Chapter 11 Orbits of Light

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- *What variety of orbits does light follow around a black hole?*
- *Can a black hole reverse the direction of a light flash?*
- *Can light go into a circular orbit around a black hole? if so, is this circular orbit stable?*
- *How many different orbits can light take from a single star to my eye?*

CHAPTER 11

Orbits of Light

Edmund Bertschinger & Edwin F. Taylor *

Then the sun god Ra emerged out of primal chaos.
—Egyptian creation story

And at once Kiho made his eyes to glow with flame—and the darkness became light.
—Tuamotuan (Polynesian) creation story

And God said, Let there be light: and there was light.
—first Biblical act of creation, Genesis 1:3

He bringeth them out of darkness unto light by His decree . . .
—Qur'an 5:16

Along with death came the Sun the Moon and the stars . . .
—Inuit creation story

11.1 ■ TURN A STONE INTO A LIGHT FLASH

Faster and faster, less and less mass

So far, observers are blind.

Thus far in this book almost all observers have been blind. Chapter 5 defined the shell observer but did not predict what he sees when he looks at stars or other objects outside his local inertial frame. The rain diver as she descends to the singularity (Chapter 7) peers in just two opposite directions—radially inward and radially outward. The explorer in her circular orbit around a black hole (Chapter 8) does not report what she sees—neither the starry heavens around her nor the black hole beneath her. In the present chapter we lay the groundwork to cure this blindness: we plot orbits of light in global map coordinates.

No local observation in this chapter

But this chapter still does not describe what any observer *sees*. Recall that we make every measurements and observation in a local inertial frame. The present chapter describes only map "starlight orbits," for example the orbit

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that connects remote Star X with an observer at (or passing through) map location Y. The following Chapter 12 will tell us in what direction an observer at Y looks to see Star X.

Seeing is *not* believing.

What can we say about the global motion of light around, past, or into a spherically symmetric, nonspinning black hole? We ask here no small question: Almost every message from events in space comes to us by way of electromagnetic radiation of different frequencies. *Exceptions:* cosmic rays, neutrinos, and gravitational waves. A starlight orbit may deflect as it passes close to a massive object. Near a black hole this deflection can be radical; starlight can even go into a circular orbit. This and the following chapter make clear that for an observer near a black hole, seeing is definitely *not* believing!

Find orbits of light.

How do we plot the global orbit of light around a black hole? This is a new question; up until now we plotted light cones with short legs that sprout from a single event. Now we want to "connect the dots," the events along an entire orbit of light that stretches from a specified distant star to a given local observer near a black hole.

Constant(s) of motion for light?

The *free stone* has two global constants of motion along its worldline: map energy E and map angular momentum L. Chapters 3 and 8 used the Principle of Maximal Aging to derive map expressions for each of these global constants of motion. Can we use the Principle of Maximal Aging to find constant(s) of motion for a light flash?

Principle of Maximal Aging does not apply directly to light.

The Principle of Maximal Aging says that a stone chooses a path across an adjoining tiny pair of segments along its worldline such that its wristwatch time is a maximum between a fixed initial event as the stone enters the pair and a fixed final event as it leaves the pair. But the Principle of Maximal Aging cannot apply directly to light, and for a fundamental reason: *The aging of a light flash along its worldline in a vacuum is automatically zero!* Aging $d\tau$ equals *zero* along every differential increment of the light flash worldline. *Question:* How can we possibly apply the Principle of Maximal Aging to light, whose aging is automatically zero?

Adapt Principle of Maximal Aging to light.

Answer: Sneak up on it! Start in flat spacetime far from a black hole. Think of a series of faster and faster stones, each stone with a smaller mass than the previous one. Let this series occur in such a way that the map energy E remains constant. Far from the black hole, map energy equals the measurable energy in a local inertial shell frame, in which the stone has squared speed v^2_{shell}. Take the limit of equation (28) in Section 1.7 as $m \to 0$ and $v_{\text{shell}} \to 1$:

$$E = \lim_{v_{\text{shell}} \to 1}^{m \to 0} \frac{m}{(1 - v^2_{\text{shell}})^{1/2}} = \text{constant} \qquad (\text{light, } r/M \gg 1) \qquad (1)$$

Stone $\to$ light as $m \to 0$ and $v \to 1$

The present chapter analyzes consequences of this limit-taking process in (1).

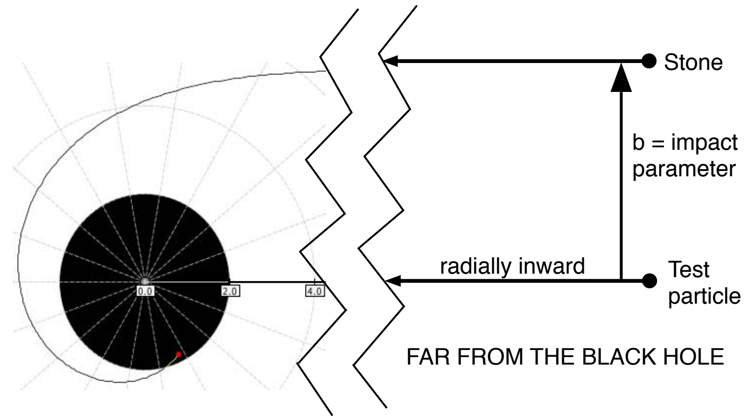

FIGURE 1 Impact parameter b of a stone that approaches the black hole from a far away. Far from the black hole, we define b as the perpendicular offset between the line of motion of the approaching stone and the parallel line of motion of a test particle that makes a dive at constant ϕ into the black hole. Values of b and M determine whether or not the black hole captures the incoming stone.

11.2 ■ IMPACT PARAMETER b

Impact parameter from map angular momentum and map energy

Chapter 8 analyzed circular orbits of a stone around the black hole. Now we want to describe more general orbits of both a stone and a light flash, so we define an orbit.

Definition: orbit

DEFINITION 1. Orbit: Stone or light flash
An orbit is the worldline of a stone or light flash described by global coordinates. An orbit need not be circular around an origin, it need not be closed, it need not even remain in a bounded region of space.

A starlight orbit is a special case of the orbit:

Definition: starlight orbit

DEFINITION 2. Starlight orbit
A starlight orbit is the orbit (Definition 1) of a light flash emitted by a star.

“Straight line” verified in local shell frame.

Think first about the orbit of a free stone far from the black hole—the right side of Figure 1. Far from the black hole this orbit is straight. How do we measure this orbit to verify that it is straight? As always, carry out measurements in a local inertial frame. We choose a shell frame (Section 5.7). Sufficiently far from the black hole this “local” shell frame can be quite large in the sense that over a significant range of r and ϕ special relativity correctly describes this orbit as a *straight line*. Now find a parallel straight line orbit that—by trial and error—moves without deflection to the center of the black hole (verified by measurement in a series of shell frames on both sides of Figure 1).

In a local inertial shell frame far from the black hole, we can measure perpendicular distances between parallel orbits. This leads to the definition of the **impact parameter**, with the symbol b. In a preliminary definition, we define the impact parameter of a stone far from the black hole:

Preliminary definition: impact parameter

DEFINITION 3. Impact parameter b of a stone (preliminary)
The impact parameter b of a stone is the perpendicular distance—measured far from the black hole—between the straight orbit of the free stone and the parallel straight orbit of a second stone (test particle) that plunges at constant ϕ into the black hole.

QUERY 1. Every moving stone has an impact parameter
Show that every distant stone that changes global coordinates r or ϕ (or both) has an impact parameter—even a stone that moves away from the black hole.

Thus far the definition of the impact parameter is purely geometric. However, the right side of Figure 1 can be used to define angular momentum. The angular momentum of the stone takes the simple form:

$$L_{\text{far}} \equiv b_{\text{far}} p_{\text{far}} \qquad \text{(stone in distant—flat—spacetime)} \tag{2}$$

Map angular momentum L

where p_{far} is the momentum of special relativity (Section 1.8). Equation (2) determines the value of L where $r/M \gg 1$, that is where spacetime is flat. However L is a map constant of motion, the same everywhere around the black hole. Therefore its value, calculated from (2) far from the black hole, is the same close to the black hole.

Recall equation (39) for a stone in Section 1.9, with p defined in (2):

$$m^2 = E^2 - p^2 = E^2 - \left(\frac{L}{b}\right)^2 \qquad \text{(stone, flat spacetime)} \tag{3}$$

Impact parameter of a stone

Solve this equation for b, in which b and L are either both positive or both negative:

$$b \equiv \frac{L}{(E^2 - m^2)^{1/2}} \qquad \text{(impact parameter for a stone, everywhere)} \tag{4}$$

Both map energy E and map angular momentum L are map constants of motion and m is an invariant quantity. Therefore equation (4) is valid close to the black hole as well as far away. Even though it was derived assuming flat spacetime, we take (4) to define b everywhere. Close to the black hole, b is no longer the perpendicular distance of Definition 3. But every orbit has an L and an E and therefore can be assigned a unique value of b.

For light, carry out the limit-taking process demanded in (1), with constant E but decreasing m. The limit $m \to 0$ defines the impact parameter for light:

Impact parameter of a light flash

$$b \equiv \frac{L}{E} \qquad \text{(impact parameter of light, everywhere)} \tag{5}$$

This leads to the final definition of the impact parameter for a stone or a light flash around a black hole:

Definition: impact parameter b

DEFINITION 4. Impact parameter b
The **impact parameter** b for a stone is given by (4) and for a light flash by (5).

?

