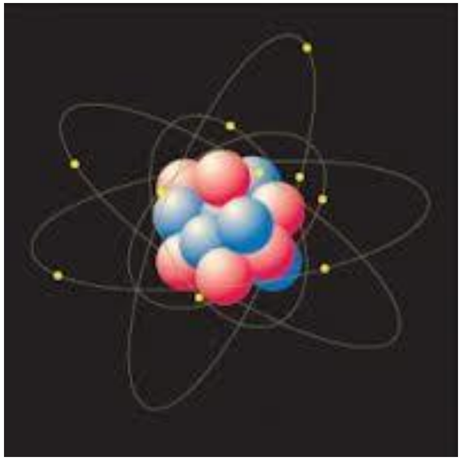
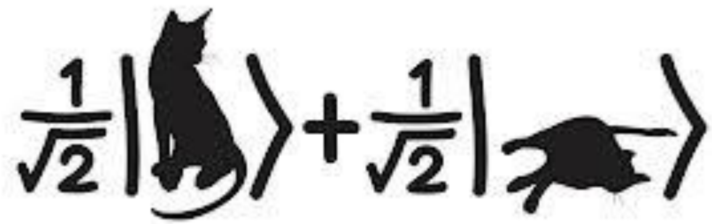
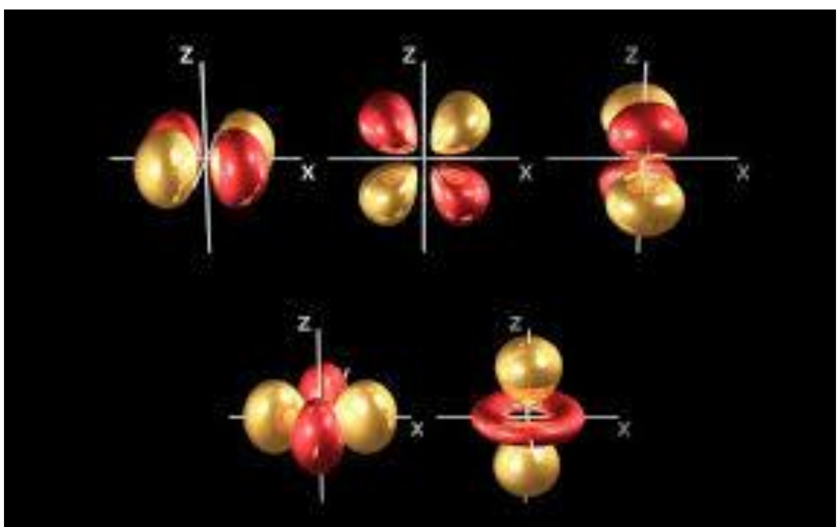
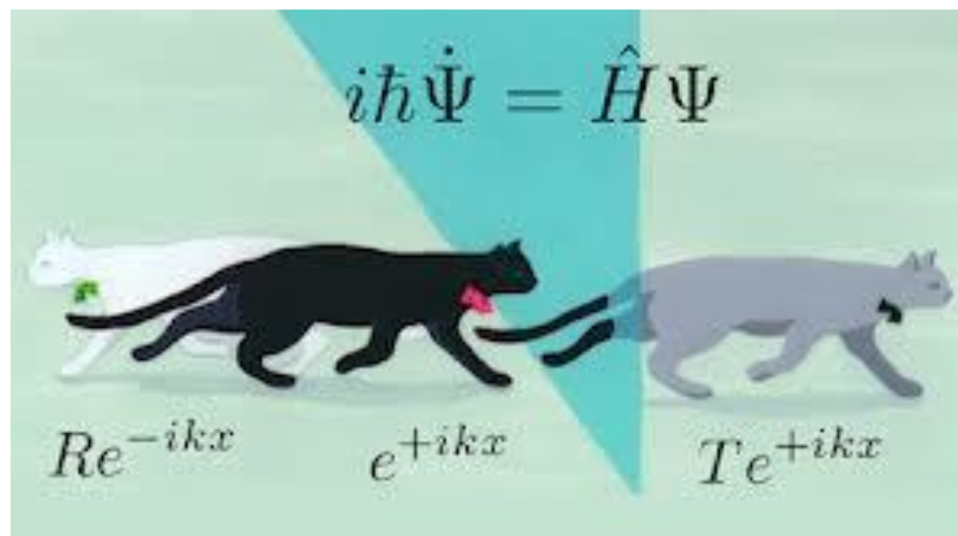
Lecture 5

Quantum physics: from atom models to quantum communication II.



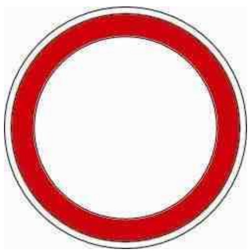










There is no king’s way to understand quantum physics !!!

The superposition of quantum states I.
/described by wave function(s)/

$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{r})+V(\vec{r})\cdot\psi(\vec{r})=E\cdot\psi(\vec{r})$$

We have seen (lecture 4)

*** Time independent Schrödinger equation ***

$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{r})+V(\vec{r})\cdot\psi(\vec{r})=-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi(\vec{r})$$

Linear equations

*** Time-dependent Schrödinger equation ***

Linear superposition of states: $\psi = c_1 \cdot \psi_1 + c_2 \cdot \psi_2$

The superposition of quantum states II.

Linear superposition of states: $\psi = c_1 \cdot \psi_1 + c_2 \cdot \psi_2$

Born's interpretation: $P(\vec{r}) \equiv |\psi(\vec{r})|^2 \equiv \psi^*(\vec{r})\psi(\vec{r})$

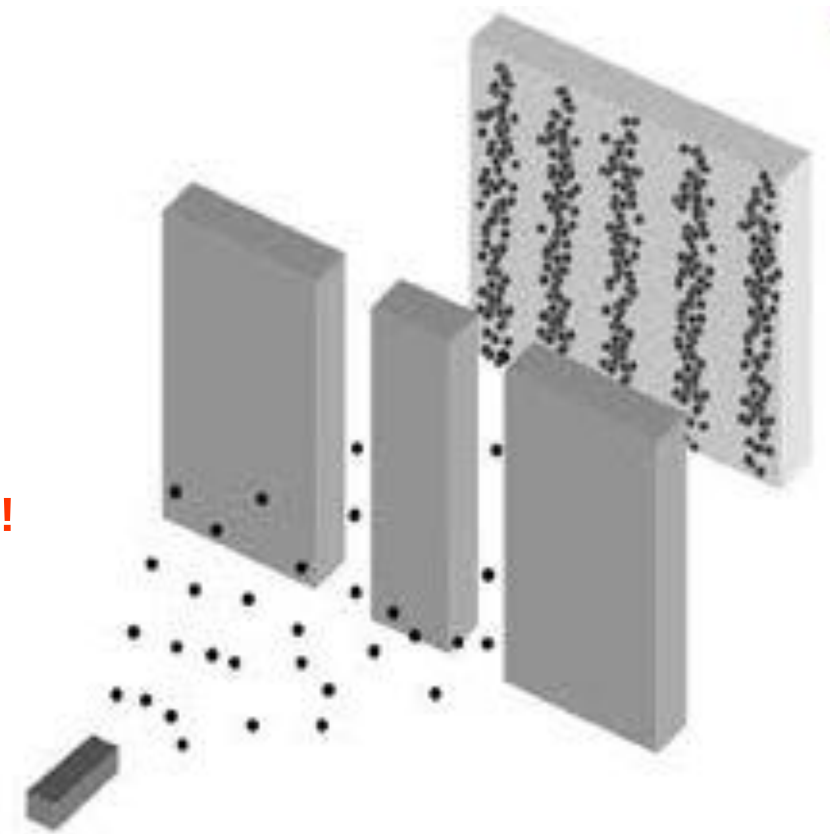
$$P = |\psi|^2 = (c_1\psi_1 + c_2\psi_2)^*(c_1\psi_1 + c_2\psi_2)$$

$$P = |c_1|^2|\psi_1|^2 + |c_2|^2|\psi_2|^2 + \mathrm{Re}\{c_1^* c_2 \psi_1^* \psi_2\}$$

$$P = |c_1|^2 P_1 + |c_2|^2 P_2 + \mathrm{Re}\{c_1^* c_2 \psi_1^* \psi_2\}$$

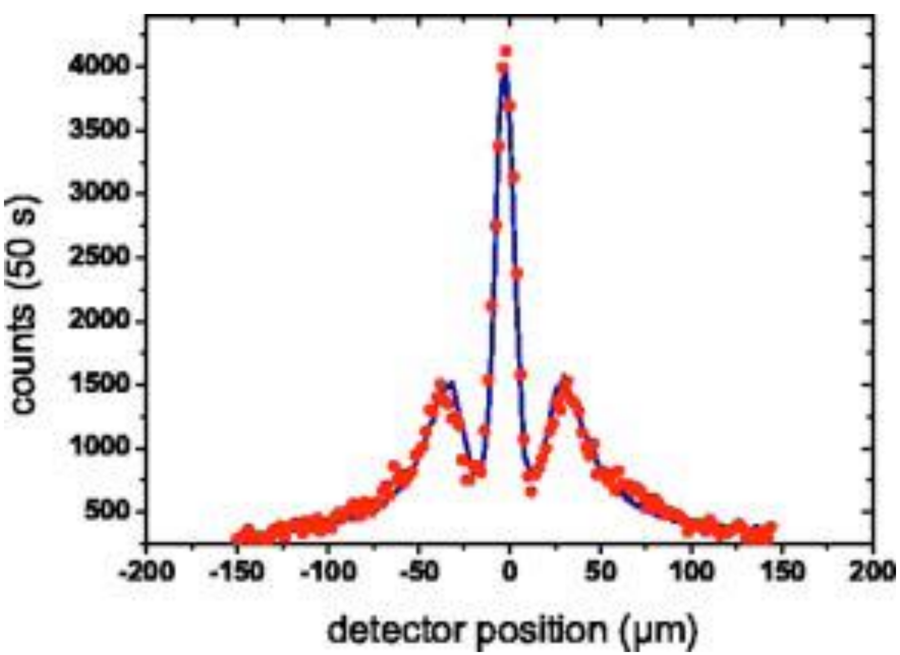
Interference!!!

Experimental verification of superposition (interference)



Double slit-experiment with C60:

Result of experiment: interference pattern!



