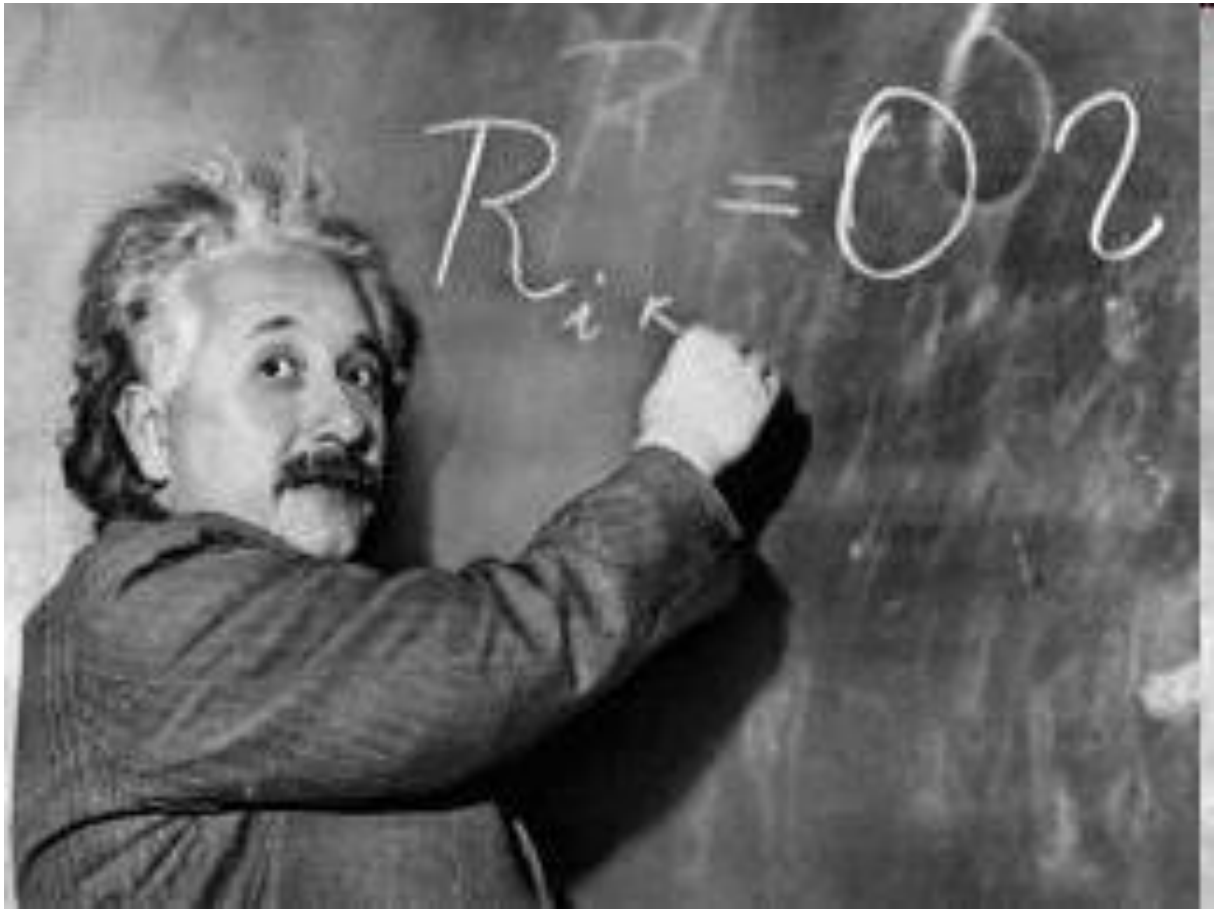
Modern fizika alkalmazásai a mérnöki gyakorlatban

4. előadás

A speciális relativitás elmélete II.



Transzformációk

Elforgatás az x – y síkon:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = x'^2 + y'^2$$

Két pont távolsága:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2$$

Elforgatás a z tengely körül:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\alpha & sin\alpha & 0 \\ -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$$

Galilei transzformáció:

x’ = x-vt és t’ = t

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

???