Objection 1. *You use two perfectly good constants of motion, L and E and give a geometric interpretation for a combination of them. So what? I can define a thousand combinations of L and E. Who cares? I didn't need any such combination for a stone. Why are you wasting my time?*

!

We introduce b because neither L alone or E alone will be helpful when $m \to 0$. Equations of motion for light derived below depend only on the fraction L/E and no other combination. Global motion of a stone depends on two constants of motion, L and E. Global motion of light is simpler, completely described by one constant of motion, $b \equiv L/E$. Rejoice!

We have defined impact parameter, but we have not yet predicted the global motion of a light flash near the black hole. To obtain equations of motion for light, we again apply the limit-taking process of equation (1), in this case to the equations of motion for a stone from Chapter 8.

11.3 ■ EQUATIONS OF MOTION FOR LIGHT

A single constant of motion for light, namely b

Flat starlight wavefront approaching the black hole . . .

Light spreads out from a star as a spherical wave. We assume that every star is so far away that as its starlight approaches our black hole—but still travels in flat spacetime—it forms a flat wavefront (right side of Figure 2).

. . . is equivalent to a bundle of parallel straight orbits.

We already have another powerful way to describe starlight in flat spacetime: as a bundle of parallel straight orbits. Figure 2 displays four starlight orbits from a single star, each with a different impact parameter b, as these orbits approach the black hole. Far from the black hole (right side of the figure) these starlight orbits remain parallel to one another. Close to the black hole (left side of the figure) they diverge: Only the orbit with $b/M = 0$ remains straight. Starlight Orbit 1 deflects but escapes; Starlight Orbit 2 enters a circular orbit; Starlight Orbit 3 plunges to the center of the black hole.

Close to the black hole, orbits from the star are neither parallel nor straight.

Critical impact paraneter

Starlight Orbit 2 in Figure 2 is unique; it enters a circular orbit at $r = 3M$. We call this orbit *critical* and its impact parameter the *critical impact parameter*, b_{critical}. In Query 3 you show that the critical impact parameter has the value $b_{\text{critical}} = (27)^{1/2}M$.

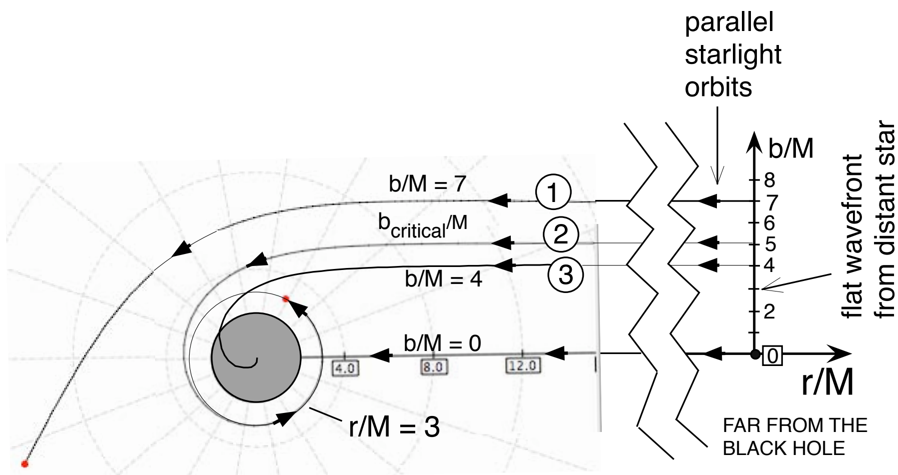

FIGURE 2 Jagged lines separate flat spacetime far from the black hole (on the right) from curved spacetime near the black hole (on the left). The right side of this plot shows two ways to visualize starlight orbits far from the black hole: first as as a set of straight parallel orbits, second as a flat wavefront. On the left side of this plot, near the black hole, only the starlight orbit with $b/M = 0$ remains straight, while starlight orbits 1 through 3, originally parallel, diverge: Starlight Orbit 1 with the impact parameter $b/M = 7$ deflects but escapes. Starlight Orbit 2 with the so-called *critical impact parameter* $b_{\rm critical}/M$, equation (28), becomes an unstable circular orbit at $r/M = 3$. Starlight Orbit 3 with $b/M = 4$ crosses the event horizon and ends at the singularity.

We need general equations of motion of light, which we now derive using the limiting process of equation (1). Start with equations of motion of a stone from Section 8.3, written in slightly altered form:

$$\frac{dr}{d\tau} = \pm\left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right)\right]^{1/2} \qquad \text{(stone)} \qquad (6)$$

$$\frac{d\phi}{d\tau} = \frac{L}{mr^2} \qquad \text{(stone)} \qquad (7)$$

$$\frac{d\tau}{dT} = \frac{\left(1 - \dfrac{2M}{r}\right)}{\dfrac{E}{m} \pm \left(\dfrac{2M}{r}\right)^{1/2}\left[\left(\dfrac{E}{m}\right)^2 - \left(1 - \dfrac{2M}{r}\right)\left(1 + \dfrac{L^2}{m^2 r^2}\right)\right]^{1/2}} \qquad (8)$$

Comment 1. Choice of signs for the motion of a stone
We choose the stone's wristwatch time to advance as the stone moves along its worldline. Therefore the upper $(+)$ sign in (6) is for a stone with increasing r and the lower $(-)$ sign is for a stone with decreasing r. The $\pm$ sign in the denominator of equation (8) has the same meaning.

In order to describe the motion of light, we need to eliminate $d\tau$ from these equations, because adjacent events along the worldline of a light flash have zero wristwatch time lapse between them: $d\tau = 0$. Multiply both sides of (6) by the corresponding sides of (8), then factor out and cancel (E/m) from the resulting numerator and denominator.

$$\begin{aligned}\frac{dr}{dT} &= \frac{dr}{d\tau}\frac{d\tau}{dT} \qquad \text{(stone)} \qquad (9)\\ &= \pm\frac{\left(1-\dfrac{2M}{r}\right)\left[1-\left(\dfrac{m}{E}\right)^2\left(1-\dfrac{2M}{r}\right)\left(1+\dfrac{L^2}{m^2r^2}\right)\right]^{1/2}}{1\pm\left(\dfrac{2M}{r}\right)^{1/2}\left[1-\left(\dfrac{m}{E}\right)^2\left(1-\dfrac{2M}{r}\right)\left(1+\dfrac{L^2}{m^2r^2}\right)\right]^{1/2}}\end{aligned}$$

Equation (1) requires that for light $m \to 0$ while E remains constant. Apply these requirements to (9). The result is our first equation of motion for light:

$$\frac{dr}{dT} = \pm\frac{\left(1-\dfrac{2M}{r}\right)\left[1-\left(1-\dfrac{2M}{r}\right)\left(\dfrac{L}{rE}\right)^2\right]^{1/2}}{1\pm\left(\dfrac{2M}{r}\right)^{1/2}\left[1-\left(1-\dfrac{2M}{r}\right)\left(\dfrac{L}{rE}\right)^2\right]^{1/2}} \qquad \text{(light)} \quad (10)$$

Carry out a similar procedure on equations (7) and (8): multiply their corresponding sides $d\phi/dT = (d\phi/d\tau)(d\tau/dT)$, factor out E/m in the denominator, cancel m with one in the numerator, then let $m \to 0$. The result is our second equation of motion for light:

$$\frac{d\phi}{dT} = \frac{\dfrac{L}{r^2E}\left(1-\dfrac{2M}{r}\right)}{1\pm\left(\dfrac{2M}{r}\right)^{1/2}\left[1-\left(1-\dfrac{2M}{r}\right)\left(\dfrac{L}{rE}\right)^2\right]^{1/2}} \qquad \text{(light)} \quad (11)$$

To construct our third equation of motion for light, combine (10) with (11):

$$\frac{dr}{d\phi} = \left(\frac{dr}{dT}\right)\left(\frac{dT}{d\phi}\right) = \pm\frac{r^2E}{L}\left[1-\left(1-\frac{2M}{r}\right)\left(\frac{L}{rE}\right)^2\right]^{1/2} \qquad \text{(light)} \quad (12)$$

Equations (10) through (12) are the equations of motion for light. The choice of signs in these equations is the same as for a stone, given in Comment 1.

Our three equations of motion for light contain a wonderful surprise: The only quantity we need to describe the orbit of light is the ratio L/E. *Meaning:* The orbit of light near a black hole is completely determined by the single value of the ratio L/E instead of by the separate values of the map constants of motion L and E. And equation (5) tells us that this ratio equals the impact parameter for light.

Light motion depends on only $L/E = b$.