C60 molecule

Average speed: 200 m/s

Separation of slits: 50nm

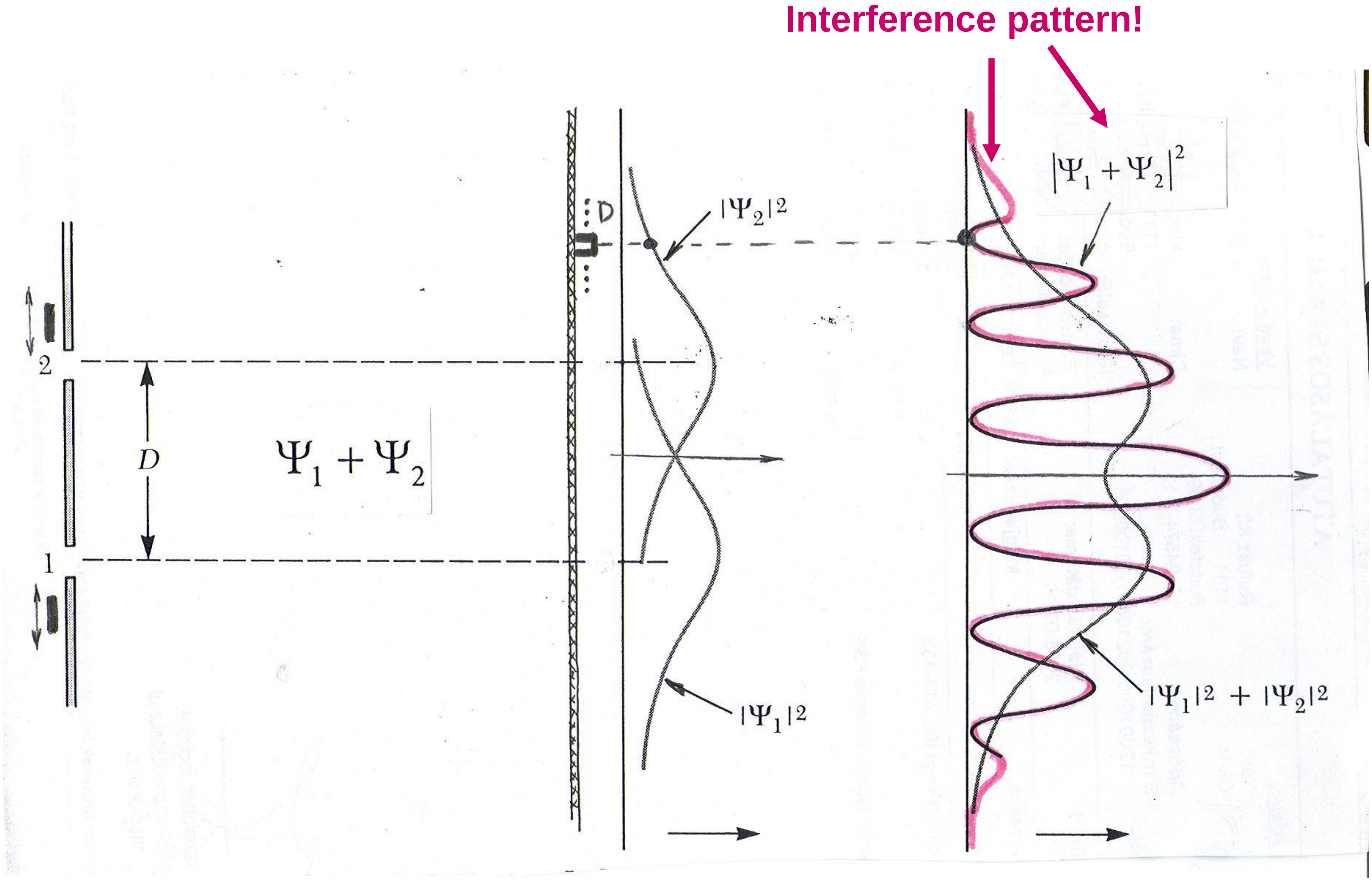
Axioms of quantum mechanics:

A. The Schrödinger-equation

B. $\left|\Psi\left(\vec{r},t\right)\right|^2 dV$ gives the probability to find the particle in a given region at a moment of ***t***

C. The superposition theorem of quantum states.

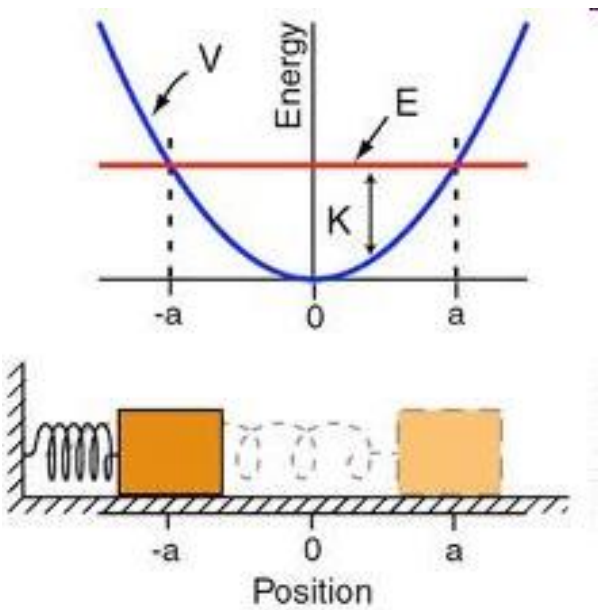
Interpretation of the double-slit experiment by quantum physics:



Harmonic oscillator I.

The ”quantum mechanics” started with **Max Planck** (1900): $\Delta E = h\nu$

Classical harmonic oscillator:



$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega^2 x^2$$

$$x(t) = a \cdot \sin(\omega t + \alpha)$$

$$\dot{x}(t) = a\omega \cdot \cos(\omega t + \alpha)$$

$$E = \frac{1}{2} m\omega^2 a^2$$

The energy of a classical harmonic oscillator can be changed fluently!!!

Harmonic oscillator II.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + \frac{1}{2}m\omega^2 x^2\psi = E\cdot\psi$$

Looking for the solution of Schrödinger-equation :

$$\varphi_1 \Rightarrow E_1$$

$$\varphi_2 \Rightarrow E_2$$

$$\varphi_3 \Rightarrow E_3$$

Harmonic oscillator : the solution

$$-\frac{\hbar^2}{2m}\frac{d}{d^2x}\varphi(x) + \frac{m\omega^2}{2}x^2\varphi(x) = E\,\varphi(x)$$

New variable & constant:

$$k = \frac{2E}{\hbar\omega} \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}x$$

$$\frac{d}{d^2\xi}\varphi(\xi) + (k - \xi^2)\varphi(\xi) = 0 \qquad (*)$$

Sommerfeld's polynomial method:

(looking for asymtotic solution)

$x \to \infty$ *és* $\xi \to \infty$

That is: $\xi >> k$

$$\frac{d}{d^2\xi}\varphi_{asz}(\xi) - \xi^2\varphi_{asz}(\xi) = 0$$

The solution:

$$\varphi_{asz}(\xi) = \exp\left(-\frac{\xi^2}{2}\right)$$

Sommerfeld's polynomial method : The solution can be multiplied with a polinomial:

(**) $$p(\xi) = \sum_{i=1}^{n} c_i \xi^i$$ ***n = ? és c_i = ?***

The final solution: $$\varphi(\xi) = \varphi_{asz}(\xi) p(\xi)$$

Substituting into (*)
(factorizing the exponential)

$$\frac{d}{d^2\xi} p(\xi) - 2\xi \frac{d}{d\xi} p(\xi) + (k-1)p(\xi) = 0$$

Substituting (**) :

$$\sum_{i=0}^{n} \{(i+2)(i+1)c_{i+2} - (2i+1-k)c_i\}\xi^i = 0$$

It is satisfied for all ξ , if:

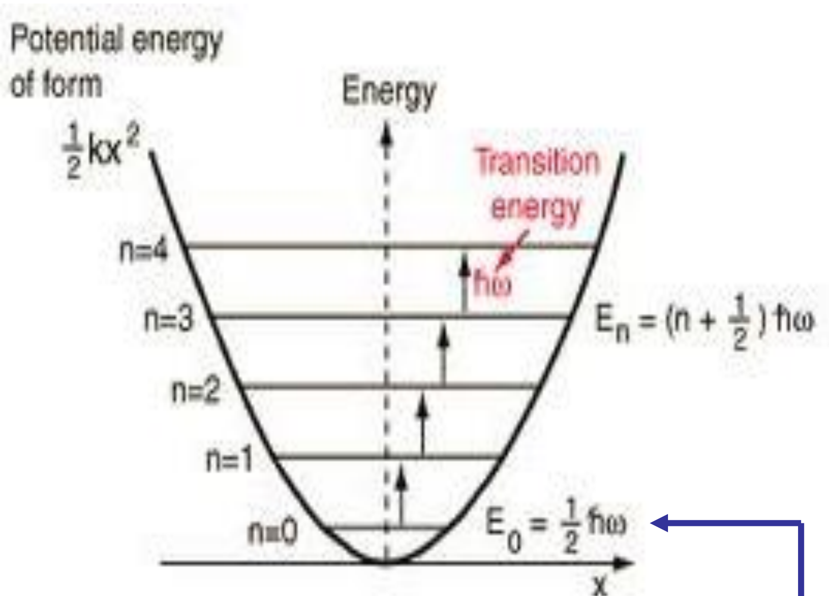
$$c_{i+2} = \frac{2i+1-k}{(i+2)(i+1)} c_i$$

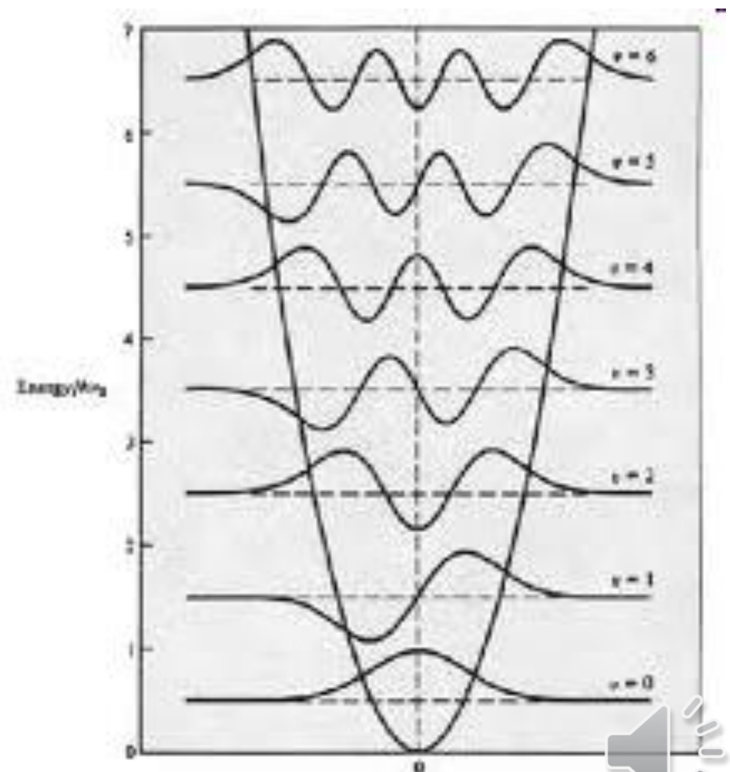
It is given only odd or even solution & from ***i = n*** all $\boldsymbol{c_i}$=0 .