Lorentz transzformáció

$x' = \Gamma(x\text{-}vt)$ és $t' = \Gamma\left(t\text{-}\frac{v}{c^2}x\right)$

$$\Downarrow \qquad \Downarrow$$

$$x' = \Gamma\left[x - \frac{v}{c}(ct)\right] \qquad ct' = \Gamma\left[ct - \frac{v}{c}x\right] \qquad \beta = \frac{v}{c}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{pmatrix} \Gamma & -\beta\Gamma & 0 & 0 \\ -\beta\Gamma & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Téridő koordináták!!!

Fényterjedésre láttuk:

$$c^2t^2 - (x)^2 - (y)^2 - (z)^2 = c^2t'^2 - (x')^2 - (y')^2 - (z')^2$$

(órák szinkronizálása)

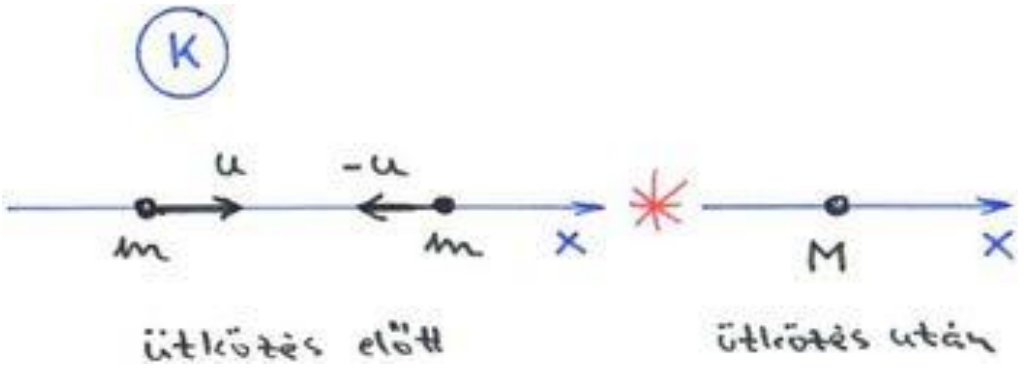
Két esemény távolsága:

$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 =$$

$$= c^2\,(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 - (y'_2 - y'_1)^2 - (z'_2 - z'_1)^2$$

Dinamika

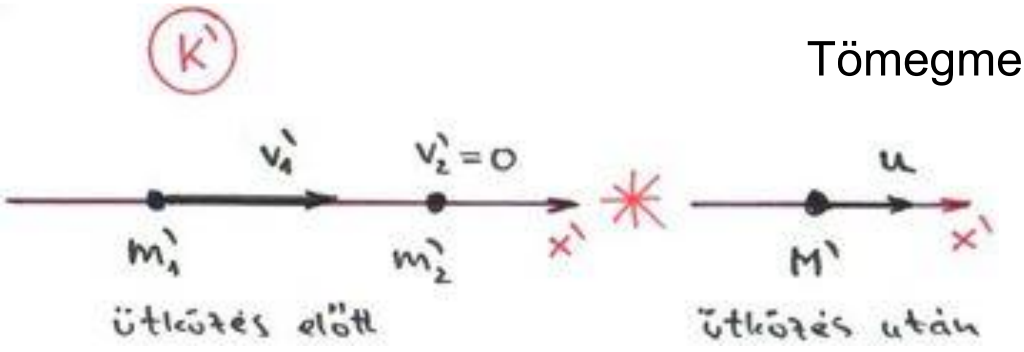
Impulzus-megmaradás I.



$$mu + m(-u) = 0$$

”Kívülről nézve…”

Rögzítsük a ***K’*** koordinátarendszerünket a ***-u*** sebességgel mozgó tömegponthoz.



$$m_1'v_1' = M'u$$

Tömegmegmaradás: $m_1' + m_2' = M'$

$\downarrow \ast u$

$$m_1'(v_1' - u) = m_2'u$$

$\downarrow$

$$\frac{m_1'}{m_2'} = \frac{u}{v_1' - u} = \frac{1}{\frac{v_1'}{u} - 1}$$

Impulzus-megmaradás II.