Substitute the expression $b = E/L$ into equations (10) through (12):

$$\frac{dr}{dT} = \pm\frac{\left(1-\dfrac{2M}{r}\right)\left[1-\left(1-\dfrac{2M}{r}\right)\left(\dfrac{b}{r}\right)^2\right]^{1/2}}{1\pm\left(\dfrac{2M}{r}\right)^{1/2}\left[1-\left(1-\dfrac{2M}{r}\right)\left(\dfrac{b}{r}\right)^2\right]^{1/2}} \qquad \text{(light)} \qquad (13)$$

$$\frac{d\phi}{dT} = \frac{\dfrac{b}{r^2}\left(1-\dfrac{2M}{r}\right)}{1\pm\left(\dfrac{2M}{r}\right)^{1/2}\left[1-\left(1-\dfrac{2M}{r}\right)\left(\dfrac{b}{r}\right)^2\right]^{1/2}} \qquad \text{(light)} \qquad (14)$$

$$\frac{dr}{d\phi} = \pm\frac{r^2}{b}\left[1-\left(1-\frac{2M}{r}\right)\left(\frac{b}{r}\right)^2\right]^{1/2} \qquad \text{(light)} \qquad (15)$$

An identical square-bracket expression appears multiple times in these equations. To simplify them, define a new function $F(b, r)$:

$$F(b,r) \equiv \left[1-\frac{b^2}{r^2}\left(1-\frac{2M}{r}\right)\right]^{1/2} \qquad \text{(light)} \qquad (16)$$

Equations of motion for light

so that equations of motion for light become:

$$\frac{dr}{dT} = \pm\frac{\left(1-\dfrac{2M}{r}\right)F(b,r)}{1\pm\left(\dfrac{2M}{r}\right)^{1/2}F(b,r)} \qquad \text{(light)} \qquad (17)$$

$$\frac{d\phi}{dT} = \frac{\dfrac{b}{r^2}\left(1-\dfrac{2M}{r}\right)}{1\pm\left(\dfrac{2M}{r}\right)^{1/2}F(b,r)} \qquad \text{(light)} \qquad (18)$$

$$\frac{dr}{d\phi} = \pm\frac{r^2}{b}F(b,r) \qquad \text{(light)} \qquad (19)$$

The $\pm$ signs in equations (17) through (19) have the same interpretation as in (6) through (8) and also (10) through (12), namely the upper $(+)$ sign describes light with increasing r and the lower $(-)$ describes light with decreasing r.

Chapters 9 and 10 use interactive software GRorbits to plot orbits of a stone. GRorbits also integrates equations (17) through (19) for light. Given the value of b and initial location, the software plots the orbit and outputs a spreadsheet with global coordinates (T, r, ϕ) of events along the orbit.

Equations of motion for light look complicated. We now derive a simple way to visualize the global r-motion of light using the effective potential, modeled after the effective potential for a stone in Section 8.4.

11.4 ■ EFFECTIVE POTENTIAL FOR LIGHT

Describe global motion of light at a glance.

The present section sets up an effective potential for a light orbit in order to visualize its r-component of motion simply and directly. Recall equation (21) in Section 8.4 that relates the r-motion of a stone to its effective potential:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(\frac{V_{\mathrm{L}}(r)}{m}\right)^2 \qquad \text{(stone)} \qquad (20)$$

The key idea of this equation is that the first term on the right is a constant of the stone's motion—independent of location—while the second term is a function of r—independent of the properties or motion of the stone. We defined the second term to be the effective potential for a stone.

To make similar predictions about the r-motion of light, we seek an equation with the same form as (20). To find this equation, square both sides of (17), rearrange the results, and multiply through by $(M/r)^2$ to obtain:

$$\left(\frac{M}{b}\right)^2 \left(1 - \frac{2M}{r}\right)^{-2} \left[1 \pm \left(\frac{2M}{r}\right)^{1/2} F(b,r)\right]^2 \left(\frac{dr}{dT}\right)^2 = \left(\frac{M}{b}\right)^2 F^2(b,r) \qquad (21)$$

On the left side of (21) we define the function

$$A^2(b,r) \equiv \left(\frac{M}{b}\right)^2 \left(1 - \frac{2M}{r}\right)^{-2} \left[1 \pm \left(\frac{2M}{r}\right)^{1/2} F(b,r)\right]^2 \qquad \text{(light)} \qquad (22)$$

and on the right side of (21) we substitute for $F^2(b,r)$ from (16).

$$\left(\frac{M}{b}\right)^2 F^2(b,r) = \frac{M^2}{b^2} - \frac{M^2}{b^2}\frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right) \qquad \text{(light)} \qquad (23)$$

effective potential for light

Substitute the left sides of (22) and (23) into (21) and write the result as:

Box 1. Use of the effective potential for a stone and for a light flash

Compare and contrast the forms and uses of effective potentials for a stone and for a light flash:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(\frac{V_L(r)}{m}\right)^2 \quad \text{(stone)} \quad (26)$$

$$A^2\left(\frac{dr}{dT}\right)^2 = \left(\frac{M}{b}\right)^2 - \left(\frac{V(r)}{m}\right)^2 \quad \text{(light)} \quad (27)$$

For a stone:

- V_L depends on both L and r.
- The turning point occurs where $V_L = \pm E$.
- $|E| < |V_L|$ is forbidden
- When $|E| \geq |V_L|$, equation (26) gives $|dr/d\tau|$ in terms of r, L, E.

For a light flash:

- V depends on r alone.
- The turning point occurs where $V = \pm 1/b = \pm E/L$, not E alone.
- $|E| < |V|$ is forbidden
- When $|1/b| \geq |V|$, equation (27) gives $|dr/dT|$ in terms of r, b.

What's the difference between the two cases?

For light, L has been removed from the effective potential and combined with E; only $b = L/E$ remains. Impact parameter b can be taken completely out of the effective potential, so V depends only on r. This makes orbits of light *simpler* than orbits of a stone. Only *one* constant of motion is needed, not two.

$$A^2(b,r)\left(\frac{dr}{dT}\right)^2 = \left(\frac{M}{b}\right)^2 - \left(\frac{V(r)}{M}\right)^2 \quad \text{(light)} \quad (24)$$

where (25) defines the square of the **effective potential for light**

$$\left(\frac{V(r)}{M}\right)^2 \equiv \frac{M^2}{r^2}\left(1 - \frac{2M}{r}\right) \quad \text{(light)} \quad (25)$$

Figure 3 plots positive values of the effective potential for light. In Query 2 you show that the coefficient $A^2(b,r)$ in equation (22) is well behaved when light descends to the event horizon, provided $b \neq 0$.

Box 1 compares and contrasts effective potentials for light and for stones.

QUERY 2. Approaching the event horizon

What happens to the left side of (24) as $r/M \to 2^+$, that is as light approaches the event horizon from above? Just above the event horizon set $r/M = 2(1+\epsilon)$ where $0 < \epsilon \ll 1$ and use our standard approximation (inside the front cover) to show that coefficient $A^2(b,r)$ in (24) is well behaved even as light descends to the event horizon, provided $b \neq 0$.

Quick predictions with the effective potential

With the effective potential we can predict—at a glance—the r-component of light motion. The first term, $(M/b)^2$, on the right side of (24) is a constant of motion, the same everywhere along the orbit. The second term is a function of r and does not include b. Figure 3 and its caption also contain a preview of *turning points*, which we analyze more fully in Section 11.4.

Same effective potential for light of EVERY energy (EVERY wavelength)

Huge payoff: The right side of (24) does not include the energy or angular momentum of light. *One effective potential applies to light orbits of every energy and every angular momentum.* In particular, it applies to electromagnetic radiation of all wavelengths: radio waves; microwaves; infrared, visible, and ultraviolet light; X-rays; and gamma rays! (This result assumes that the wavelength of light is small compared with the coordinate separations over which spacetime curvature changes appreciably.)

QUERY 3. Critical impact parameter

A. Show that the peak of the effective potential occurs at $r/M = 3$.

B. Verify that the so-called **critical value** of the impact parameter at $r/M = 3$ is

$$\frac{b_{\text{critical}}}{M} = (27)^{1/2} = 5.196\ 152\ 42 \qquad \text{(light, critical impact parameter)} \qquad (28)$$

C. From Figure 3 read off approximate values of b/M and r/M for the circular orbit. Compare these values with the analytic results of Items A and B.

Effective potentials reveals turning points.

Both the effective potential for light and effective potentials for stones enable us to find the r-coordinate at which the r-component of motion goes to zero, which occurs for a circular orbit and also at what we call a *turning point* (Section 8.4 and Section 11.5).

DEFINITION 5. Plunge Orbit, Bounce Orbit, Trapped Orbit

Figure 3 sorts all light orbits near a black hole into three categories, which we give names to simplify our analysis:

- **Plunge Orbit:** A plunge orbit is an incoming or outgoing orbit with $|b| < b_{\text{critical}}$ that passes above the peak of the effective potential curve in Figure 3. A starlight Plunge Orbit is—by definition—an incoming orbit that *plunges* through the event horizon to the singularity. Outside the event horizon light can, in principle, move in either direction along the plunge orbit shown. We call this a plunge orbit, whether r decreases or increases.
- **Bounce Orbit:** A bounce orbit is an incoming or outgoing orbit with $|b| > b_{\text{critical}}$. The bounce orbit exists only to the right of the effective potential in Figure 3 and below its peak. A starlight Bounce Orbit is—by definition—an orbit that initially moves inward, then reverses its r-component of motion—its r-coordinate *bounces*—at a turning point on the outer edge of the effective potential, while its ϕ-component of motion continues. After the bounce, the light moves outward on the same horizontal line in the figure, and escapes to infinity. A Bounce Orbit cannot reach the singularity.