$$c_{i+2} = \frac{2i + 1 - k}{(i+2)(i+1)} c_i \quad \Rightarrow \quad 2n + 1 = k = \frac{2E}{\hbar\omega}$$

Energy of harmonic oscillator:

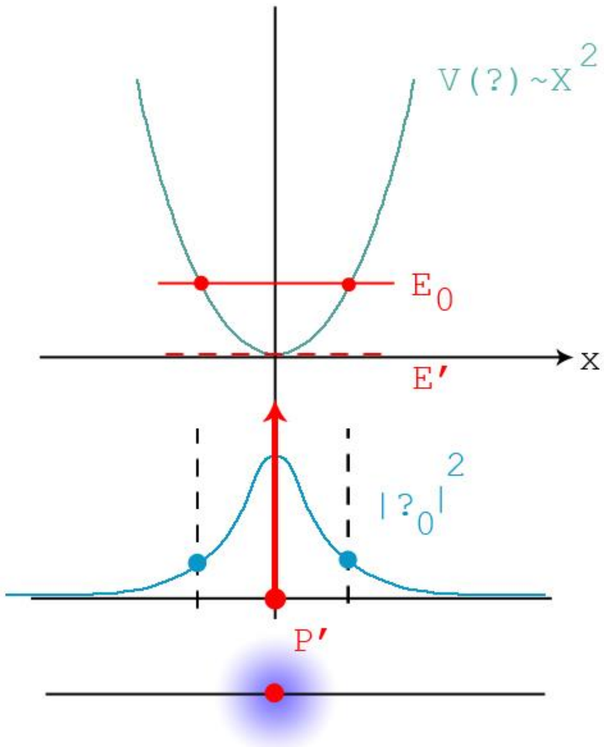
$$E_n^{SCH} = \left(n + \frac{1}{2}\right) \cdot \hbar\omega$$

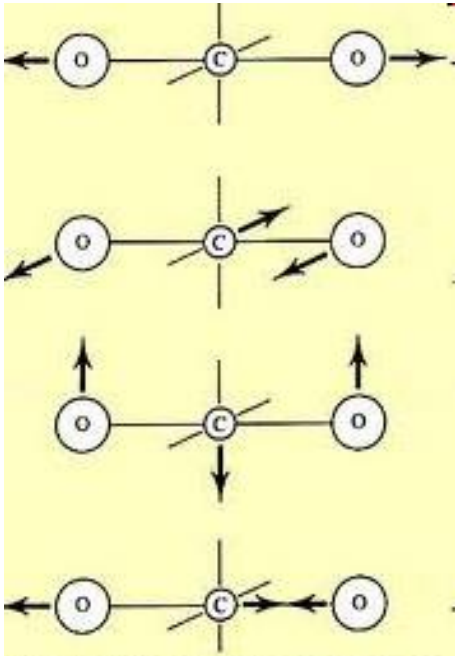




Zero point energy = energy of ground state

Harmonic oscillator III.





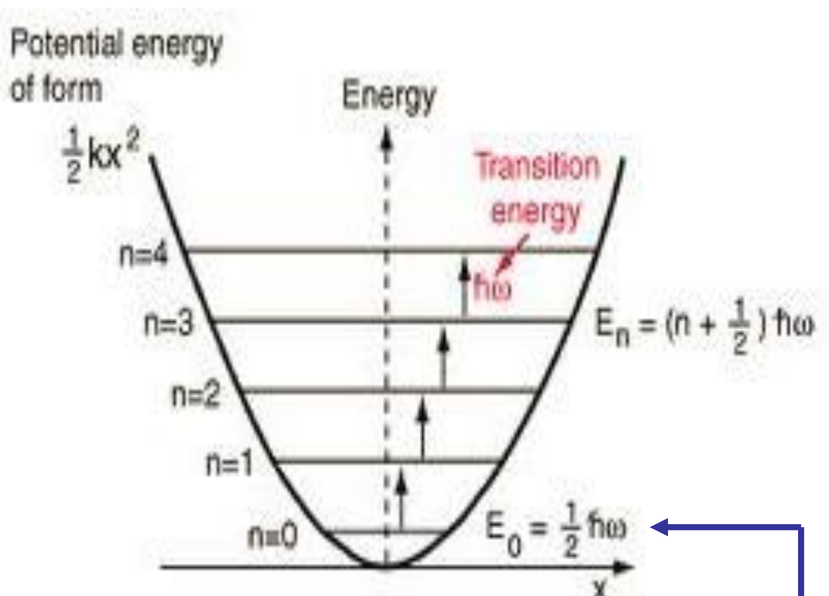
A carbon dioxide molecule shown in its four possible modes of vibration

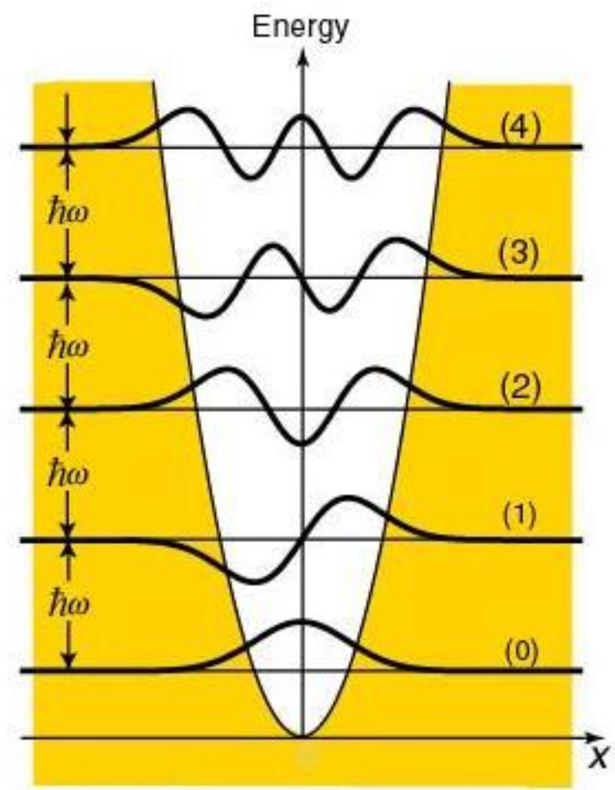
Application: molecular oscillation, solid state: crystal oscillation, stb.

Harmonic oscillator , to remember.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + \frac{1}{2}m\omega^2 x^2\psi = E\cdot\psi$$

$$E_n^{SCH} = \left(n + \frac{1}{2}\right)\cdot\hbar\omega$$



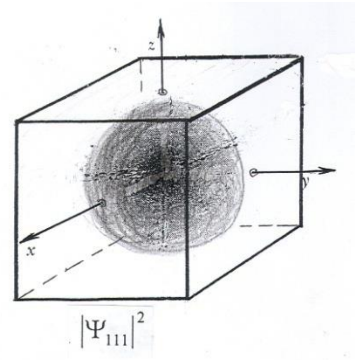


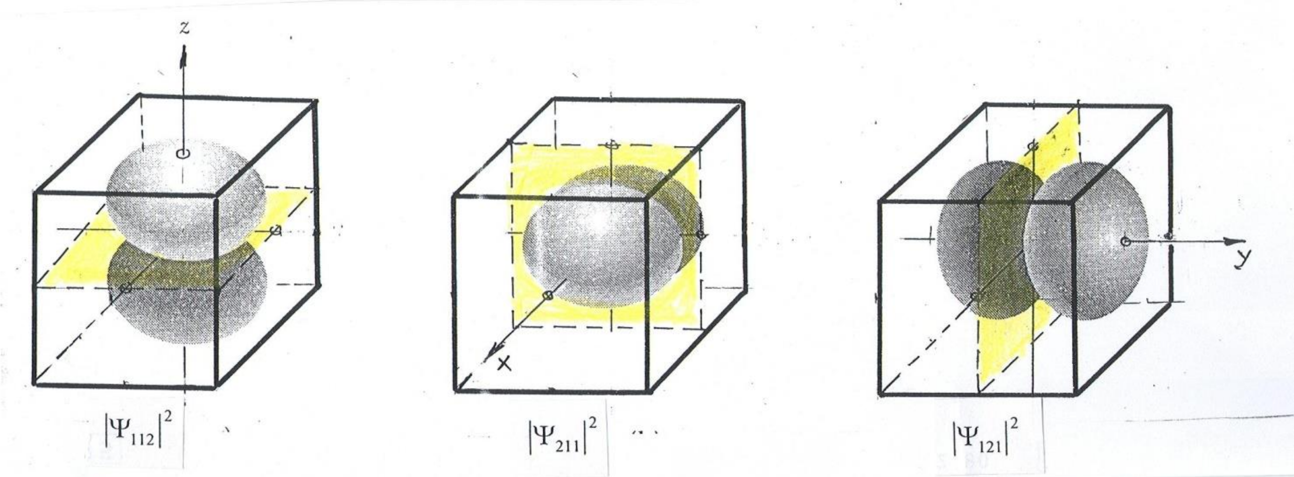
The thick solid curves are the wave functions.
(0) : the ground state
(1) (2) (3) . . . : the excited states

Zero point energy = energy of the ground state

Particle in the box (3D):

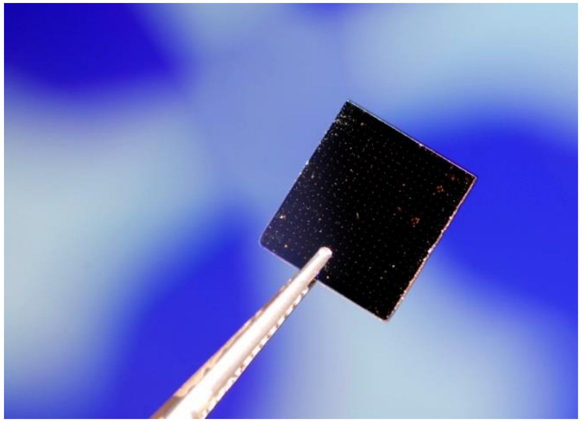
ψ	n_x	n_y	n_z	E
ψ_{111}	1	1	1	$3E_0$
ψ_{211}	2	1	1	$6E_0$
ψ_{121}	1	2	1	$6E_0$
ψ_{112}	1	1	2	$6E_0$
ψ_{122}	1	2	2	$9E_0$
ψ_{212}	2	1	2	$9E_0$
ψ_{221}	2	2	1	$9E_0$
ψ_{311}	3	1	1	$11E_0$
ψ_{131}	1	3	1	$11E_0$
ψ_{113}	1	1	3	$11E_0$
ψ_{222}	2	2	2	$12E_0$

