a stone?)
- D. From (25) show that, as a limiting case, the map angular momentum L/m increases without limit for the knife-edge circular orbit of minimum r -coordinate.

Circular orbit
of light

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Comment 5. Unlimited map angular momentum?

How can the map angular momentum possibly increase indefinitely (Item D of Query 6)? It does so only as a limiting case. According to (10), the map angular momentum is equal to $L/m = r^2 d\phi/d\tau$. The relation between wristwatch time $d\tau$ and shell time dt_{shell} is given by (27), the usual time-stretch formula of special relativity. As the stone's speed approaches the speed of light, the advance of wristwatch time becomes smaller and smaller compared with the advance of shell time. In the limit, it takes zero wristwatch time for the stone to circulate once around the black hole. Because $d\tau$ is in the denominator of the expression for angular momentum, the map angular momentum L/m increases without limit. The speed of light is the limiting speed of a stone, so the speed-of-light orbit is a limiting case, reached by a stone only after an unlimited lapse of the Schwarzschild t -coordinate. This limiting case tells us, however, that light can travel in a circular orbit at $r = 3M$ (Chapter 11).

QUERY 6. Global map energy of a stone in circular orbit

Find an expression for map energy E/m in global rain coordinates for the stone in a circular orbit, as follows:

- A. Use (25) and (15) with $dr = 0$ for a circular orbit. Show that the result is:

$$\frac{E}{m} = \frac{r - 2M}{r^{1/2}(r - 3M)^{1/2}} \quad (\text{circular orbit, } r > 3M) \quad (32)$$

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- B. Does (32) go to the values you expect in three cases: Case 1: $r \gg M$? Case 2: $r \rightarrow 3M^+$ (r decreases from above)? Case 3: $r < 3M$?

QUERY 7. Map energy and map angular momentum of a stone in the ISCO

- A. Show that the map angular momentum of the ISCO is $L_{\text{ISCO}}/(mM) = 3.464\ 101\ 615$.
 B. Show that the map energy of the ISCO is $E_{\text{ISCO}}/m = 0.942\ 809\ 042$.

QUERY 8. Shell energy of a stone in a circular orbit

- A. Use the special relativity relation $E_{\text{shell}}/m = (1 - v_{\text{shell}}^2)^{-1/2}$ for the local shell frame plus (31) for v_{shell}^2 to show that

$$\frac{E_{\text{shell}}}{m} = \left(\frac{r - 2M}{r - 3M} \right)^{1/2} \quad (\text{circular orbit, } r > 3M) \quad (33)$$

- B. From (32) and (33), verify that

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r} \right)^{-1/2} \frac{E}{m} \quad (\text{circular orbit } r > 2M) \quad (34)$$

This agrees with equation (12) in Section 6.3 for a diving stone.

- C. Far from the black hole, that is for $r \gg M$, set $\epsilon = M/r$. Use our standard approximation (inside the front cover) to show that at large r -coordinate equation (33) becomes:

$$\frac{E_{\text{shell}}}{m} \approx 1 + \frac{M}{2r} \quad (\text{circular orbit, } r \gg M) \quad (35)$$

- D. Take (31) to the same limit and show that (35) becomes:

$$E_{\text{shell}} \approx m + \frac{1}{2} m v_{\text{shell}}^2 \quad (\text{circular orbit, } r \gg M) \quad (36)$$

Would Newton be happy with your result? Would Einstein?

QUERY 9. Orbiter wristwatch time for one circular orbit

- A. From (25) and (11) verify the following wristwatch time for one circular orbit ($\Delta\phi = 2\pi$),

Section 8.6 Toy model of a quasar 8-17

$$\frac{\Delta\tau}{M} = \frac{2\pi(r/M)^2}{L/(mM)} = 2\pi \frac{r}{M} \left(\frac{r-3M}{M} \right)^{1/2} \quad (\text{one circular orbit}) \quad (37)$$