Definitions:
Plunge Orbit
Bounce Obit
Trapped Orbit

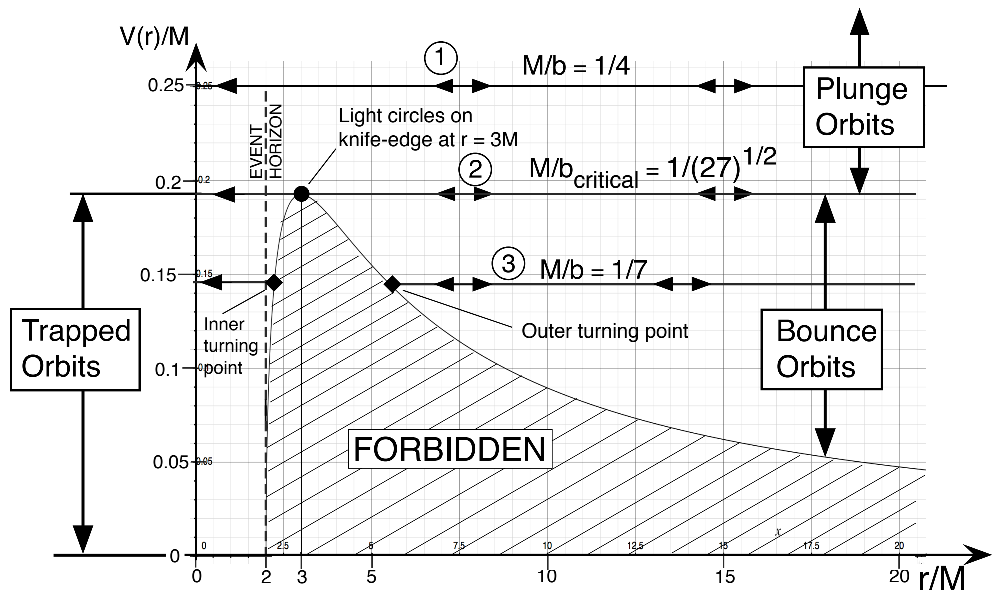

FIGURE 3 Examples of the three categories into which we sort all orbits (Definition 5). *Horizontal Line (1):* a Plunge Orbit with $M/b = 1/4$ that enters the black hole. *Horizontal Line (2):* the orbit with $M/b_{\rm critical} = 1/(27)^{1/2}$ that reaches the peak of the effective potential—marked with a little filled circle—and enters an unstable circular orbit there. *Horizontal Line (3):* a Bounce Orbit with $M/b = 1/7$ approaches the black hole, reverses its r-motion at the *outer turning point* (Section 11.6), and moves away from the black hole. The Trapped Orbit with $M/b = 1/7$ originates in the narrow horizontal region between the event horizon and the effective potential curve and moves inward through the event horizon.

- **Trapped Orbit:** A trapped orbit is an orbit with $|b| > b_{\rm critical}$ to the left of the effective potential in Figure 3 and below its peak. No starlight orbit can be a Trapped Orbit. An initially outgoing Trapped Orbit outside the event horizon reverses its r-component of motion at the inner turning point on the inner edge of the effective potential. *Every* Trapped Orbit reaches the singularity unless intercepted.

The horizontal line for $M/b_{\rm critical}$ in Figure 3 is the dividing line between these different categories of orbits. Figure 4 shows Plunge and Bounce Orbits; Figure 5 shows two Trapped Orbits.

11.5 ■ TURNING POINTS

The r-motion of light can reverse at a turning point.

At a turning point the r-component of motion goes to zero, while the ϕ-component of motion continues. Little filled squares in Figures 3 through 5 mark what we call **outer and inner turning points**.

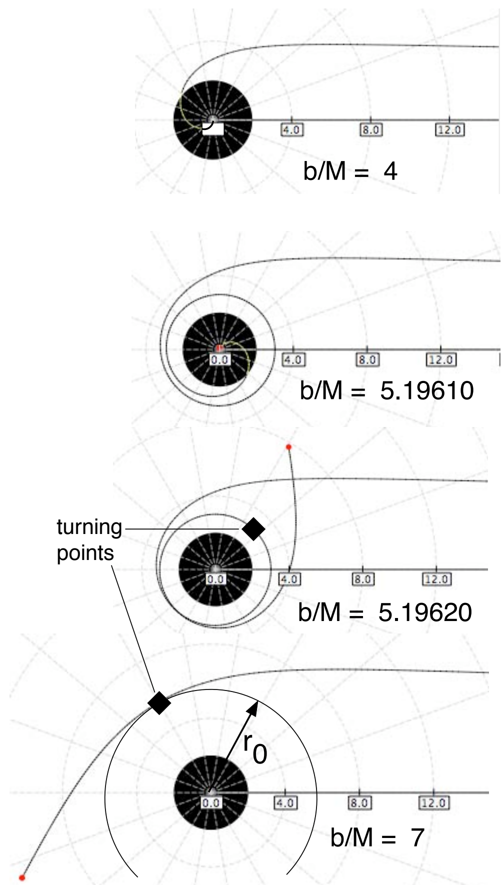

FIGURE 4 Top two panels: Plunge Orbits. Bottom two panels: Bounce Orbits, each with a little filled square at the turning point (Section 11.4). Middle two panels: b-values straddle $b_{\text{critical}}/M = 5.19615...$, for which the orbit enters a knife-edge circular orbit.

DEFINITION 6. Turning Point

A turning point is the r-value at which the right side of equation (24) equals zero, where M/b equals the value of the effective potential.

Definitions:
Turning point
Outer turning point
Inner turning point
Circular orbit poin

- An **outer turning point** is to the right and below the peak of the effective potential (see Figure 3).
- An **inner turning point** is to the left and below this peak.The peak itself is the location of the unstable (knife-edge) circular orbit of light.
- A **circular orbit point** is the r-value at which the effective potential is maximum. This is the r-location of an unstable (knife-edge) circular orbit for light.

Turning point subscript: tp

We use the subscript **tp** to label the r-coordinate of a turning point. *Example:* In Figure 3, Orbit 3 with $|b/M| = 7$ reverses its r-motion at

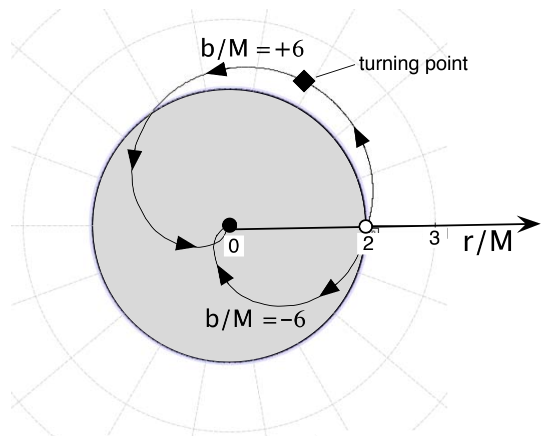

FIGURE 5 Two Trapped Orbits that originate from the same point just outside the event horizon at $r/M = 2^+$ (little open circle). One orbit has $b/M = +6$ with an inner turning point (little filled square); the other has $b/M = -6$ and no turning point. Both orbits reach the singularity at $r/M = 0$. Figure 6 adds labels to this plot.

$r_{\rm tp} = 5.617M$. Any outgoing light with $|b/M| = 7$ that arrives at the inner turning point at $r_{\rm tp,\,inner} = 2.225M$ thereafter moves with $dr < 0$ and enters the black hole.

Equations (24) and (25) tell us that the turning point $r_{\rm tp}$, the r-coordinate at which $dr/dT = 0$ and motion is purely tangential, occurs for the value of b given by:

$$b/M = \pm \frac{r_{\rm tp}/M}{\left(1 - \dfrac{2M}{r_{\rm tp}}\right)^{1/2}} \qquad \text{(given } r_{\rm tp}, \text{ find } b) \qquad (29)$$

Turning points only for $b^2 > b^2_{\rm critical}$

Comment 2. No turning point inside the event horizon
Equation (29) guarantees that there can be no turning point for light inside the event horizon, because b/M on the left side is necessarily a real quantity, while the right side of (29) is imaginary for $r_{\rm tp} < 2M$.

Derive $r_{\rm tp}$ from b.

Equation (29) gives us the value of b when we know the r-coordinate $r_{\rm tp}$ of the turning point. More often, we know the value of b and want to find the r-coordinate of the turning point. In that case, convert (29) into a cubic equation in $r_{\rm tp}$:

$$r^3_{\rm tp} - b^2 r_{\rm tp} + 2Mb^2 = 0 \qquad \text{(given } b, \text{ find } r_{\rm tp}) \qquad (30)$$

QUERY 4. *Optional:* Some consequences of turning points.

A. From equations (24) and (25) show that a light orbit with a given value of b cannot exist in a range of r-coordinates determined by the following inequality:

$$r^3 - b^2 r + 2Mb^2 < 0 \qquad \text{(region with no light orbits)} \qquad (31)$$

B. Show that inequality (31) describes the shaded region under the effective potential curve in Figure 3. In other words, light cannot penetrate the effective potential curve.

Find the turning points

Equation (30) is cubic—includes a third power of $r_{\rm tp}$. Cubic equations can be difficult to solve. Here are analytic solutions of (30). The first two yield r values of the outer and inner turning points, respectively, such as those in Figure 3. In Query 4 you show that the third solution is real but negative, so cannot represent the always-positive map r-coordinate:

$$r_{\rm tp} = 3M\left[\frac{1}{2} - \cos\left(\psi - 120^\circ\right)\right]^{-1} \tag{32}$$

(Outer turning points lie at $r > 3M$.)

$$r_{\rm tp,\,inner} = 3M\left[\frac{1}{2} - \cos\left(\psi + 120^\circ\right)\right]^{-1} \tag{33}$$

(Inner turning points lie between $r/M = 2$ and $r/M = 3$.)

$$r_{\rm NO} = 3M\left[\frac{1}{2} - \cos\psi\right]^{-1} \tag{34}$$

(Yields negative r: not physical.)

For all three solutions, ψ depends on b as follows:

$$\psi \equiv \frac{1}{3}\arccos\left(\frac{54M^2}{b^2} - 1\right) \qquad (|b| \geq b_{\rm critical},\ 0 \leq \psi \leq \pi) \tag{35}$$

We take what is called the *principle value* of the $\arccos z$, that is the angle between 0 and π radians whose cosine is z. Recall that the magnitude of the cosine is never greater than one. Therefore turning points exist only when the arccos function (35) exists, that is when $b^2 \geq b^2_{\rm critical}$ or when the horizontal line for $(M/b)^2$ in Figure 3 is at or below the peak of the effective potential. This makes graphical, as well as analytic, sense.