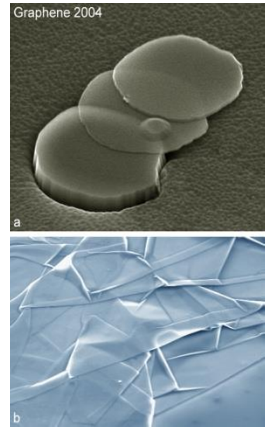


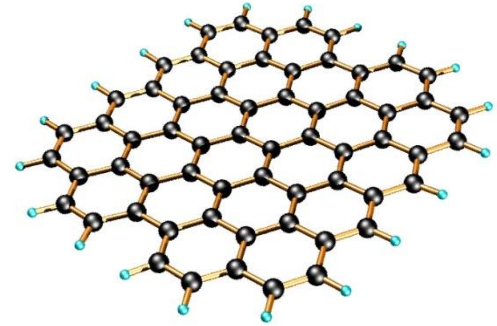


$$E = E_o\left(n_x^2 + n_y^2 + n_z^2\right)$$

The 2D electron gas

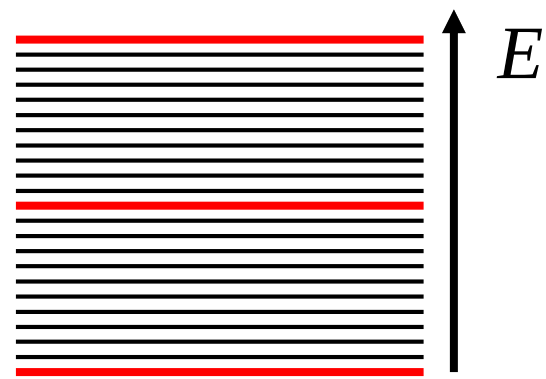






$$E = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)$$

$$c << a, b$$



Andre Geim
1958

Konstantin Novoselov
1974

Nobel-prize in physics: 2010

"for groundbreaking experiments regarding the two-dimensional material graphene"

The free particle (V=0)

Particle in a box: $\psi(x) = A\sin\left(\frac{2\pi}{\lambda}x\right) = A\sin(kx)$

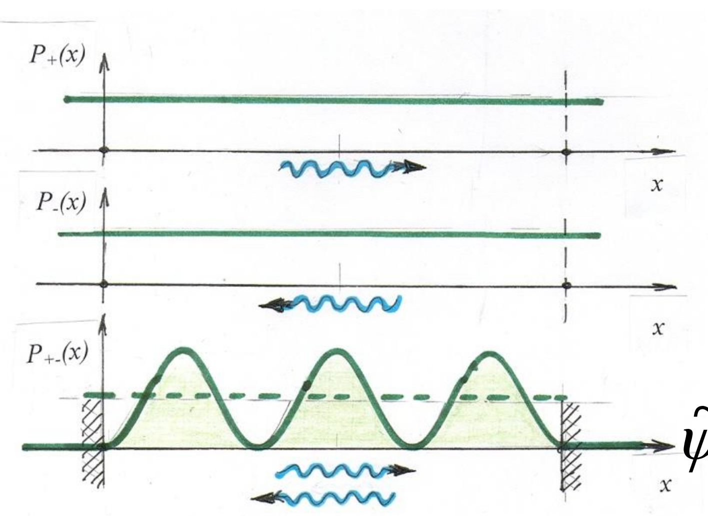
Free particle: $L \to \infty$

$$\tilde{\psi}(x,t) = \varphi(x)e^{-i\omega t} \longrightarrow \left|\tilde{\psi}(x,t)\right|^2 = \left|\varphi(x)\right|^2$$

$$\tilde{\psi}(x) = Ae^{ikx}$$

$$\tilde{\psi}_1(x,t) = Ae^{-ikx}e^{i\omega t} = Ae^{i(-kx+\omega t)}$$

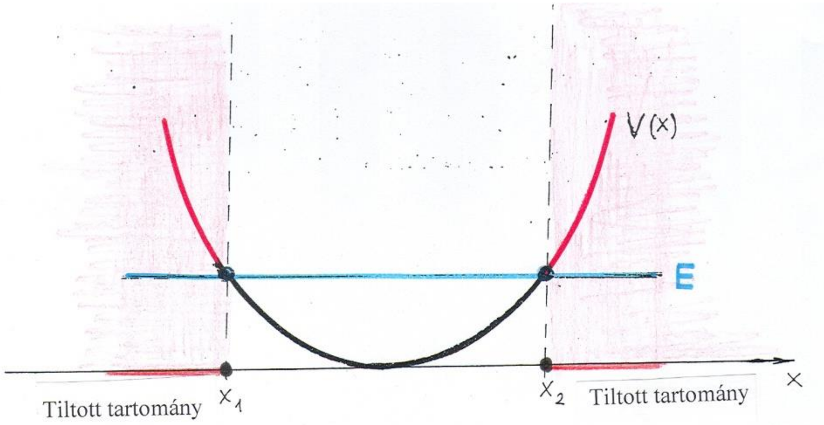
$$\tilde{\psi}_2(x,t) = Ae^{ikx}e^{i\omega t} = Ae^{i(kx+\omega t)}$$

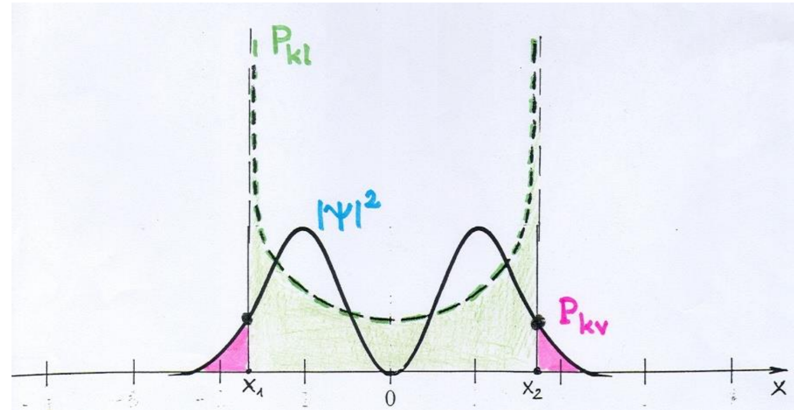


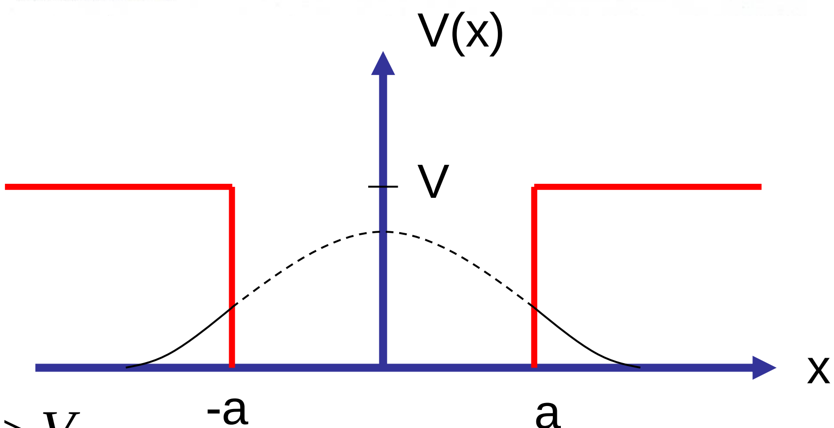
$$\tilde{\psi}(x,t) = \tilde{\psi}_1(x,t) + \tilde{\psi}_2(x,t) = Ae^{i\omega t}\left(e^{ikx} - e^{-ikx}\right)$$

$$P_{-+} = \left|\psi\right|^2 = A^2\sin^2(kx) \xrightarrow{L \to \infty}$$ Interpretation???

Probability of finding the particle in the ”forbidden” region:







$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{r})+V(\vec{r})\cdot\psi(\vec{r})=E\cdot\psi(\vec{r})$$

1D: $$-\frac{\hbar^2}{2m}\psi''=\left(E-V(x)\right)\cdot\psi$$

$E>V$

Solution : $\psi = A\sin(kx)$ or $\psi = A\cos(kx)$

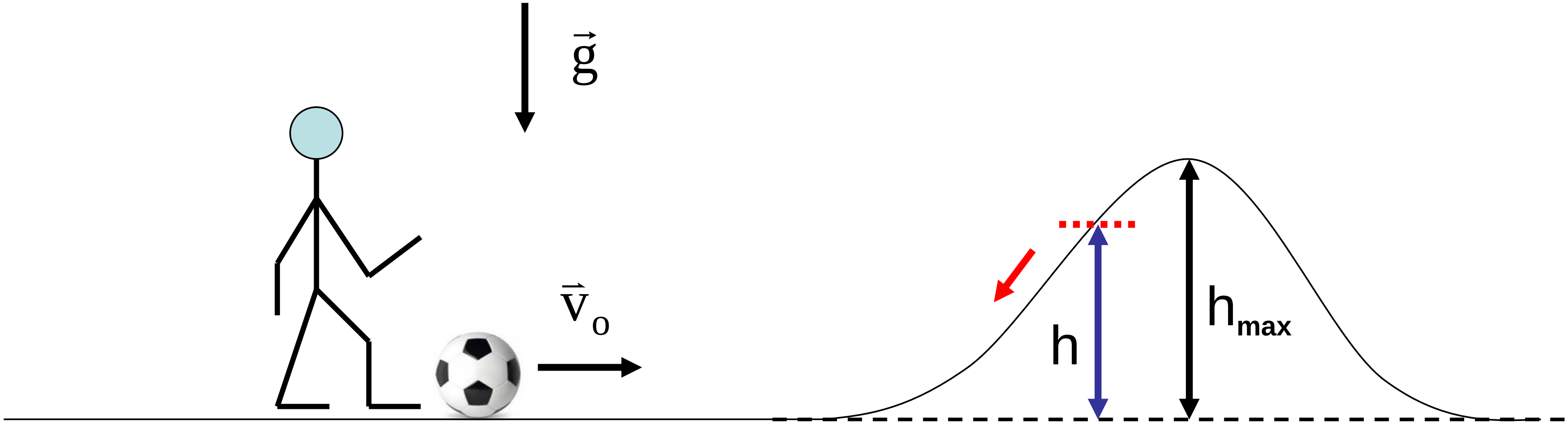
where: $$k=\sqrt{\frac{2m(E-V)}{\hbar^2}}$$

$E<V$ Solution : $\psi = Be^{\pm kx}$

Acceptable only:

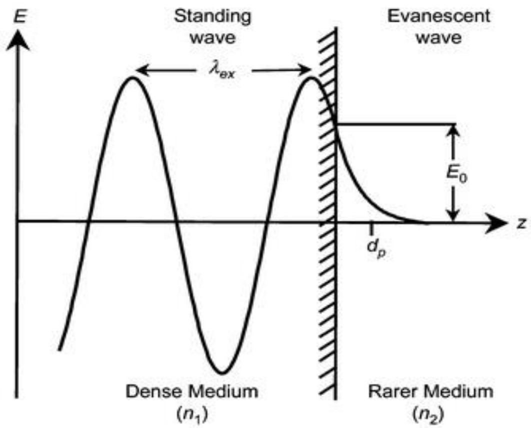
Idea $\psi = Be^{-kx}$

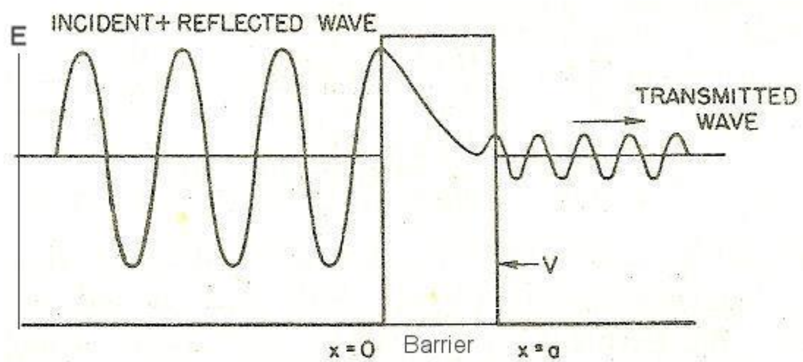
The tunnel effect 0.