- B. Explain why $\Delta\tau \rightarrow 0$ as $r \rightarrow 3M$.
- C. For a black hole with $M = 10M_{\text{Sun}}$, find the wristwatch time in seconds for one circular orbit for the three values $r/M = 10, 6, 4$.
- D. For a non-spinning black hole of mass $M \approx 4 \times 10^6 M_{\text{Sun}}$ equal to the black hole at the center of our galaxy, find the wristwatch time in seconds for one circular orbit for the three values $r/M = 10, 6, 4$.
- E. *Optional:* Solve (37) for $(r/M - 3)$ and put $r \approx 3M$ in the expression on the right side of your result. Find the value of $(r/M - 3)$ when $\tau = 1$ microsecond for a black hole of mass $M = 10M_{\text{Sun}}$. What is the numerical value of the observed distance $2\pi r$ around this circumference—in meters—a directly-measurable distance (Section 3.3). So now we have an astronaut who traverses this large, measurable circumference in a microsecond. To do this, she must move at many times the speed of light. Can this be right? Explain your answer.

QUERY 10. Shell time for one circular orbit

Verify the following expressions for the periods of one circular orbit.

- A. From equations (27), (31), and (37), show that the local shell time for one circular orbit is:

$$\Delta t_{\text{shell}} = 2\pi r \left(\frac{r-2M}{M} \right)^{1/2} \quad (\text{one circular orbit}) \quad (38)$$

For the minimum (knife-edge) orbit, with $r = 3M$, explain why the shell period is equal to the circumference of the orbit.

- B. For a circular orbit of very large r -coordinate, explain why global rain ΔT , shell Δt_{shell} , and orbiter wristwatch time $\Delta\tau$ all have the same value for one orbit, namely $2\pi r^{3/2}/M^{1/2}$.

8.6 ■ TOY MODEL OF A QUASAR

Beacon of the heavens

Quasar

A **quasar** (“quasi-stellar object”) is an astronomical object that pours out electromagnetic radiation of many frequencies at a prodigious rate. The quasar is the brightest steady source of light in the heavens, so we can see it farther away than any other steady source. At the center of a quasar is, almost certainly, a spinning black hole (Chapters 17 through 21), but here we make a first quick model of a quasar using a non-spinning black hole. This sort of rough, preliminary analysis is called a **toy model**.

Toy model

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465 A simple model of quasar emission postulates an **accretion disk**, a gas
 466 disk that swirls around the black hole in its equatorial plane. Interactions
 467 among the molecules and atoms in this gas cloud are complicated. We assume
 468 simply that interactions among neighboring atoms and ions heats the
 469 accretion disk to high temperature and that the resulting electromagnetic
 470 emission is what we observe far from the quasar. The radiated energy we
 471 observe comes from the change in orbital map energy of each atom as it moves
 472 sequentially from a large- r circular orbit to smaller- r circular orbits with
 473 smaller and smaller map energy. We also assume that significant map energy
 474 change takes place over many orbits, so sequential orbits are nearly circular,
 475 each with its nearly constant values of E/m and L/m .

Accretion disk

Radiating away
change in map
energy

476 **Comment 6. Losing map angular momentum**

477 What is the mechanism of this orbit change? The sequence of circled numbers
 478 3-2-1 in Figure 4 shows that “our circulating atom” decreases both its map
 479 energy and its map angular momentum as it occupies a set of circular orbits of
 480 decreasing r -values. Map angular momentum of an isolated system is
 481 conserved, so the lost map angular momentum of our circulating atom must be
 482 transported outward, away from the black hole. What mechanism can transport
 483 map angular momentum outward? Equation (31) tells us that atoms in adjacent
 484 circular orbits have slightly different shell speeds, with atoms in the higher orbit
 485 moving more slowly. One might think (incorrectly) that *friction* between our atom
 486 and atoms in a higher orbit increases the velocity—and therefore the map
 487 angular momentum—of atoms in the higher orbit, and so on outward. However,
 488 direct friction turns out to be far too small to account for the outward transport of
 489 map angular momentum. The mechanism may depend on our model that the
 490 accretion disk consists of highly ionized atoms, a plasma, threaded with
 491 magnetic field lines. Magnetic fields greatly increase interactions between ions,
 492 so might account for the outward transport of map angular momentum in a
 493 quasar. We simply do not know.

Transport map
angular momentum
outward

494 Eventually our atom’s circular orbit drops to $r = 6M$, the innermost
 495 stable circular orbit (Definition 6 in Section 8.5). At this point our atom
 496 continues to lose map angular momentum, so that it drops out of the last
 497 stable circular orbit and spirals inward across the event horizon. Once our
 498 atom crosses the event horizon, any further radiation moves only inward and
 499 cannot reach us, the external observers.