QUERY 5. Unphysical third solution

Show that the third solution (34) yields a negative value for r, which cannot represent the non-negative r-coordinate.

QUERY 6. Examples of turning points

A. For the outer and inner turning points of the orbit with $|b/M| = 7$, derive the numerical values $r_{\rm tp} = 5.617M$ and $r_{\rm tp,\,inner} = 2.225M$. Use Figure 3 to verify these r-coordinates approximately.

B. Show that $F(b, r) = 0$ at the turning points.

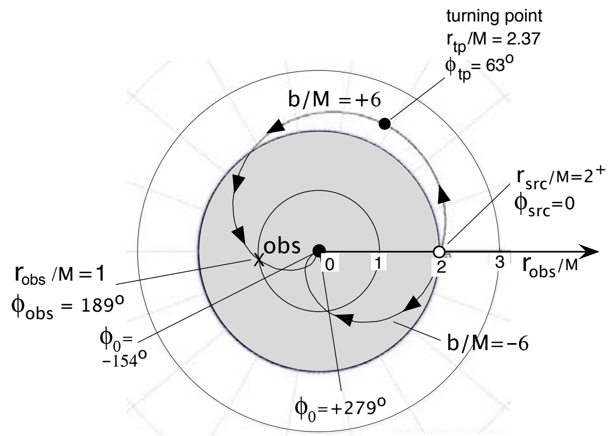

FIGURE 6 Elaboration of Figure 5. Two Trapped Orbits originate from just outside the event horizon at $r_{\rm src}/M = 2^+$, $\phi_{\rm src} = 0$. The counterclockwise orbit, with $b/M = +6$, rises to a turning point at $(r_{\rm tp}/M = 2.37, \phi_{\rm tp} = 63°)$, then falls back through the event horizon to arrive at the singularity at map angle $\phi_0 = +279°$. The clockwise orbit with $b/M = -6$ crosses the horizon immediately and reaches the singularity at the map angle $\phi_0 = -154°$. The event X locates a falling observer that intercepts the counterclockwise light orbit at $(r_{\rm obs}/M = 1, \phi_{\rm obs} = 189°)$.

C. An orbit with impact parameter $|b/M| \approx b_{\rm critical}/M = (27)^{1/2}$ circles at $r \approx 3M$ for a while. Then it "falls off the knife-edge," either spiraling inward or returning outward to $r/M \gg 1$. In the second case the turning r-coordinate is $r_{\rm tp}/M \approx 3$, but *where on that circle* is the turning point?

QUERY 7. Infinite impact parameter

A. From equation (29), find two different conditions that lead to $|b/M| \to \infty$.

B. In Figure 3, what horizontal line corresponds to $(M/b)^2 \to 0$ or $|b/M| \to \infty$? Point out two places on the graph (one a limiting case) where $(V(r)/M)^2$ reaches this line.

11.6 ■ STARLIGHT ORBIT: FROM STAR TO OBSERVER

Starlight orbit must reach me.

Which orbit(s) connect(s) the star with the observer?

Which light orbit(s) connect(s) a particular star to a given map location near the black hole? This question is important because sooner or later we want to predict in what direction one of the many possible inertial observers at that map location looks to see a particular star. But an observer cannot see light that does not reach him or her. The central goal of this chapter is to find the global path of an orbit that connects distant Star X to a given map location Y, whatever the motion may be of an observer at rest or moving through that location.

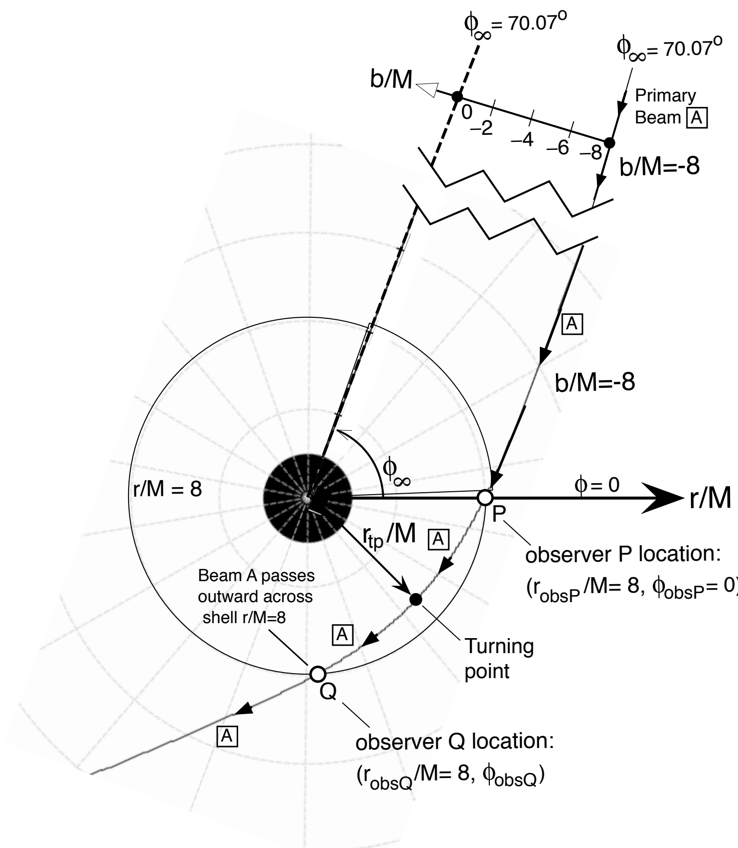

FIGURE 7 Starlight orbit A with impact parameter $b/M = -8$ moves in a clockwise direction to connect the star at map angle $\phi_\infty = 70.07°$ to observer P located at $(r_{\rm obsP}/M = 8, \phi_{\rm obsP} = 0)$. The starlight orbit proceeds to observer Q, crossing outward through the shell at the same $r_{\rm obsQ}/M = r_{\rm obsP}/M = 8$ but at a different value $\phi_{\rm obsP}$, to be determined.

?

Objection 2. *Ha, gotcha! You say that the observer can be at any coordinate $r_{\rm obs}$. But inside the event horizon nothing can stand still in global coordinates. Therefore you cannot have an observer at $r_{\rm obs} < 2M$.*

!

You are correct: No observer can remain constant r inside the event horizon. However Chapters 6, 7, and 12 describe the rain observer who starts from rest far from the black hole and drops to its center. This rain observer receives starlight even inside the event horizon. To predict the spectacular, ever-changing rain observer's pre-doom panoramas (Chapter 12), we must know which orbit(s) from every star reach(es) her there.

The orbit labeled A in Figure 7 connects a distant star to a point with map location $(r_{\rm obs}/M = 8, \phi_{\rm obs} = 0)$ where we will later place one of many possible

Map angle ϕ_∞ to a star

observers. This figure introduces the **map angle** ϕ_∞ of the distant star. The subscript infinity, ∞, reminds us that the star lies far from the black hole.

$$\phi_\infty \equiv \text{(map angle to a distant star, this angle measured counterclockwise from the direction } \phi = 0) \quad (36)$$

Primary orbit

Section 11.7 shows that many orbits—in principle an infinite number of orbits—from each star arrive at the map location of any observer. How do we choose which orbit to follow? *Answer:* We discover that there is a single most-direct orbit between star and observer, an orbit whose spatial path is the least deflected in map coordinates. We call this the **primary orbit** and give it most of our attention, often simply calling it "the orbit."

Primary orbit between star and map location of the observer

What primary orbit connects the star at given map angle ϕ_∞ most directly with the observer at map location $(r_{\rm obs}, \phi_{\rm obs} = 0)$? This is an important question with a complicated answer. So start with an example.

Figure 7 shows the interactive software GRorbits plot of a primary Bounce Orbit between a star at map angle $\phi_\infty = 70.07°$ and an observer at map location $(r_{\rm obs} = 8M, \phi_{\rm obs} = 0)$. *Result:* The orbit with impact parameter $b/M = -8$ connects this observer with the star at map angle $\phi_\infty = 70.07°$.

Incoming orbit may move out again across the same shell.

The incoming orbit in Figure 7 sweeps clockwise past the observer at $r/M = 8$, reaches a turning point at smaller r-coordinate, then crosses the $r/M = 8$ shell a second time, now in an outgoing direction. Two observers located at *different* points along the same shell can see the same orbit from the *same* star.

11.7 ■ INTEGRATE THE STARLIGHT ORBIT

An exact and immediate result

Goal: To plot $\phi_\infty - \phi_{\rm obs}$ for starlight

Our goal is to plot $\phi_\infty - \phi_{\rm obs}$ for starlight as a function of $r_{\rm obs}$ for a given value of the impact parameter b. To accomplish this, integrate $d\phi/dr$ directly. Figure 7 shows two cases. Case I: The orbit reaches the observer before the turning point. Case II: The obit reaches the observer after the turning point. Both cases integrate equation (19).

$$\phi_\infty - \phi_{\rm obs} = \int_{r=\infty}^{r_{\rm obs}} \frac{b}{r^2} F^{-1}(b,r)dr \quad (37)$$

(Case I: observer before turning point)

$$\phi_\infty - \phi_{\rm obs} = \int_{r=\infty}^{r_{\rm tp}} \frac{b}{r^2} F^{-1}(b,r)dr + \int_{r_{\rm tp}}^{r_{\rm obs}} \frac{b}{r^2} F^{-1}(b,r)dr \quad (38)$$

(Case II: observer after turning point)

Figure 8 displays the result of these integrals. The vertical axis "unrolls" the ϕ-angle.