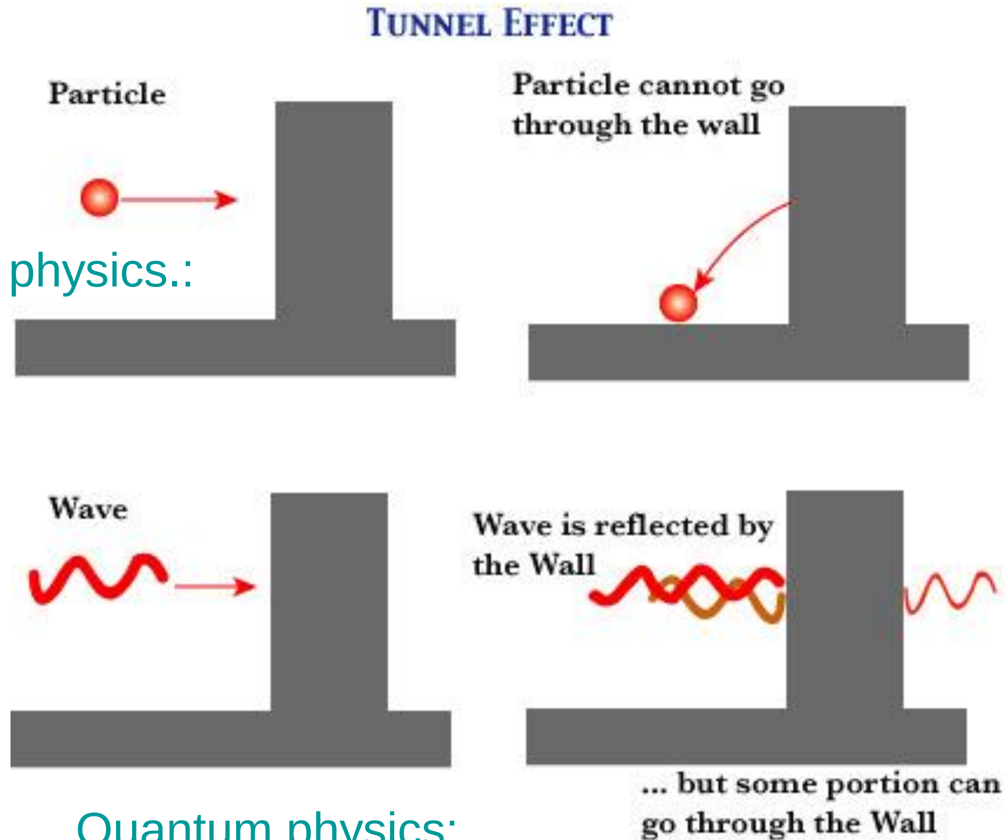


$$\frac{1}{2}mv_o^2 = \frac{p^2}{2m} = mgh < mgh_{max.}$$

The tunnel effect I. E < V !!!

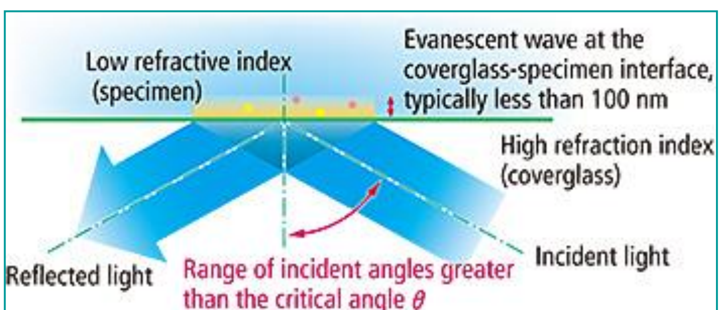






Classical physics.:

Quantum physics:

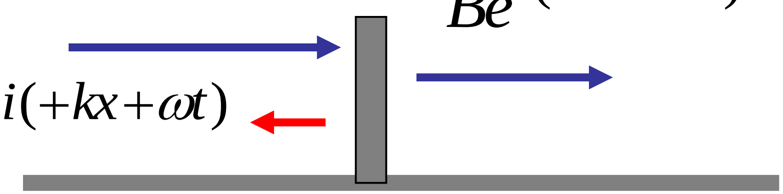


Classical physics: evanescent wave

$$Ae^{i(-kx+\omega t)}$$

$$Be^{i(-kx+\omega t)}$$

$$Ce^{i(+kx+\omega t)}$$



$$R = \left|\frac{C}{A}\right|^2$$

$$T = \left|\frac{B}{A}\right|^2$$

R: reflexion, T: transmission

The tunnel effect III.

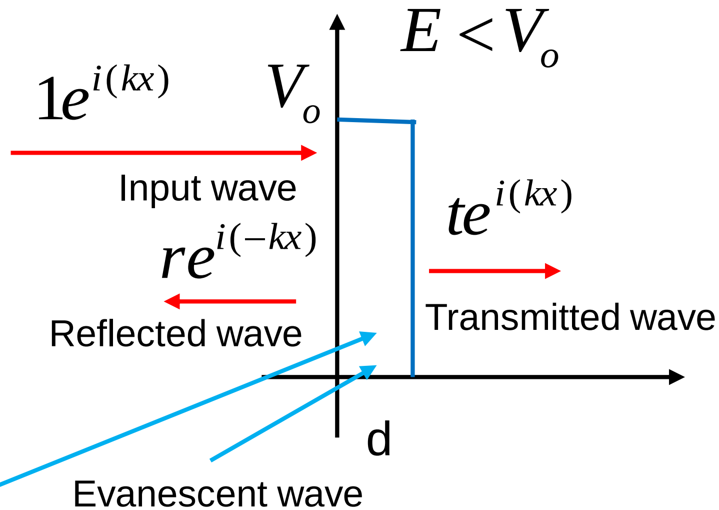
How does it work???

Example: tunneling through a potential gap

$$E = \frac{p^2}{2m} \rightarrow p = \hbar k = \sqrt{2Em} \rightarrow k = \frac{\sqrt{2Em}}{\hbar}$$

Schrödinger equation in free space:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi(x) = E \cdot \varphi(x)$$



Schrödinger equation in free space:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi(x) = E \cdot \varphi(x)$$

Schrödinger equation inside (0<x<d):

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi(x) = \left(E - V_0\right)\cdot \varphi(x) \longrightarrow Ae^{-\kappa x} + Be^{\kappa x}$$

$$\kappa = \frac{1}{\hbar}\sqrt{2m\left(V_o - E\right)}$$

Boundary condition(s):

$$1 + r = A + B$$

$$Ae^{-\kappa d} + Be^{\kappa d} = te^{\kappa d}$$

$$ik(1 - r) = \kappa(-A + B)$$

$$ikte^{-ikd} = \kappa\left(-Ae^{-\kappa d} + Be^{\kappa d}\right)$$

$$|t|^2 = \left(1 + \frac{V_o^2}{4E\left(V_o - E\right)}\sinh^2\left(\kappa d\right)\right)^{-1}$$

If $\kappa d >> 1$ $\quad |t|^2 = \frac{16E(V_o - E)}{V_o^2}e^{-2\kappa d}$

The tunnel effect IV.

Discussion

The probability of transmission:

$$|t|^2 = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(\kappa d)}$$

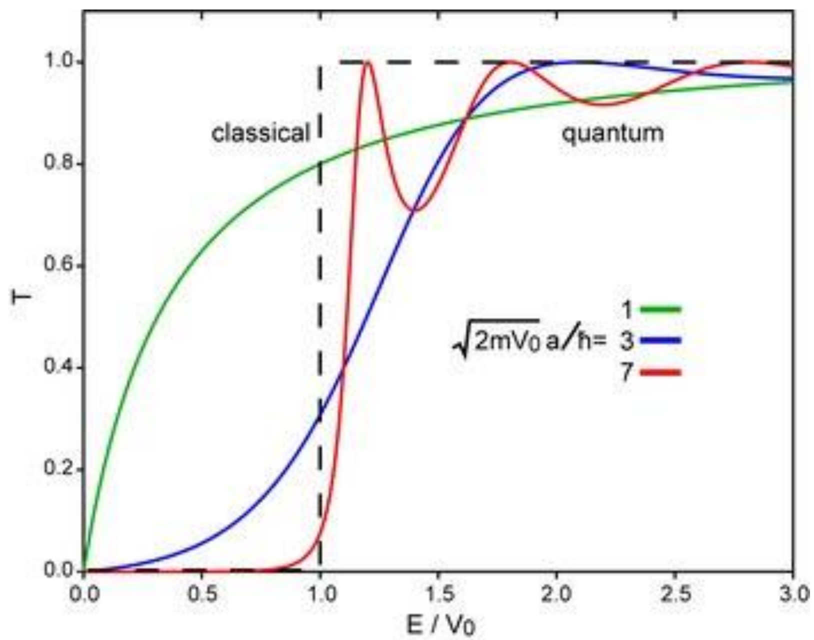


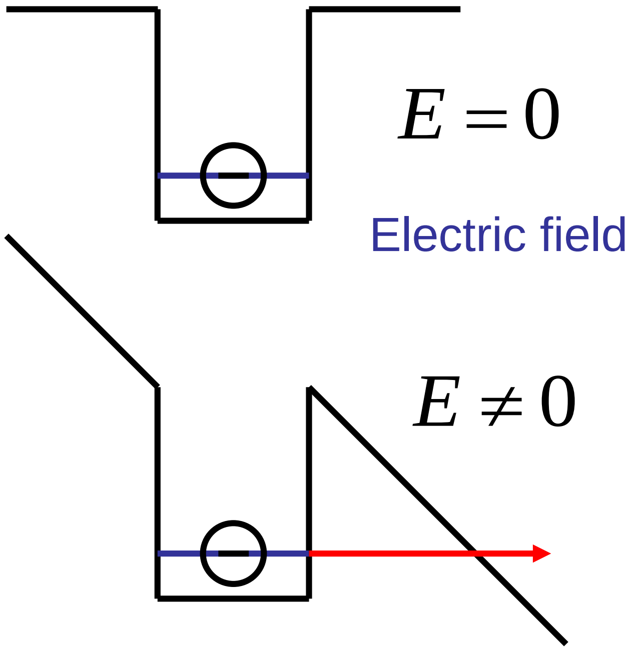
Figure: wikipedia

The probability of reflection :

$$|r|^2 = \frac{\frac{V_0^2}{4E(V_0 - E)} \sinh^2(\kappa d)}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(\kappa d)}$$