Innermost stable
circular orbit

QUERY 11. Map energy given up by “our atom.”

The prodigious radiation we observe from quasars is all emitted before orbiting atoms and ions cross the event horizon.

- A. Start with an atom in a circular orbit at large r -coordinate, moving slowly so its initial map energy is approximately equal to its mass, $E/m \approx 1$, from (32). Now think of its map energy later, as the atom moves in the stable circular orbit of minimum r -coordinate, $r = 6M$. Using (32), find the map energy E/m of the atom in this minimum- r circular orbit to three significant digits. How much map energy has the atom given up during the process of dropping gradually

from large r -coordinate to the smallest stable circular orbit? [My answers: $E_{\text{final}} = 0.943m$ so $\Delta E = 0.057m$.]

- B. Suppose that the atom emits as electromagnetic radiation all the map energy it gives up (from Item A) as it spirals down to the circular orbit at $r = 6M$. Show that the map energy of that total amount of radiation emitted is $\Delta E = 0.057m$. Since initially we had $E/m = 1$, therefore 0.057, or 5.7%, is also the fraction of initial map energy that is radiated as the atom spirals inward to the lowest stable circular orbit.

Measure map energy at far from the black hole

516
517 Map energy E/m is a constant of motion, independent of position.
518 Suppose that the map energy radiated by the atom during its descent finds its
519 way outward. Then the same map energy ΔE arrives at the distant
520 r -coordinate from which the atom departed earlier with $E/m \approx 1$. Moreover,
521 very far from the black hole spacetime is flat; so map energy is equal to shell
522 energy there, equation (34). Therefore the group of shell frame observers far
523 from the black hole see—can in principle measure—a total radiated energy of
524 $\Delta E = 0.057m$, which is 5.7 percent of the stone’s initial map energy.

525 **Comment 7. How much emitted energy?**

526 No nuclear reaction on Earth—except particle-antiparticle
527 annihilation—releases as much as one percent of the rest energy of its
528 constituents. Chapter 18 shows that for a black hole of maximum spin, the
529 fraction of initial mass radiated away by a stone that spirals down from a large
530 r -coordinate to an innermost stable circular orbit is 42 percent of its rest energy.
531 No wonder quasars are such bright beacons in the heavens!

Rate of emitted radiation

532 Now let our atom drop into the black hole from the innermost stable
533 circular orbit at $r = 6M$. How much does the mass of the black hole increase?
534 Equation (28) in Section 6.5 says that the total mass of the black hole
535 increases by the map energy E/m of the object falling into it. This allows us
536 to connect the rate of increase of the mass of a quasar and its brightness to the
537 rate at which it is swallowing matter from outside. Let dm/dT be the rate at
538 which mass falls into the black hole from far away and dM/dT be the rate at
539 which the mass of the black hole increases. Then Item B in Query 11 tells us
540 that the rate of radiated energy is

$$\text{Rate of radiated energy} \approx 0.057 \frac{dm}{dT} \quad (dm = \text{mass falling in}) \quad (39)$$

541 so that the mass M of the black hole increases at the rate:

$$\frac{dM}{dT} = (1 - 0.057) \frac{dm}{dT} = 0.943 \frac{dm}{dT} \quad (M = \text{mass of black hole}) \quad (40)$$

542 **QUERY 12. Power output of a quasar**

During every Earth-year, a distant quasar swallows $m = 10M_{\text{Sun}}$ = ten times the mass of our Sun. Recall that watts equals joules/second and, from special relativity, $\Delta E[\text{joules}] = \Delta m[\text{kilograms}]c^2[\text{meters}^2/\text{second}^2]$.

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- A. How many watts of radiation does this quasar emit, according to our toy model?
- B. Our Sun emits radiation at the rate of approximately 4×10^{26} watts. The quasar is how many times as bright as our Sun?
- C. Compare your answer in Item B to the total radiation output of a galaxy, approximately 10^{11} Sun-like stars.

QUERY 13. How long does a quasar shine?