Objection 3. *How do you carry out these integrals? Function $F(b, r)$ in (16) is complicated; these integrations must be difficult.*

Modern numerical methods evaluate these integrals to high accuracy. We do not pause here to describe these methods.

Plunge Orbit has small $|b|$. Bounce Orbit has large $|b|$.

Figure 3 previewed the summary message of Figure 8: An incoming orbit with small magnitude of $|b|$ plunges through the event horizon to the singularity. An incoming orbit with a large magnitude of $|b|$ deflects and returns outward again. An incoming orbit with the particular intermediate value $\pm b_{\text{critical}}$ circles temporarily at $r = 3M$, then either continues ingoing or becomes outgoing.

Objection 4. *You are not telling us the whole story! Orbits in most figures of this chapter have arrows on them. Every arrow tells us the direction of motion of light at that place along the orbit. But motion involves increments in the T-coordinate. Your equations that lead to these figures do not contain global T. Therefore these equations can give us only the curves themselves, without arrows.*

Yes and no. Equation (5) defines b as L/E, so the sign of the impact parameter is the same as the sign of L. This means that the motion of light is counterclockwise for positive values of b and clockwise for negative values. So equations (38) and (39) do give us the directions of motion (arrow directions) simply from the signs of b/M in those equations. Indeed, these equations do not tell us the map position of each light flash as a function of the T-coordinate. But we are interested in the plot of a steady starlight orbit, which does not vary with T.

Sample Problems 2 illustrate uses of Figure 8.

Comment 3. Every black hole redirects to every observer multiple orbits from every star.
You can use Figure 8 to find the value b of an orbit that connects *any* distant star $(-180° < \phi_\infty \leq +180°)$ to a map location on *some* circle of *any* r-coordinate around the black hole. Whoa! Does this mean that the black hole never obscures any star in the heavens for an observer near it? Yes, and more: The following section and Figure 10 show that every black hole in the visible Universe redirects multiple orbits from every single star in the heavens to an observer at every single map location.

11.8 ■ MULTIPLE STARLIGHT ORBITS FROM EVERY STAR

An infinite number of orbits that appear fainter and fainter to an observer.

One star: Infinite images?

It is remarkable that every map location near a black hole receives multiple orbits—in principle an infinite number of orbits—from a single star, and thus

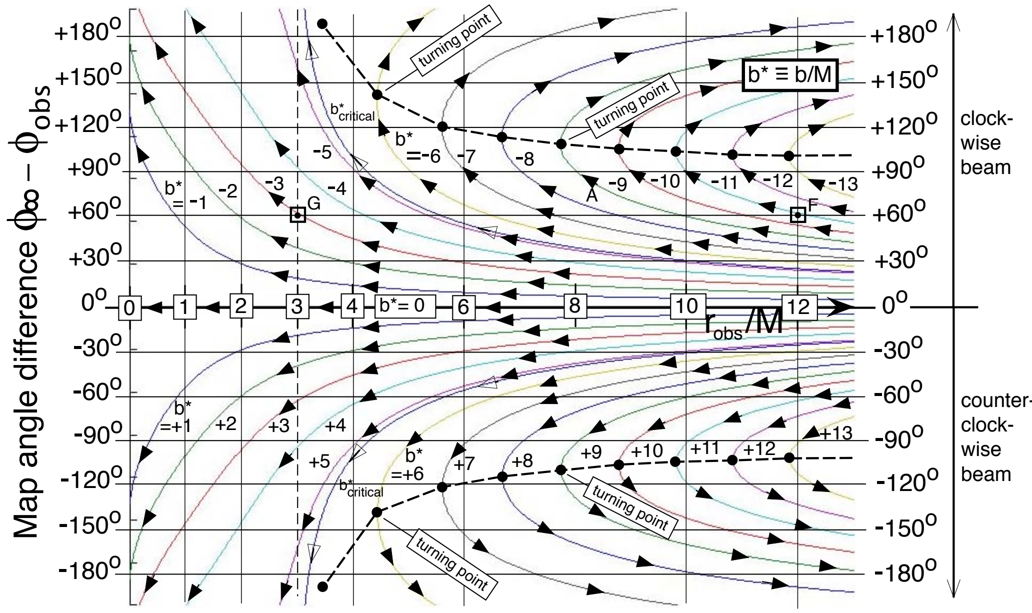

FIGURE 8 Difference in map angles between a distant star and the observer at map location $(r_{obs}/M, \phi_{obs})$ derived for an orbit of impact parameter b/M from that star. To reduce clutter, we define $b^* \equiv b/M$. Arrows on the curves tell whether the starlight is incoming or outgoing; at a turning point the orbit changes from incoming to outgoing.

from *every* star in the heavens. Figure 9 replots the primary orbit of Figure 7 and adds two additional orbits, called **higher-order orbits** from the same star. By trial and error, the interactive software program GRorbits finds values $b/M = +5.4600$ and $b/M = -5.2180$ for these additional orbits from the same star.

Higher-order orbits

In Figure 9, the higher-order orbit with $b/M = +5.4600$ moves around the black hole counterclockwise and approaches the map location $(r/M = 8,\ \phi = 0)$ from below. This orbit lacks $70.07°$ of making a complete circuit around the black hole. Therefore the *total* angle to the same star is $\phi_\infty = -(360° - 70.07°) = -289.93°$.

The next higher-order orbit with $b/M = -5.2180$ moves around the black hole clockwise and approaches the map location $(r/M = 8,\ \phi = 0)$ from above. This orbit makes a complete circuit around the black hole, *plus* $70.07°$, for a total of $430.07°$. Therefore the *total* angle to the same star is $\phi_\infty = +(360° + 70.07°) = +430.07°$.

Each observer receives many orbits from every star.

Figure 10 extends the vertical scale of Figure 8 to show orbits with b-values close to the critical value that circle several times around the black hole before they either escape outward or plunge on inward. The upward and downward vertical scales in Figure 10 extend indefinitely, leading to more and more orbits with b-values on either side of $b_{\text{critical}}/M = (27)^{1/2} = 5.196152....$ *Conclusion:* An observer at each r-coordinate r_{obs} receives multiple orbits—in principle an infinite number of orbits—from every star in the heavens.

Sample Problems 1. Orbits that reach $r/M = 3$

Think of orbits with different b-values that reach the observer map location at $(r_{\rm obs}/M = 3, \phi_{\rm obs} = 0)$. Use Figure 8 to provide approximate answer the following questions.

A. What is the b-value of the orbit that comes from the star at map angle $\phi_\infty = +60°$? **Solution A:** Look at the vertical dashed line at $r_{\rm obs}/M = 3$. This line intersects with the horizontal line $\phi_\infty = +60°$ very close to the curve $b/M = -3$, at the point marked G. So this is the b-value of the Plunge Orbit that connects the star at map angle $\phi_\infty = +60°$ with the observer at $(r_{\rm obs}/M = 3, \phi_{\rm obs} = 0)$.

B. What is the b-value of the orbit that comes from the star at map angle $\phi_\infty = +90°$? **Solution B:** The vertical dashed line at $r_{\rm obs}/M = 3$ intersects the horizontal line $\phi_\infty = +90°$ very close to the Plunge Orbit $b/M = -4$.

C. What is the b-value of the orbit that comes from the star at map angle $\phi_\infty = +30°$? **Solution C:** The vertical dashed line $r_{\rm obs}/M = 3$ intersects with the horizontal line $\phi_\infty = +30°$ about six-tenths of the separation between the curves $b/M = -1$ and $b/M = -2$. Therefore the Plunge Orbit with $b \approx -1.6$ connects the star at map angle $\phi_\infty = +30°$ with the map location $(r_{\rm obs}/M = 3, \phi_{\rm obs} = 0)$.

D. What is the b-value of the orbit that comes from the star at negative map angle $\phi_\infty = -90°$? **Solution D:** The vertical dashed line $r_{\rm obs}/M = 3$ intersects the horizontal line $\phi_\infty = -90°$ very close to the curve $b/M = +4$. The positive b-value means that the orbit moves counterclockwise around the black hole.

E. an orbit comes from the opposite side of the black hole, at $\phi_\infty = 180°$. What is the b-value of this orbit? **Solution E:** Both $\phi_\infty = +180°$ and $\phi_\infty = -180°$ are map angles to a star on the other side of the black hole. The vertical dashed line $r_{\rm obs}/M = 3$ intersects the horizontal lines $\phi_\infty = \pm 180°$ approximately half way between $b/M = \pm 5$ and $b/M = \pm(27)^{1/2} = \pm 5.196$. Therefore the b-values of these two Plunge Orbits are approximately $b \approx \pm 5.1$. *Optional:* Sketch this orbit.

Sample Problems 2. Orbits from a single star that reach observers at different r-coordinates

Orbits with different b-values from the star at map angle $\phi_\infty = +60°$ reach observers at different r-coordinates along the line $\phi = 0$. What are these b-values at r-coordinates $r_{\rm obs}/M = 12, 8, 4, 2, \text{and } 1$? In each case say whether the orbit is a Plunge Orbit, a Bounce Orbit, or a Trapped Orbit.

Solution: All of the orbits are from a star; therefore none of them can be a Trapped Orbit. In Figure 8, look at the intersections of horizontal line $\phi_\infty = +60°$ with vertical lines at these different r-coordinates. We estimate the b-values to one decimal place.