$$\frac{m_1'}{m_2'} = \frac{u}{v_1' - u} = \frac{1}{\frac{v_1'}{u} - 1} \qquad (*)$$

$$m_1' = m_2' = m \longrightarrow 1 = \frac{u}{v_1' - u} \rightarrow v_1' = 2u$$

Azonban:

$$v_1' = \frac{2u}{1 + \frac{u^2}{c^2}} = \frac{2u}{1 + \beta^2}$$

Newtoni tömegfogalom: $m_1' = m_2' = m \rightarrow M = 2m$

$$mv_1' = Mu \longrightarrow m\frac{2u}{1 + \beta^2} \neq 2mu$$

?

Impulzus-megmaradás III.

$$\frac{v_1'}{u}=\frac{2}{1+\beta^2}$$

Beírjuk (∗)-be:

$$\frac{m_1'}{m_2'}=\frac{1}{\frac{v_1'}{u}-1}=\frac{1}{\frac{2}{1+\beta^2}-1}=\frac{1+\beta^2}{1-\beta^2}$$

$$\beta_1=\frac{v_1'}{c}=\frac{2u/c}{1+\beta^2}=\frac{2\beta}{1+\beta^2}$$

$$1-\beta_1^2=1-\frac{4\beta^2}{\left(1+\beta^2\right)^2}=\frac{1+2\beta^2+\beta^4-4\beta^2}{\left(1+\beta^2\right)^2}=\frac{\left(1-\beta^2\right)^2}{\left(1+\beta^2\right)^2}$$

Nyugalmi tömeg

$$\frac{m_1'}{m_2'}=\frac{1}{\sqrt{1-\beta_1^2}}\longrightarrow m_1'=\frac{m_2'}{\sqrt{1-\left(v_1'/c\right)^2}}\longrightarrow \boxed{m=\frac{m_0}{\sqrt{1-\left(v/c\right)^2}}}$$

Relativisztikus tömeg

Relativisztikus impulzus (relativisztikus tömeg nélkül !!!)

K'-ben: $a = \frac{F}{m} = const.$

$$m dv_o = F d\tau$$

K'-ben:

$$m dv_o = F d\tau = F\sqrt{1-\frac{v^2}{c^2}}dt$$

$$dv : dv_o = \frac{d\ell}{dt} : \frac{d\ell_o}{d\tau}$$

$$d\ell = d\ell_o\sqrt{1-\frac{v^2}{c^2}}$$

és

$$dt = \frac{d\tau}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$dv = dv_o\left(1-\frac{v^2}{c^2}\right)$$

$$\frac{dv}{dt} = \left(1-\frac{v^2}{c^2}\right)^{3/2}\frac{F}{m}$$

$$\frac{m}{\left(1-v^2/c^2\right)^{3/2}}\frac{dv}{dt} = F$$

$$\frac{dp}{dt} = F$$

$$\boxed{p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}}$$

Relativisztikus mozgásegyenlet

$$\dot{\vec{p}} = \vec{F}$$

ahol

$$p = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Példa: tömegpont mozgása állandó erő hatására I.

$$m_o \ddot{x} = F_o$$

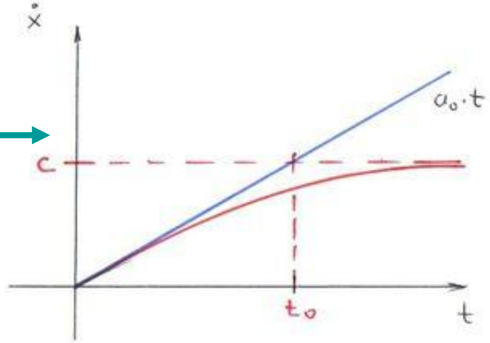
$$\ddot{x} = \frac{F_o}{m_o} \equiv a_o$$

Példa: tömegpont mozgása állandó erő hatására II.