We see most quasars with large redshifts of their light, which means they began emission not long after the Big Bang, about 14×10^9 years ago. A typical quasar is powered by a black hole of mass less than 10^9 solar masses. Explain, from the results of Query 12, what this says about the lifetime during which the typical quasar shines.



561 **Objection 3.** *We have talked about t and T global coordinates and*
 562 *different kinds of local times near a black hole: shell time, diver time,*
 563 *orbiter time. Is it possible for me to travel to a black hole and use it, in*
 564 *some way, to live longer than I can live here on Earth?"*



565 As with many profound questions, the answer is both "Yes," and "No." With
 566 or without a black hole, you may live longer—on your wristwatch—between
 567 any two events than your twin—on her wristwatch—who takes a different
 568 worldline through spacetime between these two events (Twin "Paradox,"
 569 Section 1.6). However, you cannot escape the iron rule that *your aging is*
 570 *identical to the total time lapse on your wristwatch*, no matter where you
 571 travel or at what rates you move along the way. When your wristwatch says
 572 100 years after birth, you have aged 100 years. This is the total time lapse
 573 that you *experience*. In this sense, relativity does not provide a way for you
 to burst the bonds of human aging. Sorry!

8.7 ■ EXERCISES**1. Shell time for one orbit**

575 An observer in a circular orbit at a given map r -coordinate moves at speed
 576 v_{shell} past the shell observer. Equation (31) gives the value of this shell speed.
 577 Query 9 gives the wristwatch time for one orbit. What is the shell time for one
 578 orbit?
 579

580 A. Show that this shell time for one orbit is

$$\frac{\Delta t_{\text{shell}}}{M} = \frac{2\pi r/M}{v_{\text{shell}}} = 2\pi \frac{r}{M} \left(\frac{r-2M}{M} \right)^{1/2} \quad (\text{one circular orbit}) \quad (41)$$

Section 8.7 Exercises **8-21**

581 (*Hint:* Recall the definition in Section 3.3 of r —the “reduced
582 circumference”—as the measured circumference of a concentric shell
583 divided by 2π .)

- 584 B. Compare Δt_{shell} for one orbit in (41) with $\Delta\tau_{\text{shell}}$ for one orbit from
585 (37). Which is longer at a given r -value? Give a simple explanation.
- 586 C. What is the map angular momentum L of the orbiter, written as r
587 times an expression involving v_{shell} ? (The answer is *not* $mr v_{\text{shell}}$.)
- 588 D. The text leading up to Definition 4 in Section 8.5 shows that the
589 smallest r -coordinate for a stable circular orbit is $r = 6M$; equation (31)
590 determines that in this orbit the orbiter’s shell speed $v_{\text{shell}} = 0.5$, half
591 the speed of light. Assume the central attractor to be Black Hole Alpha,
592 with $M = 5000$ meters. The following equation gives, to one significant
593 digit, the values of some measurable quantities for the innermost stable
594 circular orbit. Find these values to three significant digits.

$$\Delta t_{\text{shell}} \approx 4 \times 10^5 \text{ meters} \quad (\text{shell time for one orbit}) \quad (42)$$

$$\Delta\tau_{\text{orbiter}} \approx 3 \times 10^5 \text{ meters} \quad (\text{wristwatch time for one orbit})$$

$$L/m \approx 2 \times 10^4 \text{ meters}$$

- 595 E. The orbiter of Item D completes one circuit of the black hole in
596 approximately one millisecond on her wristwatch. If you ignore tidal
597 effects, does this extremely fast rotation produce *physical discomfort* for
598 the orbiter? If she closes her eyes, does she get dizzy as she orbits?

599 **2. When are Newton’s Circular Orbits Almost Correct?**

600 Your analysis of the Global Positioning System (GPS) in Chapter 4 calculated
601 values of r -coordinate and orbital speed of a GPS satellite in circular orbit
602 using Newton’s mechanics, with the prediction that the general relativistic
603 analysis gives essentially the same values of r -coordinate and speed for this
604 application. Under what circumstances are circular orbits predicted by Newton
605 indistinguishable from circular orbits predicted by Einstein? Answer this
606 question using the following outline or some other method.

