SUPPLEE'S PARADOX

Relativistic buoyancy

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Because of relativistic length concentration, the buoyant force on a submerged projectile depends on its velocity. How this affects the motion of a submerged projectile is considered here. The case considered is highly idealized, yielding a tractable conceptual problem in relativity.

grav. kiiktatása [hogy csak SR legyen használható]: ekvivalencia-elvvel

I. INTRODUCTION

Consider this relativistic situation: A certain bullet has rest density equal to the density of water (ρ_0). The bullet would therefore be neutrally buoyant if it were at rest in water. But here the bullet is fired horizontally through water, so that its energy/c^2 (sometimes called "relativistic mass") is increased by the Lorentz factor γ, and its length in the direction of motion (hence its volume) is contracted by $1/\gamma$. This makes the bullet's density $\gamma^2\rho_0$, and thus it is denser than water and sinks. (This article addresses a conceptual exercise in relativity and idealizes the problem by ignoring viscosity and the wake.)

What makes this problem interesting is that at first glance there seems to be the following paradox: Changing to the inertial frame in which the bullet's initial speed is zero results in the bullet having its rest density and the now moving water having density $\gamma^2\rho_0$, and thus the bullet floats instead of sinking. Actually, there can be no paradox, and calculations in the two frames agree, as is shown below.

NINCS GRAVIT.!

II. UNPRIMED INERTIAL FRAME

For convenience, picture a rectangular lake (Fig. 1). Evoking the equivalence principle, the lake is taken as accelerating upward with acceleration g. The frame is a local frame; all distances relevant to this problem are much less than c^2/g. Using the equivalence principle (in Secs. II and III) allows for completing the calculation with no reference to gravitational forces. The coordinates in Fig. 1 use the "unprimed" inertial frame in which the lake was instantaneously at rest when the bullet was fired; the origin is coincident with the lower left corner of the lake at $t = 0$.

a sűrűsége ezért ρ_0 marad

In the unprimed frame, the buoyant force on the projectile is the mass of the displaced water times g,

$$f_b = (V_0/\gamma)\rho_0 g = m_0 g/\gamma, \tag{1}$$

where V_0 and m_0 are the bullet's rest volume and rest mass, respectively. The y component of Newton's second law is

$$f_y = \frac{d}{dt}(\gamma m_0 u_y), \tag{2}$$

$$f_y = m_0(\dot{\gamma} u_y + \gamma \dot{u}_y),$$

where $\mathbf{u}$ is the bullet's velocity, f_y is the vertical component of the usual three-force,[1,2] and the dot denotes differentiation with respect to time. For constant u_x, the derivative of the Lorentz factor is

$$\dot{\gamma} = (\gamma^3/c^2)u_y\dot{u}_y. \quad \text{(ld. hátul)} \tag{3}$$

Using Eq. (3) in the second of Eqs. (2) and rearranging yields

$$f_y = m_0\gamma\dot{u}_y\left(\frac{1-(u_x/c)^2}{1-(u_x/c)^2-(u_y/c)^2}\right) \approx m_0\gamma\dot{u}_y, \tag{4}$$

(ld. hátul)

where the approximation holds well since[3] $u_y \ll c$. It is interesting that Eq. (4) is just what would have resulted from ignoring $\dot{\gamma} u_y$ (the product of two very small terms) in Eq. (2). But that leap would have been inappropriate at that time since the size of du_y/dt (which is what is ultimately sought here) was unknown.

Combining Eqs. (1) and (4) yields

$$\boxed{\frac{du_y}{dt} = \frac{g}{\gamma^2} = g\left(1 - \frac{u^2}{c^2}\right)} \tag{5}$$

This upward acceleration of the projectile is less than the upward acceleration of the lake g so the projectile "sinks" with relative acceleration,

$$A = g - g/\gamma^2 = g\beta^2, \tag{6}$$

where $\beta \equiv v/c$. The bullet therefore strikes the bottom at time[3]

$$t = \sqrt{2h/A} = (1/\beta)\sqrt{2h/g} \tag{7}$$

and travels a total horizontal distance

$$x = vt = (v/\beta)\sqrt{2h/g} = c\sqrt{2h/g}. \tag{8}$$

III. PRIMED INERTIAL FRAME

Now, reconsider the entire problem from the inertial frame in which the initial speed of the projectile is zero (the primed frame). Choose the origin so that $(x',y',z',t') = (0,0,0,0)$ is coincident with $(x,y,z,t) = (0,0,0,0)$. In this frame, the upward acceleration of any fixed point (constant x) on the lake floor (for example, the lower left corner of the lake) is