- At $r_{\rm obs}/M = 12$, $b/M \approx -10.9$, the point marked F in the figure; a Bounce Orbit
- At $r_{\rm obs}/M = 8$, $b/M \approx -7.3$, a Bounce Orbit
- At $r_{\rm obs}/M = 4$, $b/M \approx -3.8$, a Plunge Orbit
- At $r_{\rm obs}/M = 2$, $b/M \approx -2.0$, a Plunge Orbit
- At $r_{\rm obs}/M = 1$, $b/M \approx -1.2$, a Plunge Orbit

Look at the little square white boxes on the vertical line at $r/M = 8$ in Figure 10. Three of the little white boxes on the vertical line at $r/M = 8$ correspond to the three starlight orbits displayed in Figure 9. Other little boxes represent more of the multiple higher-order orbits between this star and this observer. Each little box is offset vertically by $\pm 360°$ from its nearest neighbor.

QUERY 8. *Optional:* Classify primary and higher-order orbits from a star.

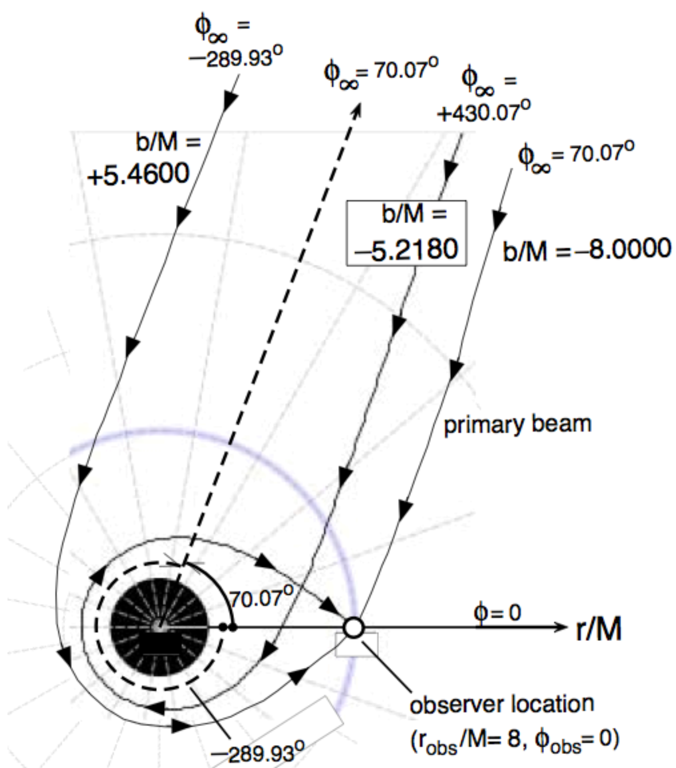

FIGURE 9 Three of the infinite number of orbits of light that, in principle, arrive at the same observer from a single star. For the primary orbit with $b/M = -8$, the star angle is $\phi_\infty = 70.07^\circ$ (as in Figure 7). For the second orbit, with $b/M = +5.4600$, the star angle (dashed arc) is $\phi_\infty = -(360^\circ - 70.07^\circ) = -289.93^\circ$. For the third orbit, with $b/M = -5.2180$, the star angle (angle-arc not shown) is $\phi_\infty = (360^\circ + 70.07^\circ) = +430.07^\circ$. All three orbits come from the same star, but the observer sees three different images in three different directions.

Classify the primary and higher-order starlight orbit as a Plunge Orbit or a Bounce Orbit. Figure 10 may be useful. *Reminder:* This analysis says nothing about the state of motion of the observer at that map location: he may be at rest there; she may dive or orbit past that map location.

A. Show that for every observer inside $r/M = 3$, *all* starlight orbits are Plunge Orbits.
B. Show that for every observer outside $r/M = 3$, starlight orbits are either Plunge Orbits or Bounce Orbits.
C. At any $r/M > 3$, what is the value of b/M that divides Plunge Orbits from Bounce Orbits?
D. Find an equation for the *maximum* magnitude of the impact parameter b/M of a Bounce Orbit that an observer on the shell of a given r-coordinate $r/M > 3$ can see?
E. Show that for every observer at $r/M > 3$, every higher-order orbit is an outgoing Bounce Orbit.
F. Can a primary or higher-order starlight orbit be a Trapped Orbit? Explain your answer.

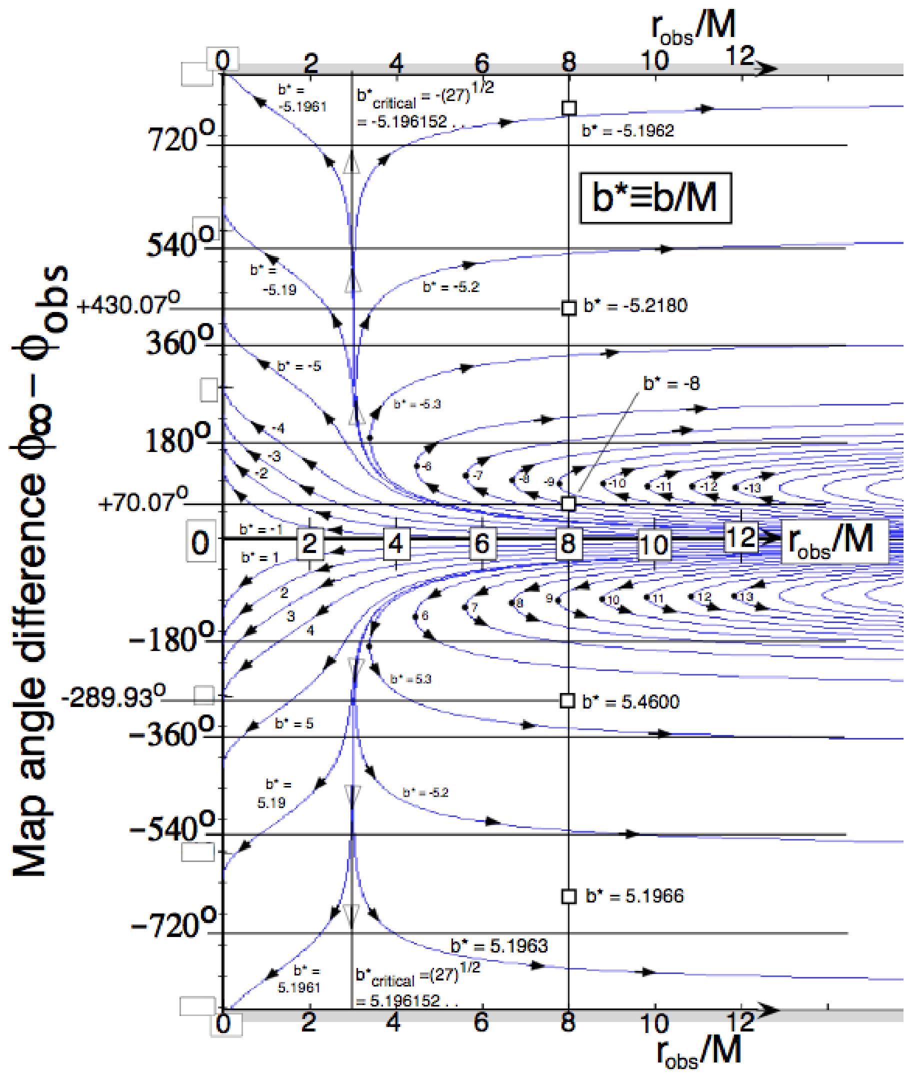

FIGURE 10 Expanded vertical scale for starlight orbits of Figure 8. The observer is at map location $(r_{\text{obs}}/M, \phi_{\text{obs}})$. *New feature of this plot:* Orbits with $b^* \approx \pm b_{\text{critical}}/M$ follow the vertical line at $r/M = 3$ (they circulate at $r/M = 3$) before they either return to $r/M \gg 1$ or plunge into the black hole. *Result:* Multiple orbits—in principle an infinite number of orbits—from every star arrive at each observer, cross every possible vertical line in the figure. *Example:* Three of the little white boxes on the vertical line at $r/M = 8$ correspond to the three starlight orbits displayed in Figure 9.

Higher-order orbits have fainter, smeared images.

Higher-order orbits that go around the black hole more and more times are less and less intense when they arrive at the observer. There is always *some* spread in the orbit, so the more times an orbit circles the black hole, the more it spreads out transverse to its direction of motion and the smaller the

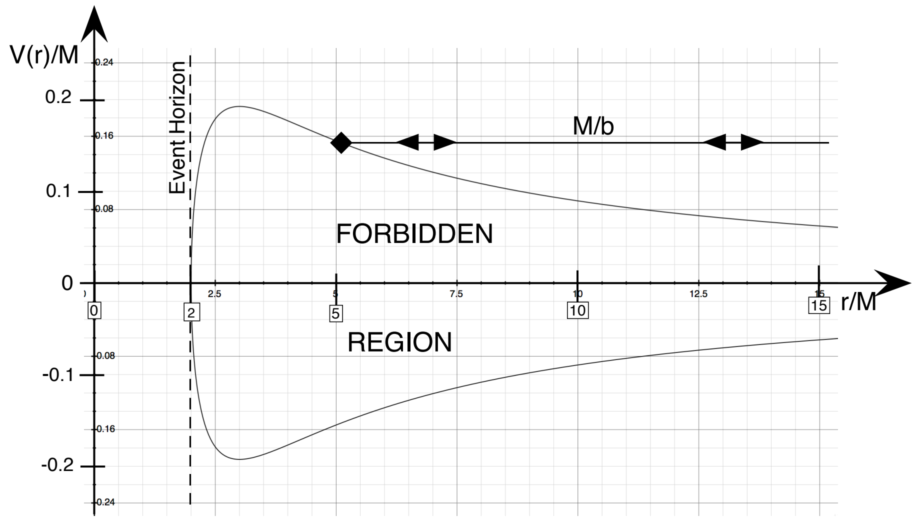

FIGURE 11 Forbidden region for light. Near the non-spinning black hole, this forbidden region separates our world, above the forbidden region, from another world, below the forbidden region.

fraction of photons in the initial orbit that enter the detector at the final map location. Chapter 12 shows that the shell observer also sees higher-order orbits bunched closer and closer together in the observed direction. *Overall result:* Higher and higher order orbits lead to images that get fainter and fainter and smear into one another. As a result, an observer sees separately only a few of the infinite number of orbits that, in principle, arrive from each star.