- 607 A. Find Newton’s expression similar to equation (26) for the r -coordinate
608 of a stable circular orbit, starting with equation (23).
- 609 B. Recast equation (26) for the general-relativistic prediction of r for
610 stable orbits in the form

$$r = r_{\text{Newt}}(1 - \epsilon) \quad (43)$$

611 where r_{Newt} is the r -coordinate of the orbit predicted by Newton and ϵ
612 is the small fractional deviation of the orbit from Newton’s prediction.
613 This expression neglects differences between the Newtonian and

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- 614 relativistic values of L when expressed in the same units. Use the
 615 approximation inside the front cover to derive a simple algebraic
 616 expression for ϵ as a function of r_{Newt} .
- 617 C. Set your expression for ϵ equal to 0.001 as a criterion for good-enough
 618 equality of the r -coordinate according to both Newton and Einstein.
 619 Find an expression for r_{min} , the smallest value of the r -coordinate for
 620 which this approximation is valid.
- 621 D. Find a numerical value for r_{min} in meters for our Sun. Compare the
 622 value of r_{min} with the r -coordinate of the Sun's surface.
- 623 E. What is the value of ϵ for the r -coordinate of the orbit of the planet
 624 Mercury, whose orbit has an average r -coordinate 0.387 times that of
 625 Earth?
- 626 F. What is the value of ϵ for the r -coordinate of a 12-hour orbit of GPS
 627 satellites around Earth?

3. Map ΔT for one orbit

628 Convert lapse of wristwatch time $\Delta\tau$ for one circular orbit from (37) to lapse
 629 ΔT for one circular orbit using the following outline or some other method:
 630

- 631 A. Show that for a circular orbit, equation (13) becomes:

$$\frac{\Delta T}{\Delta\tau}(\text{one orbit}) = \frac{E}{m} \left(1 - \frac{2M}{r}\right)^{-1} = \frac{E}{m} \frac{r}{r - 2M} \quad (44)$$

- 632 B. Into this equation, substitute for E/m from (32) to obtain

$$\frac{\Delta T}{\Delta\tau}(\text{one orbit}) = \left(\frac{r}{r - 3M}\right)^{1/2} \quad (45)$$

- 633 C. Use this result plus (37) to show that

$$\Delta T(\text{one orbit}) = \Delta\tau \frac{\Delta T}{\Delta\tau} = 2\pi \frac{r^{3/2}}{M^{1/2}} \quad (46)$$

634 Does any observer measure this lapse Δt for one orbit?

4. Kepler's Laws of Planetary Motion

635 Johannes Kepler (1571-1630) provided a milestone in the history of astronomy:
 636 his **Three Laws of Planetary Motion**, deduced from a huge stack of
 637 planetary observations made by his mentor Tycho Brahe (1546-1601) and
 638 expressed in our notation.
 639

- 640 1. A planet orbits around the Sun in an elliptical orbit with the
 641 Sun at one focus of the ellipse.

- 642 2. The r -coordinate vector from the Sun to the planet sweeps out
643 equal areas in equal lapses of T -coordinate.
- 644 3. The square of the period of the planet is proportional to the
645 cube of the planet's mean r -coordinate from the Sun.
- 646 A. Show by a simple symmetry argument that Kepler's Second Law
647 describes circular orbits around a black hole.
- 648 B. From equation (46) show that Kepler's Third Law is also valid for
649 *circular* orbits around a black hole (when expressed in global rain
650 coordinates).
- 651 C. Kepler's Third Law is sometimes called the **1-2-3 Law** from the
652 exponents in the following equation. Use equation (46) to show that for
653 circular orbits, in our regular notation using meters,

$$M \equiv M^1 = \omega^2 r^3 \quad (47)$$

654 where $\omega \equiv 2\pi/\Delta T$, with ΔT for one orbit.

655 **Comment 8. Is Kepler's First Law Valid?**

656 Figure 4 in Section 9.3 shows that Kepler's First Law is definitely *not* valid for
657 non-circular orbits near a non-spinning black hole. Chapter 11 shows that the
658 orbit of the planet Mercury differs *slightly* from the planetary orbit analyzed by
659 Newton. The predicted value of this deviation of Mercury's orbit was an early
660 validation of Einstein's general relativity.