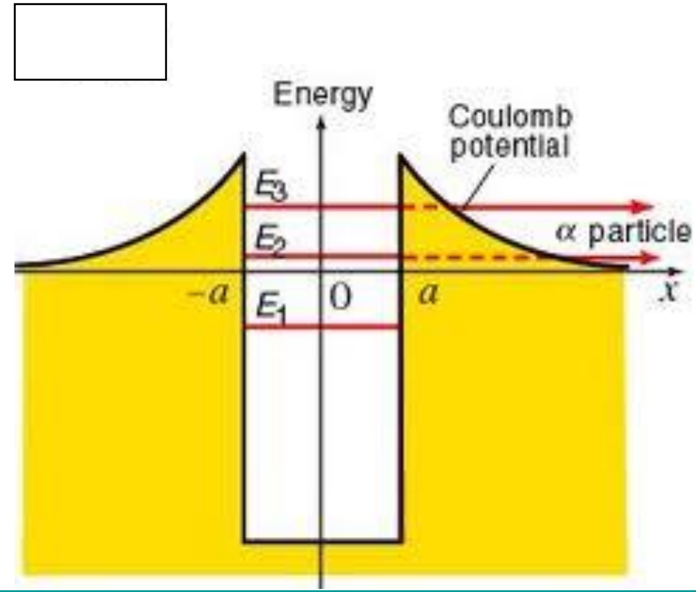
Condition: $|t|^2 + |r|^2 = 1$

The tunnel effect V.

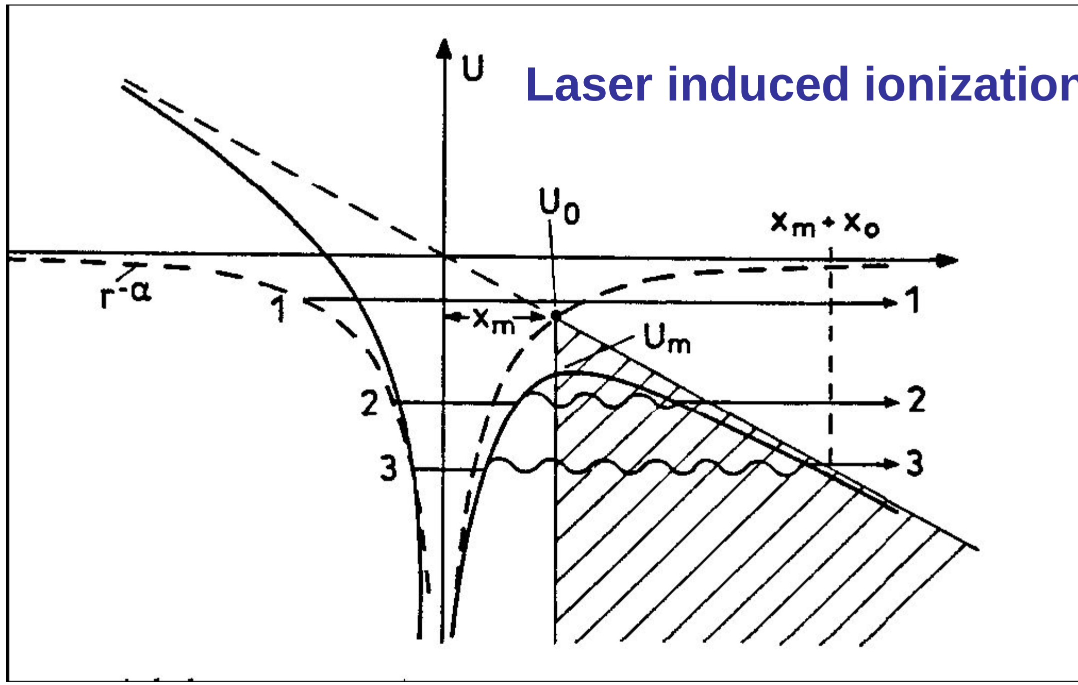


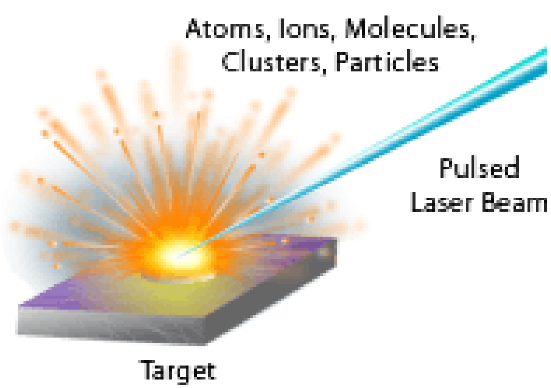
Cold emission

***α* decay**



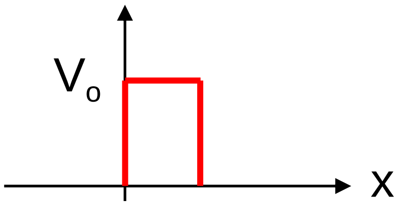
Laser induced ionization





The tunnel effect VI.

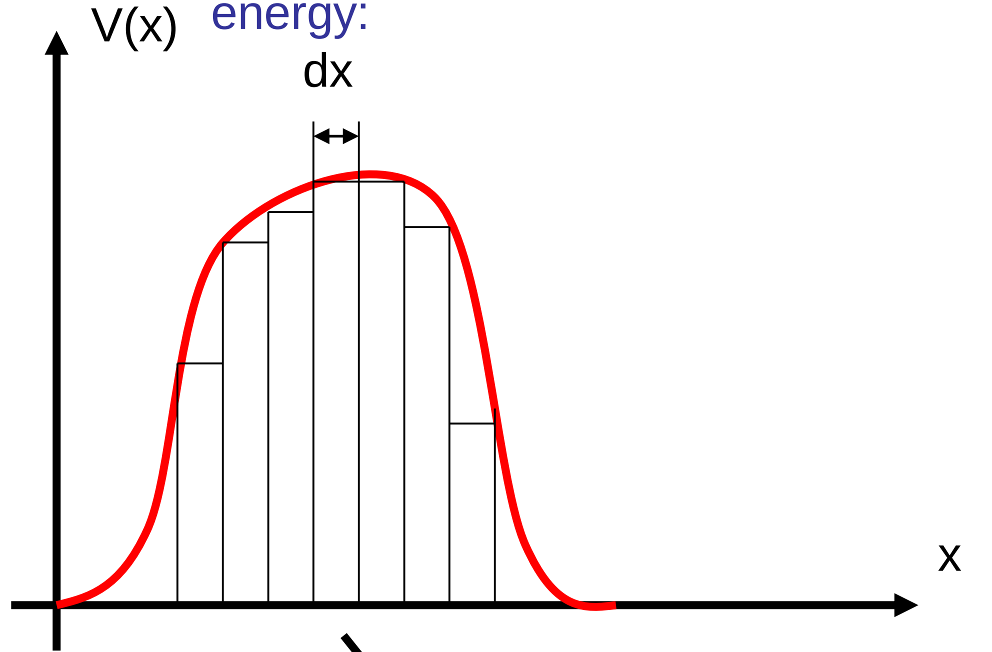
- Method of approximation:



$$-\frac{\hbar^2}{2m}\psi'' = \left(E - V(x)\right)\cdot\psi$$

$$\psi = Be^{-kx} \qquad k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

At the case of not constant potential energy:



$$t \approx \frac{\Psi_2(d)}{\Psi_2(0)} \approx \exp\left\{-\sqrt{\frac{2m(V_0 - E)}{\hbar^2}}d\right\}$$

Generalization

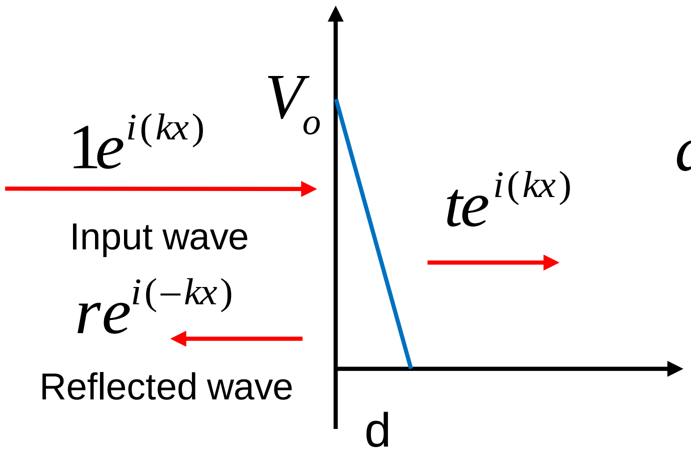
$$t \approx \exp\left\{-\sqrt{\frac{2m(V(x) - E)}{\hbar^2}}dx\right\}$$

$$t = \prod_i t_i \approx \exp\left\{-\int_{x_1}^{x_2}\sqrt{\frac{2m(V(x) - E)}{\hbar^2}}dx\right\}$$

The tunnel effect VII.