$$a' = g/\gamma^2. \quad \text{(ld. hátul)} \tag{9}$$

Equation (9) can be convincingly derived by considering two briefly separated events at the lower left corner of the lake, Lorentz transforming these events and the lake corner's velocity at these events to the primed frame, and then simply using the definition of acceleration (see, e.g., Ref. 4). Equation (9) can also be obtained from the requirement that the y component of the four-acceleration[5]

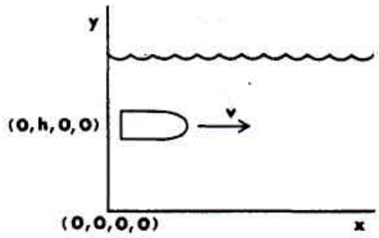

Fig. 1. Lake and bullet in the unprimed frame.

of the lake corner be invariant under the standard Lorentz transformation.

In the primed frame, the water's density is

$$\rho' = \gamma^2 \rho_0. \tag{10}$$

The buoyant force on the projectile is

$$f_b' = \rho' V' a'. \tag{11}$$

Noting that the projectile has its rest volume[3] in the primed frame and using Eqs. (9) and (10), Eq. (11) becomes

$$f_b' = m_0 g. \tag{12}$$

Comparing Eqs. (1) and (12) verifies that this three-force has Lorentz transformed as it should.[6] (Readers preferring four-forces can equivalently verify that the y component of the four-force is invariant.) Using[3] $\gamma' = 1$ and using Eq. (12) in the primed equivalent of the first of Eqs. (2) yields

$$\frac{du_y'}{dt'} = g. \tag{13}$$

Note that the upward acceleration of the bullet (13) is greater than the upward acceleration of the corner of the lake (9); it still seems that the bullet floats in the primed frame.

This seeming paradox can be resolved by noting that in the primed frame the lake bottom is no longer flat. The equation of the lake floor is given by

$$y = \tfrac{1}{2} g t^2, \tag{14}$$

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which can be immediately Lorentz transformed to

$$y' = \tfrac{1}{2} g [\gamma(t' + vx'/c^2)]^2. \quad \boxed{\text{LAKE FLOOR}} \tag{15}$$

Now one can check whether, and if so where, the bullet strikes the lake floor: Standard kinematics[3] gives the bullet's location as

$$y_b' = h + \frac{1}{2}\left(\frac{du_y'}{dt'}\right) t'^2. \tag{16}$$

Using Eq. (13) in (16) gives

$$y_b' = h + \tfrac{1}{2} g t'^2. \quad \boxed{\text{BULLET}} \tag{17}$$

The impact event has

$$x' = 0 \tag{18}$$

(since the bullet always has $x' = 0$). And impact occurs when the y coordinates of the bullet and lake floor match. So the impact time is found by equating Eqs. (15) and (17) and solving for t' to obtain

$$t' = \sqrt{2h/g}\sqrt{1/(\gamma^2 - 1)} = (1/\gamma\beta)\sqrt{2h/g}. \tag{19}$$

Lorentz transforming Eqs. (18) and (19) to the unprimed frame does give Eqs. (8) and (7) as expected; the calculations in the two frames agree.

A second method for arriving at the same result while working in the primed frame can be stated concisely by contrasting it with the calculation above: Equation (9) above is the acceleration of the lake corner as measured in the primed frame. Since the lake corner has fixed x, Eq. (9) could be written

$$\left.\frac{\partial^2 y'}{\partial t'^2}\right|_{x = \text{const.}} = \frac{g}{\gamma^2}, \quad \text{(vgl. mit (9))} \tag{20}$$

as can be directly verified using Eq. (15) plus the Lorentz transformations. Since the bullet eventually strikes bottom at a different x, the curved nature of the lake bottom must be considered. In contrast, one can "stay with the bullet" by considering constant x'. Differentiating Eq. (15) then gives

$$\boxed{\left.\frac{\partial^2 y'}{\partial t'^2}\right|_{x' = \text{const.} = 0} = g\gamma^2.} \tag{21}$$

([illegible])

This upward acceleration of the lake floor is greater than the upward acceleration of the bullet [Eq. (13)], so the bullet sinks with relative acceleration

$$\Delta = g(\gamma^2 - 1). \tag{22}$$

Since this calculation is done with $x' = 0$, the curved bottom need not be addressed separately; simple kinematics yields

$$t' = \sqrt{2h/\Delta} = (1/\gamma\beta)\sqrt{2h/g}, \tag{23}$$

in immediate agreement with Eq. (19).