$$\frac{d}{dt}\left[\frac{m_o\dot{x}}{\sqrt{1-(\dot{x}/c)^2}}\right]=F_o \longrightarrow \left[\frac{m_o\dot{x}}{\sqrt{1-(\dot{x}/c)^2}}\right]_0^{\dot{x}}=F_o t$$

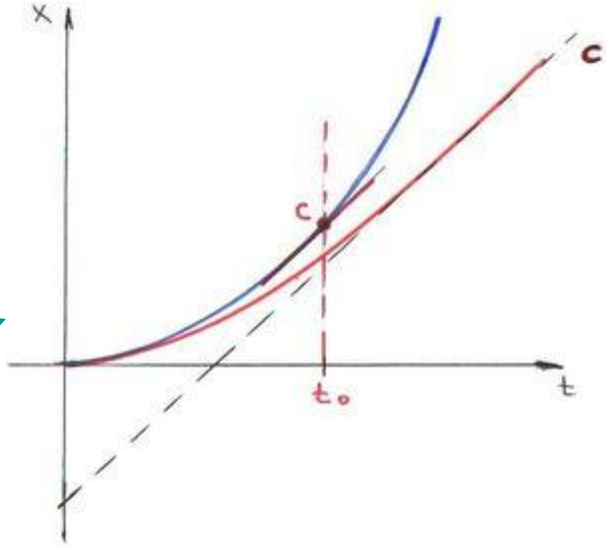
$$\dot{x}=a_o t\sqrt{1-(\dot{x}/c)^2} \longrightarrow \dot{x}=\frac{a_o t}{\sqrt{1+(a_o t/c)^2}}$$

$$\lim_{t\to\infty}\dot{x}=c$$



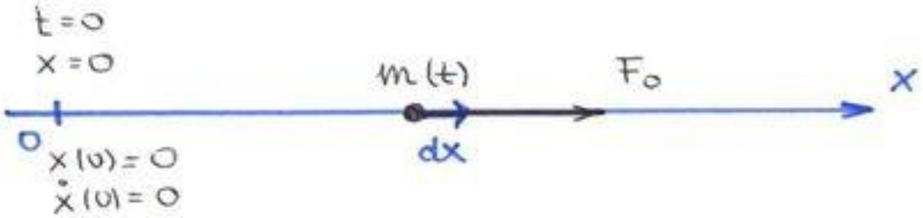
$$[x]_0^t=\int_0^t\frac{a_o t}{\sqrt{1+(a_o t/c)^2}}dt=\left[\frac{c^2}{a_o}\sqrt{1+(a_o t/c)^2}\right]_0^t$$

$$x(t)=\frac{c^2}{a_o}\left(\sqrt{1+(a_o t/c)^2}-1\right)$$



A tömeg-energia ekvivalencia

$$W=\int_0^x F\,dx$$



$$\frac{dp}{dt}=F \rightarrow p=mv=\frac{m_o v}{\sqrt{1-(v/c)^2}}$$

$$W=\int_0^x F\,dx=\int_0^x \frac{dp}{dt}dx=\int_0^p \frac{dx}{dt}dp=\int_0^p v\,dp=\int_0^v v\frac{dp}{dv}dv \longrightarrow W=[pv]_0^v-\int_0^v p\,dv$$

$$W=[pv]_0^v-m_o\int_0^v \frac{v}{\sqrt{1-(v/c)^2}}dv \rightarrow W=m_o\frac{v^2}{\sqrt{1-(v/c)^2}}-m_o\left[-c^2\sqrt{1-(v/c)^2}\right]_0^v$$

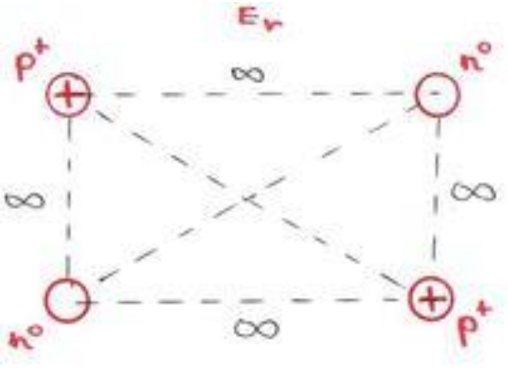
$$W=m_o\frac{v^2}{\sqrt{1-(v/c)^2}}+m_oc^2\sqrt{1-(v/c)^2}-m_oc^2$$