661 **5. Longest Life Inside the event Horizon**

662 Objection 12 in Section 7.8 asked, "Can I increase my lifetime inside the
663 event horizon by blasting rockets in either ϕ direction to add a ϕ -component
664 to my global velocity?" You are now able to answer this question using your
665 new knowledge of map angular momentum. Suppose that you ride on a stone
666 that moves between the event horizon and the singularity.

- 667 A. What equation in the present chapter leads to the following expression
668 for your wristwatch lifetime inside the horizon?

$$\tau [2M \rightarrow 0] = \int_0^{2M} \left[\left(\frac{E}{m} \right)^2 + \left(\frac{2M}{r} - 1 \right) \left(1 + \frac{L^2}{m^2 r^2} \right) \right]^{-1/2} dr \quad (48)$$

669 Note, first, that the square-bracket expression on the right side of (48)
670 is in the denominator of the integrand. Second, note that this equation
671 describes any motion of the observer whatsoever, free-fall or not.
672 Free-fall motion has constant E and L . For motion that is not free-fall,
673 the value of E or L (or both) can change along the worldline of the
674 stone.

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- 675 B. Can any non-zero value of L along your worldline increase your
676 wristwatch lifetime inside the event horizon?
- 677 C. What value of E gives you the maximum wristwatch lifetime inside the
678 event horizon?
- 679 D. By what practical maneuvers can you achieve the value of E
680 determined in Item C?
- 681 E. Show that the maximum value of wristwatch time from the event
682 horizon to the singularity is πM meters. *Hint:* Make the substitution
683 $(r/2M)^{1/2} = \sin \theta$.
- 684 F. Chapter 7 found the mass of a “20-year black hole” for a raindrop. Find
685 the numerical value of (*fraction*) in the following equation:

$$\begin{aligned} &(\text{mass of “20-year black hole” in Item E}) && (49) \\ &= (\textit{fraction}) \times (\text{mass of “20-year black hole” for a raindrop}) \end{aligned}$$

686 **6. Forward Time Travel Using a *Stable* Circular Orbit**

687 You are on a panel of experts asked to evaluate a proposal from the Space
688 Administration to “travel forward in time” using the difference in rates
689 between a clock in a stable circular orbit around a black hole and our clocks
690 remote from the black hole. Give your advice about the feasibility of the
691 scheme, based on the following analysis or one of your own.

- 692 A. Consider two sequential ticks of the clock of a satellite in a stable
693 circular orbit around a black hole. Use a result of Exercise 1 to show
694 that

$$\frac{\Delta\tau_{\text{orbiter}}}{\Delta T} = \left(\frac{r - 3M}{r} \right)^{1/2} \quad (50)$$

- 695 B. What is the value of the ratio $\Delta\tau_{\text{orbiter}}/\Delta T$ in the stable circular orbit
696 of smallest r -coordinate, $r = 6M$?
- 697 C. What rocket speed in flat spacetime gives the same ratio of rocket clock
698 time to “laboratory” time as the stable circular orbit of smallest
699 r -coordinate?
- 700 D. Does the proposed time travel method require rocket fuel to put the
701 rocket in orbit and to escape the black hole?
- 702 E. Based on this analysis, do you recommend in favor of—or against—the
703 Space Administration’s proposal for forward time travel using stable
704 circular orbits around a black hole?

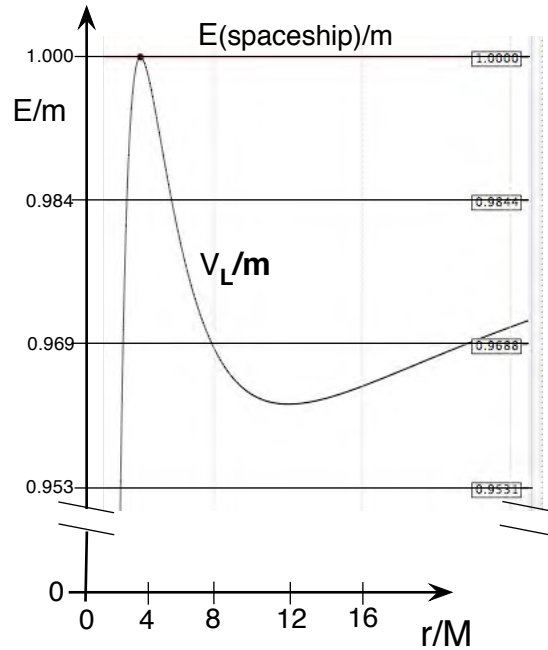


FIGURE 6 Insertion into a knife-edge orbit at $r = 4M$ with map energy $E/m \approx 1$, equal to that of a spaceship moving slowly at large r -coordinate in a direction chosen to give it the value of L/m required to establish the peak value for V_L/m .

