<u>Description of motion in non-inertial frames. Inertia forces.</u>

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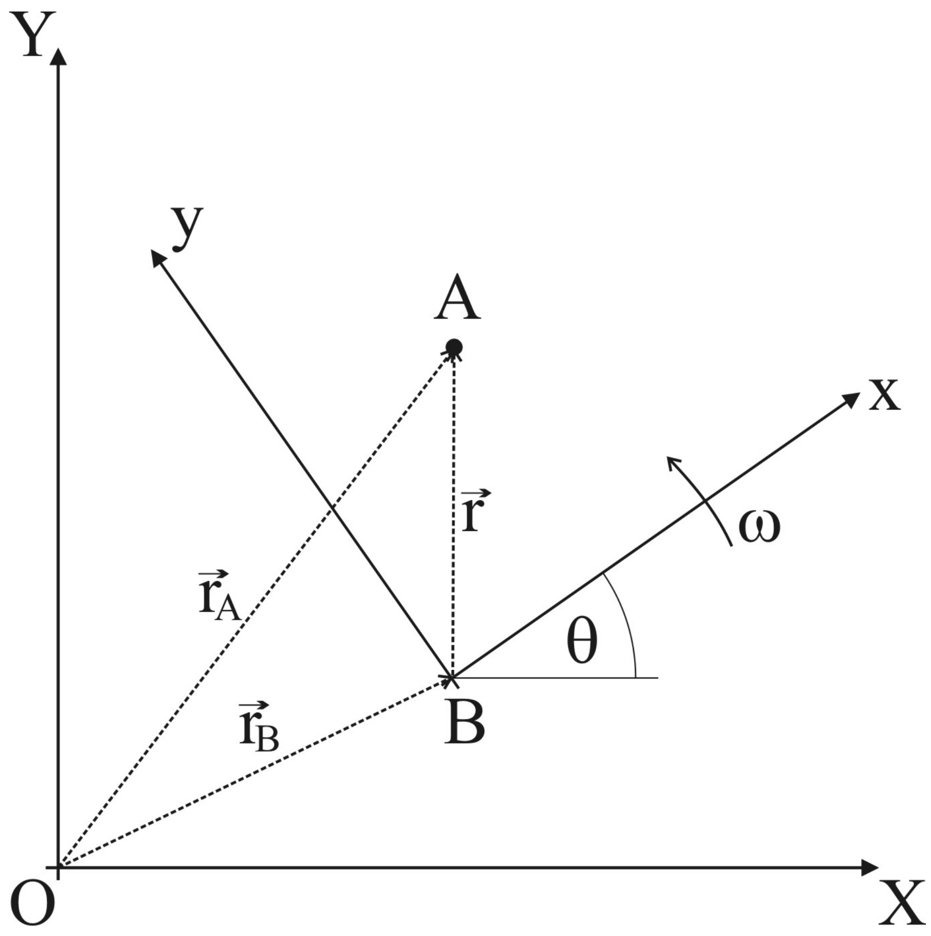
(X,Y) is an *inertial frame*.

(x,y) is an *accelerating frame*. It is *non-inertial*, because:
(1) its origin B performs *translational acceleration* with respect to (X,Y),
(2) the (x,y) coordinate axes are *rotating* around origin B.



The particle whose motion we want to describe is located at point A. Its position vector in the (X,Y) inertial frame can be written as

$$\vec{r}_A = \vec{r}_B + \vec{r} = \vec{r}_B + \left(x\vec{i} + y\vec{j}\right), \tag{1}$$

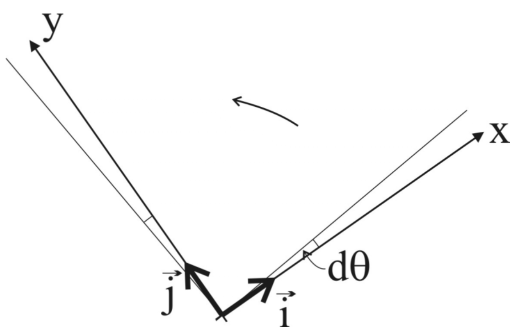
where $\vec{r}$ is the position vector in the (x,y) accelerating frame, and $\vec{i}$ and $\vec{j}$ are the x and y unit vectors in the accelerating frame, respectively.
As the figure below shows, the time derivatives of these unit vectors are:

$$\frac{d\vec{i}}{dt} = \frac{d\theta \cdot \vec{j}}{dt} = \omega \cdot \vec{j} \tag{2}$$

and

$$\frac{d\vec{j}}{dt} = -\frac{d\theta \cdot \vec{i}}{dt} = -\omega \cdot \vec{i} \ , \tag{3}$$

where $\omega = \dot{\theta}$ is the angular speed of the (x,y) axes.