Two worlds, separated for the non-spinning black hole

Strange results follow from equation (24), which expresses $(dr/dT)^2$ in terms of the difference $(M/b)^2 - (V(r)/M)^2$. Differentials dr and dT are both real, so dr/dT must be real. In other words $(dr/dT)^2$ must be positive. *Conclusion:* $(M/b)^2 - (V(r)/M)^2$ must be positive. A consequence of this condition is that either $M/b > +V(r)/M$ or $M/b < -V(r)/M$. The result is a forbidden region where light cannot exist, as shown in Figure 11. Compare corresponding Figure 5 in Section 8.4 for the stone and review the text that accompanies that figure. Near the black hole the forbidden region for light separates our world (above the forbidden region) from another world (below the forbidden region). We can move between these worlds only by entering and then exiting the event horizon—not possible for a non-spinning black hole. However, we will find that for the spinning black hole a trip from the corresponding upper region to the corresponding lower region may be possible. John Archibald Wheeler's radical conservatism says,"Follow the equations wherever they lead, no matter how strange the result."

11.9 ■ EXERCISES

Note: In the exercises the word *approximately* means that the requested number may be estimated from a figure in this chapter.

1. Thought question: Shadow of a Black Hole?

According to legend, a vampire has no reflection in a mirror and casts no shadow. When illuminated from one side by a distant incoming flat wave, does a black hole cast a shadow on the other side? Think of a possible shadow on a flat plane located far away from the black hole where spacetime is flat.

2. Values of b for orbits that arrive at $r_{\rm obs}/M = 6$.

Repeat parts A through E of Sample Problems 2 for orbits that reach the observer at map location $(r_{\rm obs}/M = 6, \phi_{\rm obs} = 0)$. Classify each orbit as incoming, outgoing, or tangential.

3. Orbits that reach observers at different r-coordinates from the star at map angle $\phi_\infty = -120°$.

Repeat Sample Problems 2 for a star at map angle $\phi_\infty = -120°$.

4. The visual size of a black hole

Figure 10 shows the b-values of beams that escape or are captured by the black hole. The smallest b-value of a beam that can escape is $|b_{\rm critical}| = (27)^{1/2}M$. Some light from every star circles temporarily on this unstable orbit at $r = 3M$. Because this is a knife-edge orbit, it continually sheds light beams that "fall off" to move either inward or outward.

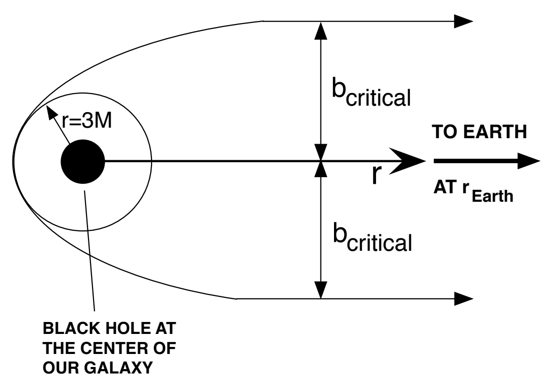

FIGURE 12 Schematic diagram showing the visual size of the black hole Sagittarius A* located at the center of our galaxy, assumed (incorrectly) to be non-spinning. The text shows that all possible parallel straight beams form a three-dimensional cylinder directed toward the observer on Earth.

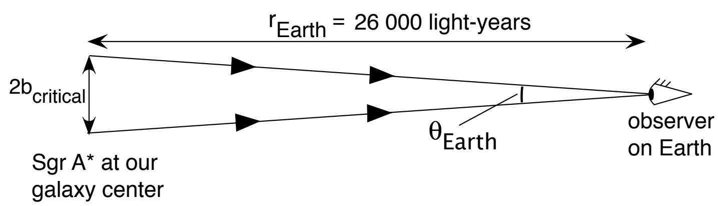

FIGURE 13 Critical beams from Sgr A* form a long cone as seen from Earth

Consider outward light beams that enter the eye of a distant observer on Earth. Figure 12 shows two such beams on one $[r,\phi]$ slice through the center of the black hole. But the same distant observer sees a similar pair of beams that lie on each of an infinite number of similar slices rotated around the r-axis in Figure 12. The resulting set of beams form a cylinder observed by the Earth observer.

To speak more carefully, the beams we see on Earth do not move *exactly* on a cylinder, but rather on a very long cone with its apex at the Earth (Figure 13). As a result, we on Earth see the black hole as a ring. What angle does this ring subtend at our eye on Earth?

Answer this question for the monster black hole called Sagittarius A* (abbreviation: SgrA*) with mass $M_{\text{SgrA}} \approx 4\times 10^6 M_{\text{Sun}}$ that lies at the center of our galaxy, about 26 000 light-year from Earth. Label this distance r_{Earth}. Assume (incorrectly) that SgrA* is a nonspinning black hole. Derive and justify an expression for the angular size θ_{Earth} of this black hole observed from Earth. (An exercise in Chapter 20 carries out a more realistic analysis that takes account of the spin of this black hole.)

A. From Figure 13, derive the following expression for the very small angle θ_{Earth}.

$$\theta_{\text{Earth}} \approx \frac{2(27)^{1/2} M_{\text{SgrA}}}{r_{\text{Earth}}} \qquad (r \gg M_{\text{SgrA}}) \qquad (39)$$

B. Insert into (39) values for M_{SgrA} and Earth's r-coordinate separation from the black hole of r_{Earth} light years. The following are results to one significant digit. Find each result to two significant digits:

$$\begin{aligned}\theta_{\text{Earth}} &\approx 2\times 10^{-10} && \text{radian} \qquad (40)\\ &\approx 1\times 10^{-8} && \text{degree}\\ &\approx 5\times 10^{-5} && \text{arcsecond}\\ &\approx 50 && \text{microarcseconds}\end{aligned}$$

Comment 4. Microwaves, not visible light
Dust between Earth and the spinning black hole at the center of our galaxy absorbs visible light. Microwaves pass through this dust, so our detectors on Earth are microwave dishes distributed over the surface of Earth.

5. The "incoming map floodlight"

Define an **incoming map floodlight** as a lamp at a given r-coordinate $r_{\rm inlamp}$ that emits all light beams that are ingoing at that r—that is, all beams with a negative r-coordinate differential, $dr < 0$.

A. An incoming map floodlight at $r_{\rm inlamp}/M = 12$ emits light that might have come from stars with approximately what range of map angles ϕ_∞?
B. An incoming map floodlight at $r_{\rm inlamp}/M = 6$ emits light that might have come from stars with approximately what range of map angles ϕ_∞?
C. An incoming map floodlight at $r_{\rm inlamp}/M = 3$ emits light that may have come from stars with approximately what range of map angles ϕ_∞?
D. An incoming map floodlight at $r_{\rm inlamp}/M = 1$ emits light that may have come from stars with approximately what range of map angles ϕ_∞?
E. Can the incoming map floodlight at $r_{\rm inlamp}/M = 6$ be at rest in global coordinates? Can the incoming map floodlight at $r_{\rm inlamp}/M = 1$ be at rest in global coordinates?

6. The "outgoing map floodlight"

Define an **outgoing map floodlight** as a lamp at a given r-coordinate, $r_{\rm outlamp}$, that emits all light beams that are outgoing at that r-coordinate—that is, all beams with a positive r-coordinate differential, $dr > 0$.

A. An outgoing map floodlight at $r_{\rm outlamp}/M = 8$ emits light that might have come from stars with approximately what range of map angles ϕ_∞?
B. An outgoing map floodlight at $r_{\rm outlamp}/M = 5$ emits light that may have come from stars with approximately what range of map angles ϕ_∞?
C. An outgoing map floodlight at $r_{\rm outlamp}/M = 3$ emits light that may have come from stars with approximately what range of map angles ϕ_∞?
D. Is there a range of r-coordinates in which the outgoing map floodlight is useless? *Hint:* look at Figure 10.

7. Newton's plot of map angle difference.

Make a *rough* sketch (don't sweat the details) of Figure 8 for orbits of light in Newtonian mechanics, in which spacetime is flat around the center of attraction and light is fast particle. What "Newtonian assumptions" do you make about the path of light under this attraction? (We have no record that Newton himself made any prediction about the effect of his "gravitational force" on the orbits of light.)

11.10 ■ REFERENCES

Initial quotes:

Egyptian creation quote from M=http:/www.aldokkan.com/religion/creation.htm/=

Tuamotuan creation quote from *The Myths of Creation* by Charles H. Long, George Braziller, New York, 1963, pages 173 and 179.

Inuit creation quote from *The Power of Stars—How Celestial Observations Have Shaped Civilization* by Bryan E. Penprasem, New York, Springer 2011, page 97.

The interactive GRorbits program that plots orbits of light is available at website http://stuleja.org/grorbits/