705 7. Forward Time Travel Using a *Knife-Edge* Circular Orbit

706 Whatever your own vote on the forward time travel proposal of Exercise 6, the
 707 majority on your panel rejects the proposal because it requires extra rocket
 708 thrust for insertion into and extraction from the circular orbit at $r = 6M$. The
 709 Space Administration returns with a new proposal that uses a knife-edge
 710 circular orbit, assuming that an automatic device can fire small rockets to
 711 balance the satellite safely on the knife-edge of the effective potential. The
 712 Space Administration notes that such an orbit can be set up to require *very*
 713 *small* rocket burns, both for insertion into and extraction from a knife-edge
 714 circular orbit. As an example, they present Figure 6 for the case of
 715 nonrelativistic distant velocity, so that the map energy of the satellite is
 716 $E/m \approx 1$. While still far from the black hole, the spaceship captain uses
 717 rockets to achieve the value of L required so that $V_L(r)/m = E/m = 1$ on the
 718 peak shown in Figure 6. They boast that the time stretch factor is increased
 719 enormously by high satellite shell speed in the knife-edge orbit without the
 720 need for rocket burns to achieve that speed.

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- 721 A. The condition shown in Figure 6 means that $V_L(r)/m = 1$ at the peak
 722 of the effective potential (18). This equation plus equation (26) are two
 723 equations in the two unknowns r and L . Solve them to find $r = 4M$
 724 and $L/m = 4M$. *Optional:* Describe in words how the commander of
 725 the spaceship sets the desired value of L while still far away, without
 726 changing the remote non-relativistic speed v_{far} .
- 727 B. What is the factor $d\tau/dt_{\text{shell}}$ for the spaceship in this orbit? What
 728 speed in flat spacetime gives the same time-stretch ratio?
- 729 C. Does the spaceship require a significant rocket burn to leave its
 730 knife-edge circular orbit and return to a remote position? What will be
 731 its shell speed at that distant location?

732 **8. “Free” data-collection orbit**

733 After its long interstellar trip, the spaceship approaches the black hole at
 734 relativistic speed, that is $E/m > 1$. The commander does not want to use a
 735 rocket burn to change spaceship map energy, but rather only its direction of
 736 motion (hence its map angular momentum) to enter a knife-edge circular orbit
 737 with the same map energy it already has.

- 738 A. Draw a figure similar to Figure 6 for this case.
- 739 B. Show that the astronauts can find a knife-edge circular orbit on which
 740 to perch, no matter how large the incoming far-away speed with respect
 741 to the black hole.

742 Once in an unstable circular orbit, small rocket thrusts keep the spaceship
 743 balanced at the peak of the effective potential. After they finish collecting
 744 data, the astronauts push-off outward and return toward home base at the
 745 same speed at which they approached, even if this speed is relativistic. In
 746 summary, once launched toward a black hole the explorers need little rocket
 747 power to go into an unstable circular orbit, to balance in that orbit while they
 748 study the black hole, then to return home. Further details in Chapter 9.

749 **9. Nandor Bokor disproves relativity.**

750 Nandor Bokor looks at Exercise 1 and shouts, “Aha, now I can disprove
 751 relativity!” Parts A through D below are steps in Nandor’s reasoning, not
 752 separate questions to be answered. Resolve Nandor’s disproof without
 753 criticizing him.