This angular *speed* can be expressed vectorially (as angular *velocity*) in the following way:

$$\vec{\omega} = \omega \cdot \vec{k} \ , \tag{4}$$

where $\vec{k}$ is the z unit vector (pointing out of the plane of the figure). The direction of $\vec{\omega}$ thus obeys the so-called „right hand rule".

From (4) it is easy to see that the following relations hold:

$$\vec{\omega} \times \vec{i} = \omega \cdot \vec{j} \tag{5}$$

and

$$\vec{\omega} \times \vec{j} = -\omega \cdot \vec{i} \ . \tag{6}$$

Comparing (2) with (5) and (3) with (6), we obtain

$$\dot{\vec{i}} = \vec{\omega} \times \vec{i} \tag{7}$$

and

$$\dot{\vec{j}} = \vec{\omega} \times \vec{j} \ . \tag{8}$$

To obtain the velocity of the particle we have to differentiate (1) with respect to time:

$$\dot{\vec{r}}_A = \dot{\vec{r}}_B + \frac{d}{dt}\left(x\vec{i} + y\vec{j}\right) = \dot{\vec{r}}_B + \left(x\dot{\vec{i}} + y\dot{\vec{j}}\right) + \left(\dot{x}\vec{i} + \dot{y}\vec{j}\right). \tag{9}$$

The second term on the right hand side of (9) can be re-written using (7) and (8):

$$x\dot{\vec{i}} + y\dot{\vec{j}} = \vec{\omega} \times \left(x\vec{i} + y\vec{j}\right) = \vec{\omega} \times \vec{r} , \tag{10}$$

and the third term on the right hand side of (9) is the velocity of the particle *as measured in the accelerating frame* („relative velocity"):

$$\vec{v}_{rel} = \dot{x}\vec{i} + \dot{y}\vec{j} . \tag{11}$$

Substituting (10) and (11) into (9), we obtain the following expression for the velocity of the particle:

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r} + \vec{v}_{rel} , \tag{12}$$

where $\vec{v}_B$ is the velocity (as measured in the inertial frame) with which point B moves.

The acceleration (as measured in the inertial frame) of the particle at point A can be obtained by differentiating (12) with respect to time:

$$\vec{a}_A = \vec{a}_B + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}} + \dot{\vec{v}}_{rel} . \tag{13}$$

Using the formulas above, the third term on the right hand side of (13) can be written as

$$\vec{\omega} \times \dot{\vec{r}} = \vec{\omega} \times \left(\vec{\omega} \times \vec{r} + \vec{v}_{rel}\right) = \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + \vec{\omega} \times \vec{v}_{rel} , \tag{14}$$

and the fourth term on the right hand side of (13) can be written as

$$\dot{\vec{v}}_{rel} = \frac{d}{dt}\left(\dot{x}\vec{i} + \dot{y}\vec{j}\right) = \dot{x}\dot{\vec{i}} + \dot{y}\dot{\vec{j}} + \ddot{x}\vec{i} + \ddot{y}\vec{j} = \vec{\omega} \times \left(\dot{x}\vec{i} + \dot{y}\vec{j}\right) + \vec{a}_{rel} = \vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel} , \tag{15}$$

where $\vec{a}_{rel}$ is the acceleration of the particle *as measured in the accelerating frame* („relative acceleration").

Hence

$$\vec{a}_A = \vec{a}_B + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel} . \tag{16}$$

Let us now multiply both sides by the mass *m* of the particle, and rearrange the equation so that it *looks like* Newton's 2nd law (but as written from the viewpoint of the accelerating observer):

$$m\vec{a}_{rel} = \vec{F}_{net} - m\vec{a}_B + m\vec{r} \times \dot{\vec{\omega}} + m\vec{\omega} \times \left(\vec{r} \times \vec{\omega}\right) + 2m\vec{v}_{rel} \times \vec{\omega} , \tag{17}$$

where $\vec{F}_{net}$ could be written instead of $m\vec{a}_A$, according to Newton's 2nd law in inertial frames.

Thus, in order to be able to use an equation similar to N2, the accelerating observer must introduce additional „forces" on the right hand side (in addition to the *real, physical* forces represented by $\vec{F}_{net}$). These additional terms are called inertia forces. They don't represent any real physical interaction. They are just mathematical terms, needed for the accelerating observer in order to be able to use an equation that looks like N2. The inertia forces on the right hand side of (17) are called:

$-m\vec{a}_B$:	translational inertia force
$m\vec{r} \times \dot{\vec{\omega}}$:	Euler force
$m\vec{\omega} \times (\vec{r} \times \vec{\omega})$:	centrifugal force
$2m\vec{v}_{rel} \times \vec{\omega}$:	Coriolis force