Fowler-Northeim tunneling

Potential gate: electric field



$$d = \frac{V_o}{E}$$

Schrödinger equation (stationary state):

$$-\frac{\hbar^2}{2m}\psi'' = \left(\varepsilon - V(x)\right)\cdot\psi$$

Wentzel-Kramers-Brillouin

Slowly changing potential:

Probe function (WBK appr.): $Ae^{-u(x)}$

$$\frac{\partial^2 u(x)}{\partial x^2} = 0 \quad \text{, but} \quad \frac{\partial u(x)}{\partial x} \neq 0$$

$$\frac{\hbar^2}{2m}\left(\frac{\partial u(x)}{\partial x}\right)^2 = \left(V(x) - \varepsilon\right) \rightarrow u(x) = \frac{\sqrt{2m}}{\hbar}\int_o^x \left(V(x') - \varepsilon\right)^{1/2} dx' = \frac{\sqrt{2m}}{\hbar}\int_o^x \left(V_o - Ex' - \varepsilon\right)^{1/2} dx'$$

Probability of transmission:

$$|t|^2 = |\Psi(d)|^2 = \exp\left\{-2\frac{\sqrt{2m}}{\hbar}\int_o^x \left(V_o - Ex' - \varepsilon\right)^{1/2} dx'\right\} = \exp\left\{\frac{4\sqrt{2m}}{3\hbar E}\left(V_o - \varepsilon\right)^{3/2}\right\}$$

Tunneling current can be increased by E: $\sim \exp\left\{-\frac{1}{E}\right\}$

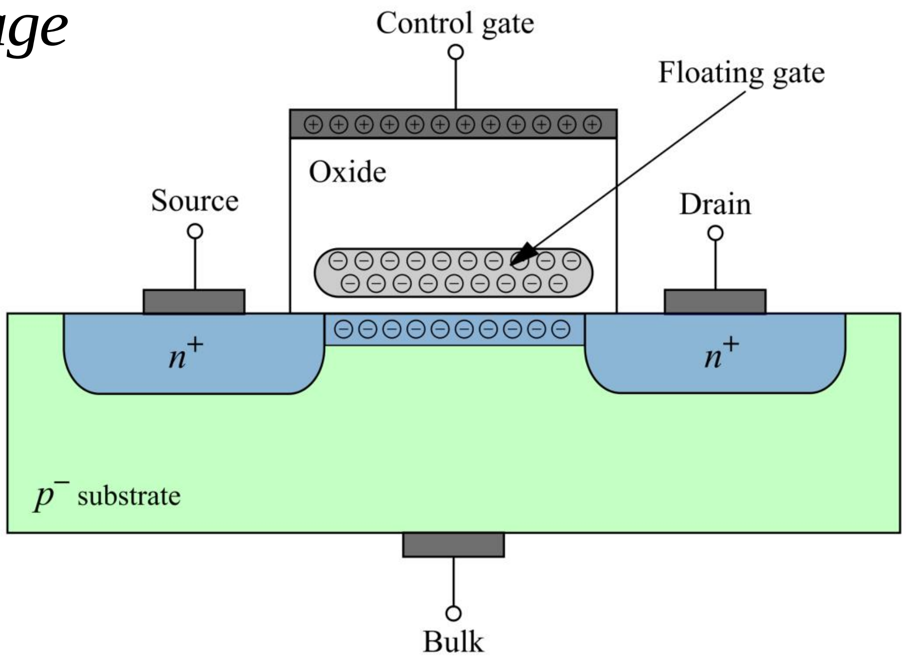
The tunnel effect VIII.

Application: Flash memory

https://en.wikipedia.org/wiki/Floating-gate_MOSFET

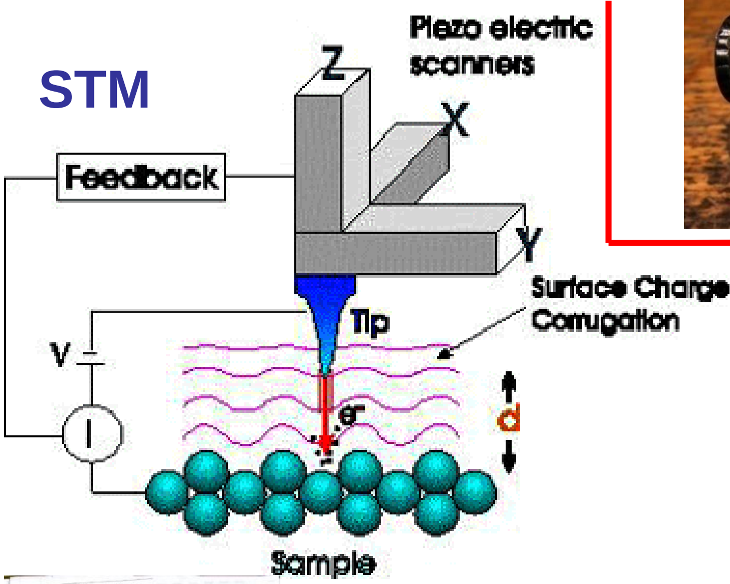
Reliability: > 1000 years, under data transfer of 20 GB/day

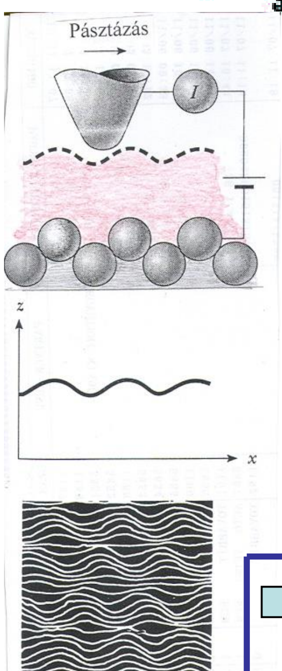
$$10^5 \quad \textit{electrons} \quad \textit{gate} \quad \textit{voltage}$$

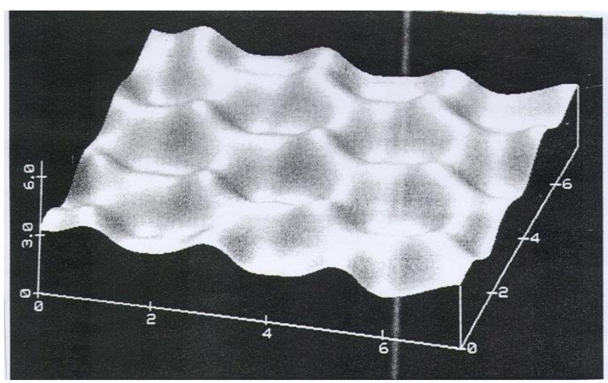


The tunnel effect IX.

STM



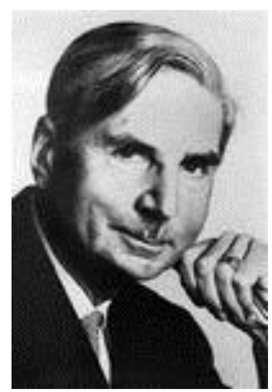




Metal-semiconductor junction (a diode)

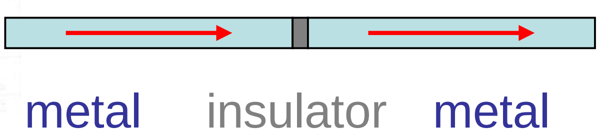
1938

Walter Hermann Schottky
German physicist
(1886-1976)



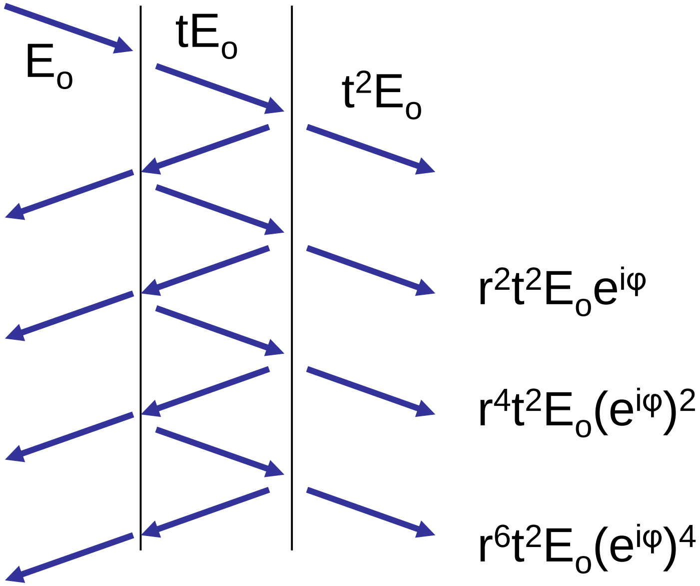
Leo Esaki (1925-)
Nobel-prize:1973

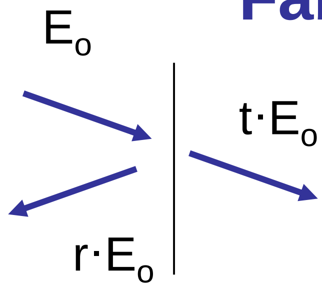
…where his research on heavily-doped Ge and Si resulted in the discovery of the ***Esaki tunnel diode***; this device constitutes the first quantum electron device.



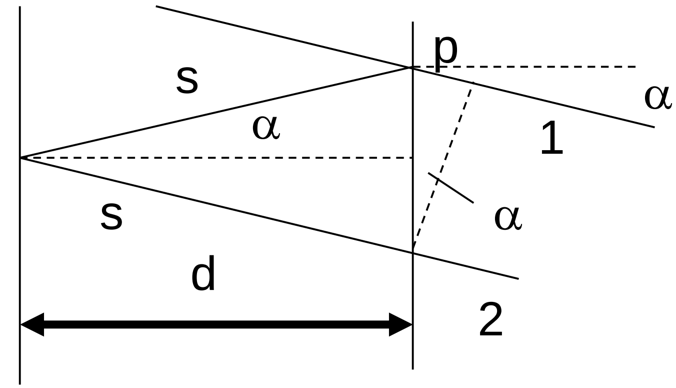
Barrier = insulator

Fabry-Perot interferometer I.





T=t² és R=r²



Path difference: $\Delta=2s-p$

$\Delta=2d\cos\alpha$

$$E_T = E_o t^2\left(1+r^2e^{i\varphi}+r^4e^{i2\varphi}+r^6e^{i3\varphi}+\ldots\right)$$