- 754 A. Nandor Bokor says, “Before I begin my disproof of relativity, recall that
 755 we have always had a choice about the shell frame. *First choice:* In
 756 order to be inertial, the local shell frame must be in free fall. In this
 757 case we drop the local shell frame from rest as we begin the experiment
 758 and must complete the experiment so quickly that the shell frame’s

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- 759 r -coordinate changes a negligible amount. *Second choice:* The local
 760 shell frame is at rest and therefore has a local gravitational
 761 acceleration. In that case we must complete our experiment or
 762 observation so quickly that local gravity does not affect the outcome.
 763 Usually our choice does not change the experimental result, but I am
 764 being super-careful here and will take the first choice, so that shell and
 765 orbiter frames are both inertial.
- 766 B. “Assume, then, that the shell frame is inertial,” Nandor continues.
 767 “Equation (42) says that during one revolution of the orbiter its
 768 measured time lapse is $\Delta t_{\text{orbiter}} \approx 3 \times 10^5$ meters, while the measured
 769 shell clock time lapse is $\Delta t_{\text{shell}} \approx 4 \times 10^5$ meters. Note that these are
 770 both observed readings—measurements—and they are *different*. When
 771 the orbiter returns after one orbit the two inertial frames—orbiter and
 772 shell—overlap again.
- 773 C. “Now we have two truly equivalent inertial reference frames that
 774 overlap twice so we can compare their clock readings directly. (This is
 775 different from special relativity, in which one of the two frames—in the
 776 Twin Paradox, Section 1.6—is not inertial during their entire
 777 separation.) In the present orbiting case, neither observer can tell which
 778 of the two inertial frames s/he is in from inside his or her inertial
 779 frame.”
- 780 D. Nandor concludes, “You tell me, Dude, which of the two equivalent
 781 inertial clocks—the orbiter’s frame clock or the shell observer’s frame
 782 clock—runs slow compared with the clock in the other frame. You
 783 can’t! Equation (42) claims a difference where no difference is possible.
 784 Good-bye relativity!”

785 **10. Equations of motion in Schwarzschild global coordinates**

786 Start with the Schwarzschild metric, equation (5) in Section 3.1, and show that
 787 equations (11), are (15) are the same in both global coordinate systems, but
 788 (16) takes the simpler form:

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \frac{E}{m} \quad (\text{stone, Schwarzschild}) \quad (51)$$

789 **Comment 9. Why not Schwarzschild?**

790 Why don’t we take advantage of the simpler equation (51) by using
 791 Schwarzschild coordinates to describe the motion of the free stone? Because we
 792 already know—equation (22) in Section 6.4—that neither light nor a stone moves
 793 inward through the event horizon in a finite lapse of the Schwarzschild
 794 t -coordinate. In theory, Schwarzschild coordinates would not cause a problem
 795 with circular orbits in the present chapter because these orbits exist only outside
 796 the event horizon—indeed, only in the region $r > 3M$. But Chapter 9 treats
 797 more general trajectories of a stone, some of which move inward across the
 798 event horizon.

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799 **11. Life under the forbidden map energy region**

800 If we could find some way to travel from our normal upper, positive map
801 energy region in Figure 5 to the lower, negative map energy region (which
802 extends outward far from the black hole), could we live a normal life there?
803 What does “normal life” mean? We reduce “normal life” to essentials: that the
804 equations of motion for a stone are real! Limit attention to motion outside the
805 event horizon:

- 806 A. Show that the first two equations of motion (11) and (15) are the same
807 for E/m under the forbidden region as for E/m above the forbidden
808 region.
- 809 B. Show that the third equation of motion (16) tells us that $dT/d\tau$ is
810 negative under the forbidden region, so that global T runs backward
811 along the worldline of the stone. But T is a unicorn, not a measured
812 quantity, so the third equation of motion is also valid under the
813 forbidden region.

814 *Where* are we when we are under the forbidden map energy region in Figure
815 5? This is our first hint that our everyday lives may not have access to all
816 regions of spacetime. Alice had it right: Wonderland—and black
817 holes—become “curiouser and curiouser.”

8.8 ■ REFERENCES

- 819 Initial Emily Dickinson poem from R. W. Franklin, *The Poems of Emily*
820 *Dickinson, Variorum Edition* 1998, The Belknap Press of Harvard
821 University. This poem is variation E of the poem with Franklin number
822 1570, written about 1882. Reprinted and modified with permission of
823 Harvard University.
- 824 GRorbits interactive software program that displays orbits of a stone and light
825 flash is available at <http://stuleja.org/grorbits/>
- 826 Last sentence of the final exercise: *Alice in Wonderland* by Lewis Carroll, first
827 sentence of Chapter 2.