IV. USING GRAVITATIONAL FORCE

Since this problem is fraught with subtleties, it is appropriate to reconsider it using the gravitational force; the equivalence principle is not used in this section. The force acting on a particle in a constant gravitational field is[7]

$$\mathbf{f} = \gamma m_0 c^2 [-\nabla \ln\sqrt{g_{00}} + \sqrt{g_{00}}(\mathbf{v}/c) \times \text{curl}\,\mathbf{g}]. \tag{24}$$

For the weak uniform field of this problem

$$\text{curl}\,\mathbf{g} = 0, \tag{25}$$

$$g_{00} = 1 + 2\varphi/c^2, \tag{26}$$

and

$$\varphi = gy \tag{27}$$

will be used. Here, φ is the gravitational potential. The gradient operator is just the usual Cartesian operator,

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}. \tag{28}$$

Using Eqs. (25)–(28) in Eq. (24) yields

$$\mathbf{f} = -\mathbf{j}(\gamma m_0 g)/(1 + 2gy/c^2). \tag{29}$$

Here, $2gy/c^2 \ll 1$ so

$$\mathbf{f} = -\mathbf{j}\gamma m_0 g. \tag{30}$$

That is, gravity pulls down on the total energy/c^2, not just on the rest mass. One might have expected this since gravity pulls on photons.

The net force on the bullet is the buoyant force (weight of the water displaced) minus the gravitational force;

$$f_{\text{total}} = m_0 g/\gamma - \gamma m_0 g = -m_0 g(\gamma - 1/\gamma). \tag{31}$$

Setting this equal to the time rate of change of the vertical component of momentum yields

$$\frac{d}{dt}\left(\gamma m_0 \frac{dy}{dt}\right) = -m_0 g\left(\gamma - \frac{1}{\gamma}\right), \tag{32}$$

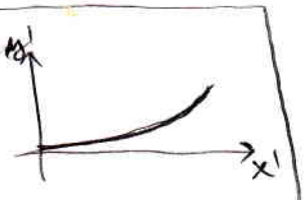
$$\frac{d^2 y}{dt^2} = -g\beta^2. \tag{33}$$

This agrees with Eq. (6); the bullet therefore strikes the bottom with the coordinates given by Eqs. (7) and (8) as before.

V. A COMMENT

This problem involves a number of subtleties and traps. It also includes some unrealistic idealizations. The author's communications with referees and colleagues suggests that there are varying points of view regarding this problem;

$t' = \text{konst}: y'(x')$: Parabel!



further consideration of some of the issues raised here could be interesting. A full treatment involving relativistic hydrodynamics[8] would be revealing, but probably difficult.

ACKNOWLEDGMENTS

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[1] See, for example, W. Rindler, *Essential Relativity* (Van Nostrand, New York, 1969), pp. 111–113.

[2] See, for example, A. P. French, *Special Relativity* (Norton, New York, 1986), pp. 214–219.

[3] Here and throughout the mathematics is simplified because all vertical velocity components are very much less than c.

[4] Reference 2, p. 154.

[5] See, for example, Ref. 1, p. 86.

[6] See, for example, Ref. 2, p. 217, Eq. (7-24). Or see W. P. Ganley, Am. J. Phys. **31**, 510 (1963).

[7] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), p. 253.

[8] See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), pp. 562–566.

Complex analysis and quantum mechanics: A perturbative approach for the evolution operator

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A simple way of obtaining an explicit expression for the perturbative expansion of the evolution operator to any order is shown.