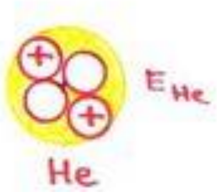
$$\Delta E_k=W=\frac{m_oc^2}{\sqrt{1-(v/c)^2}}-m_oc^2$$

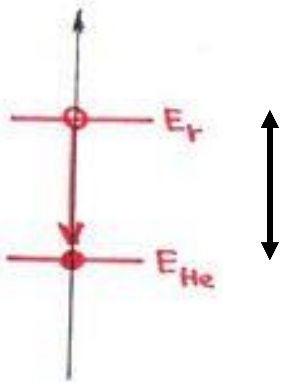
$$\boxed{E=mc^2}$$

$$\frac{v}{c}<<1\rightarrow E_k=\frac{1}{2}mv^2+\ldots$$

A tömegdefektus



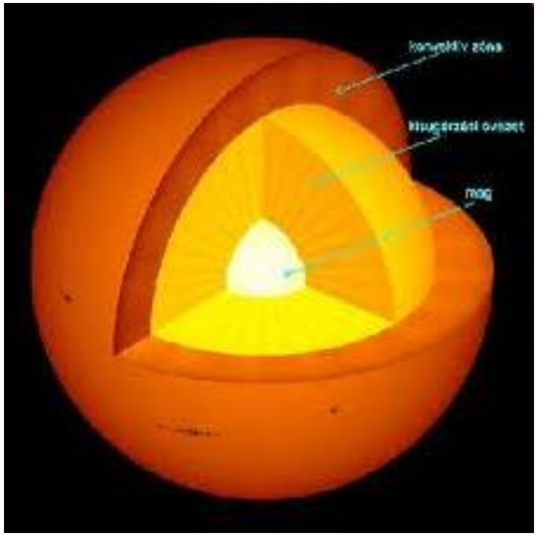


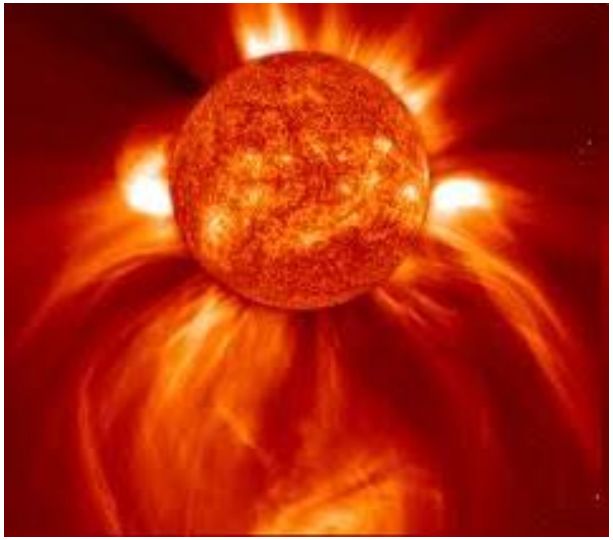


$$2m_p + 2m_n > m_{He}$$

$$\Delta E = 28MeV$$

$$1\,mól \rightarrow 10^{11}\mathrm{J}$$





Az energia, mint a sebesség és az impulzus függvénye

Klasszikus fizika: $E=\frac{1}{2}mv^2$

$p=mv$

$$E=\frac{p^2}{2m}$$

$$E=mc^2=\frac{m_oc^2}{\sqrt{1-(v/c)^2}}$$

$$p=\frac{m_ov}{\sqrt{1-(v/c)^2}}$$

?

Belátható: $E^2-p^2c^2=m_o^2c^4$

$$\boxed{E=\sqrt{m_o^2c^4+p^2c^2}} \rightarrow E_k=\sqrt{m_o^2c^4+p^2c^2}-m_oc^2\approx\frac{p^2}{2m} \text{ ha } p<<m_oc$$