In the following, I would like to show a simple method of calculating the quantum mechanical evolution operator. A usual problem occurring in actual calculations is the evaluation of transition rates, for which Dirac's procedure of "variation of constants"[1,2] is mainly adopted. Let us briefly recall it: If H_0 denotes the unperturbed Hamiltonian, $|k\rangle$ and E_k denote its eigenvectors and eigenvalues, respectively, and V denotes the perturbation suddenly starting at $t=0$, the eigenfunctions of the total Hamiltonian $H=H_0+V$ are expanded into eigenfunctions $|k\rangle$ of H_0, the amplitudes of which are time dependent; from the Schrödinger equation, a system of coupled differential equations is obtained for these amplitudes, and a perturbative solution is worked out. The explicit form of this solution, if the time interval $(0, t)$ goes to infinity, can be given to all orders.[3,4]

This method does not work so well if we search for a solution at a finite time t: Convolution theorems for Fourier transforms cannot be used and the multiple integrals occurring in higher-order terms become more and more cumbersome. We will see, however, that the calculation can be performed without difficulty by complex analysis methods. A very beautiful application of these methods to quantum mechanics was given by Merzbacher,[5] who, starting from a generalized Cauchy's formula, recovered in a very elegant and compact form explicit expressions for an arbitrary function of a matrix (until then, this subject was treated only in specialized mathematical publications). Therefore, the problem of the temporal evolution of a physical system, described by a finite number of basis states, has become, in principle, a straightforward one.

The same method can be used in a perturbative approach (the basic formulation is extensively discussed in the literature[5–7]); the generalized Cauchy's formula is written in the form

$$f(L)=(2\pi i)^{-1}\int_{\gamma_\infty} f(z)(z-L)^{-1}\,dz, \tag{1}$$

expressing the function of a general Hermitian operator L as a contour integral, and γ_∞ is a closed contour enclosing all the nonzero eigenvalues of L. The total wavefunction $\psi(t)$ is related to $\psi(0)$ by means of the evolution operator $U(t)$,[8]

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle, \tag{2}$$

with

$$U(t)=\exp[-(i/\hbar)Ht], \quad H=H_0+V,$$
$$H_0|k\rangle = E_k|k\rangle. \tag{3}$$

If $|\psi(0)\rangle = |m\rangle$, on the basis of the eigenvectors of H_0, we have

$$|\psi(t)\rangle = \sum_k a_k(t)|k\rangle, \quad a_k(t)=\langle k|U(t)|m\rangle. \tag{4}$$

In this case, Eq. (1) becomes

$$f(H_0+V)=(2\pi i)^{-1}\int_{\gamma_\infty} f(z)G(z)\,dz, \tag{5}$$

where

$$f(z)=\exp[-(i/\hbar)zt], \quad G(z)=[z-(H_0+V)]^{-1} \tag{6}$$

and γ_∞ encloses all the eigenvalues of (H_0+V) as well as the eigenvalues of H_0; setting

$$G_0(z)=[z-H_0]^{-1}, \tag{7}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{u_x^2}{c^2}-\frac{u_y^2}{c^2}}} \qquad u_x = \text{const.},\; u_y = u_y(t)$$

$$\boxed{\dot\gamma = \frac{-1}{\cancel{2}\left(1-\frac{u^2}{c^2}\right)^{3/2}}\cdot\left(-\cancel{2}\frac{u_y}{c^2}\right)\cdot \dot u_y = \frac{\gamma^3}{c^2}u_y\dot u_y} \qquad (3)$$

$$\boxed{f_y} = m_0\left(u_y^2\frac{\gamma^3}{c^2}\dot u_y + \gamma\dot u_y\right) = m_0\gamma\dot u_y\left(1+\frac{u_y^2}{c^2}\gamma^2\right) = m_0\gamma\dot u_y\left(1+\frac{\frac{u_y^2}{c^2}}{1-\frac{u^2}{c^2}}\right) =$$

$$\boxed{= m_0\gamma\dot u_y\left(\frac{1-\frac{u_x^2}{c^2}}{1-\frac{u^2}{c^2}}\right)} \qquad (4)$$

Lorentz:

$$u_y' = \frac{u_y\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{u_x v}{c^2}}$$

$(u_x = 0 = \text{konst.})$

$(u_y$ változik$)$

$$\boxed{a_y' = \frac{du_y'}{dt'}} = \frac{\frac{du_y\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{u_x v}{c^2}}}{\frac{dt - v\,dx/c^2}{\sqrt{1-\frac{v^2}{c^2}}}} = \frac{a_y\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}{\left(1-\cancel{\frac{u_x v}{c^2}}\right)^2} = \boxed{a_y\left(1-\frac{v^2}{c^2}\right)} =$$

$$= \boxed{g/\gamma^2} \qquad (9)$$