$$\varphi=\frac{2\pi}{\lambda}2d\cos\alpha$$

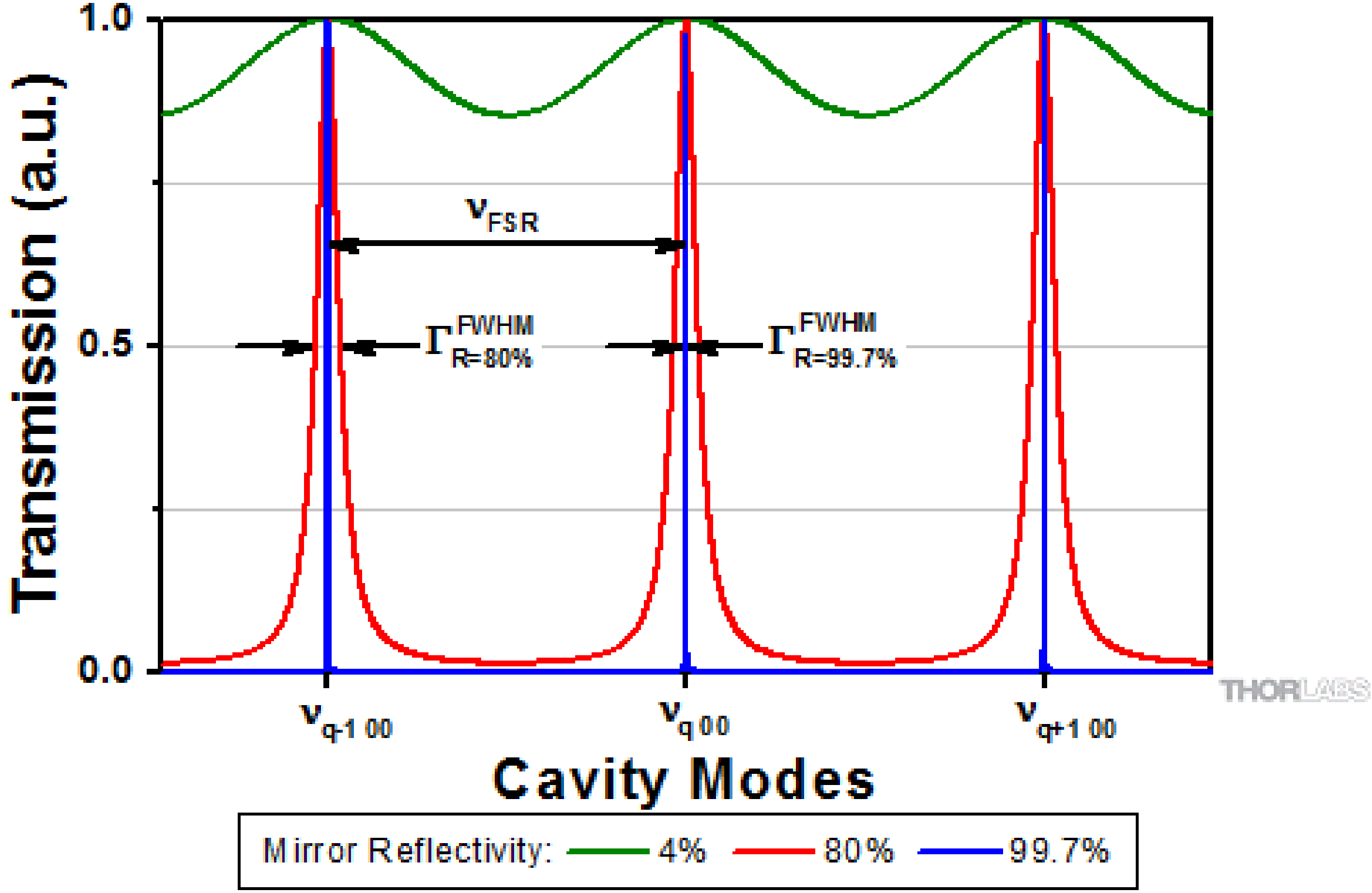
$$E_T=\frac{E_o t^2}{1-r^2e^{i\varphi}} \quad \text{és} \quad I_T=\frac{1}{2}EE^*$$

$$I_T=\frac{\frac{1}{2}t^2E_o^2}{1-2r^2\cos\varphi+r^4}$$

$$I_T=\frac{I_o}{1+\frac{4R\sin^2(\varphi/2)}{(1-R)^2}}$$

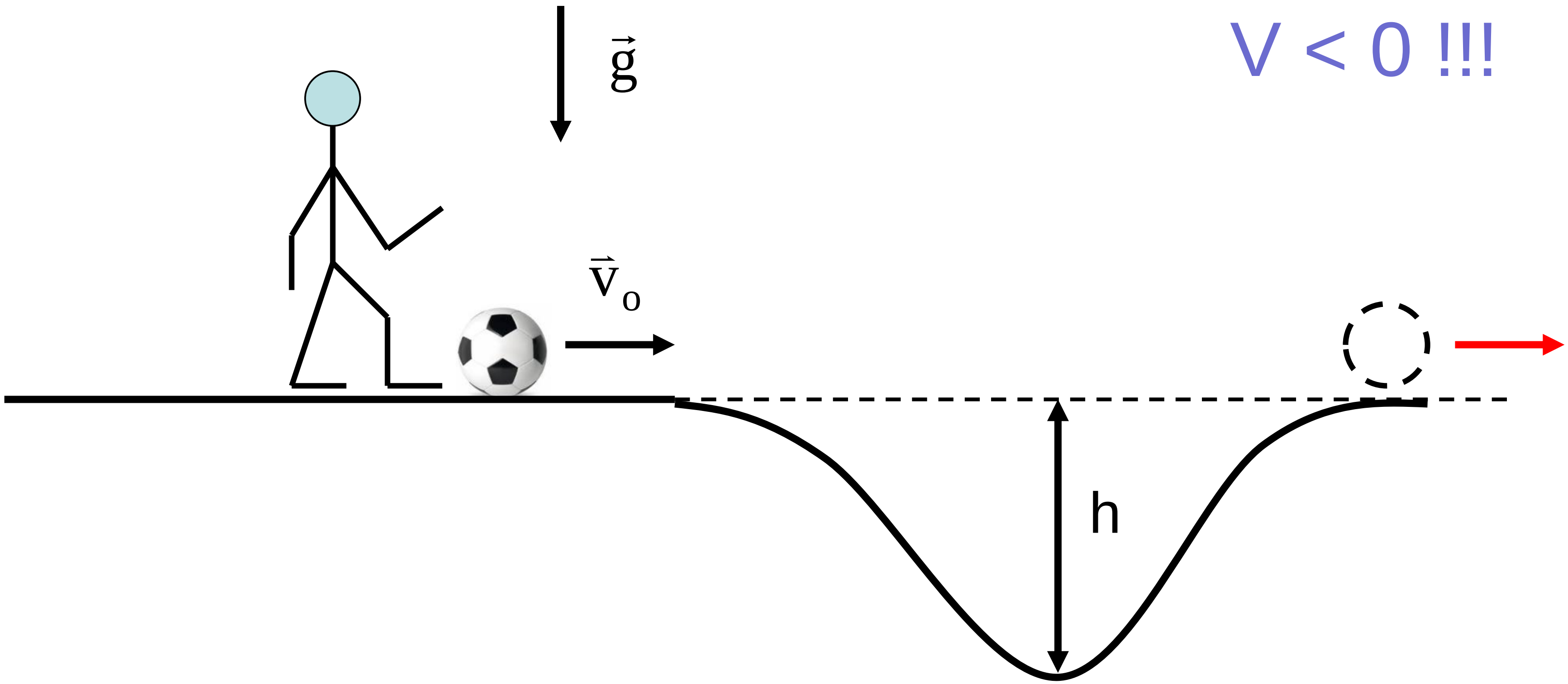
$$\sin^2\frac{\varphi}{2}=\frac{1-\cos\varphi}{2} \quad \text{és} \quad T=t^2 \quad \text{és} \quad R=r^2$$

Fabry-Perot interferométer II.



Thorlabs – catalog

V < 0 !!!

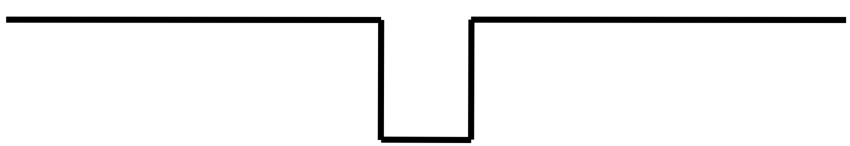


$Ae^{i(-kx+\omega t)}$

$Be^{i(-kx+\omega t)}$

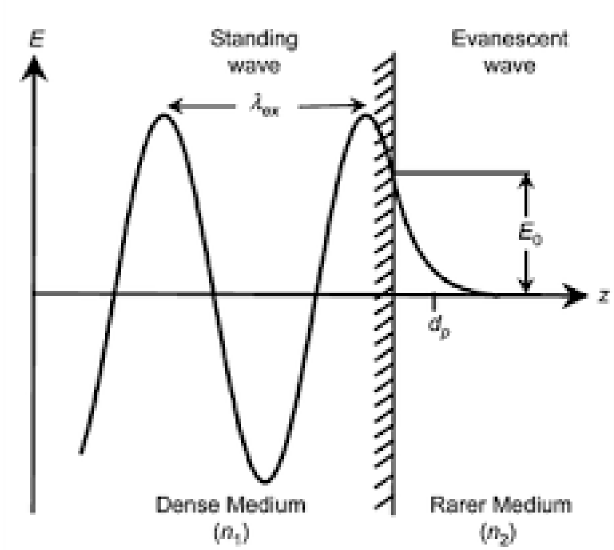
$Ce^{i(+kx+\omega t)}$

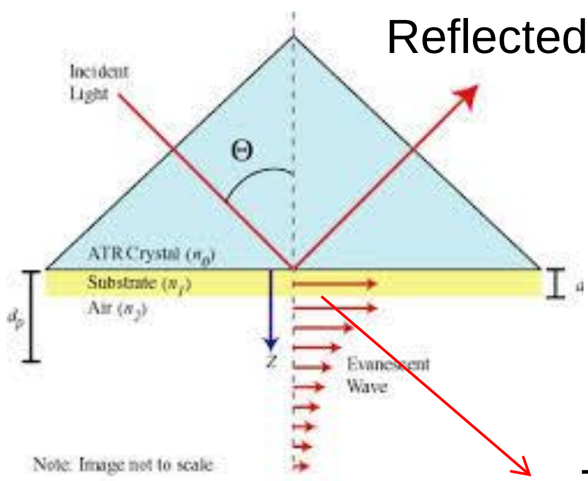
$$R = \left|\frac{C}{A}\right|^2$$



$$T = \left|\frac{B}{A}\right|^2$$

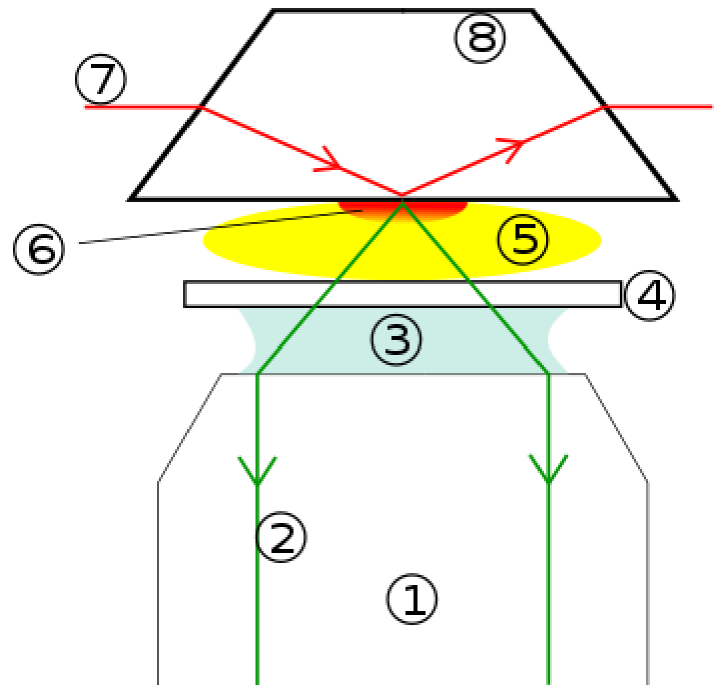
Evanescent waves (analogy)





Reflected wave

Transmitted wave



Microscope of total reflection (wikipedia)

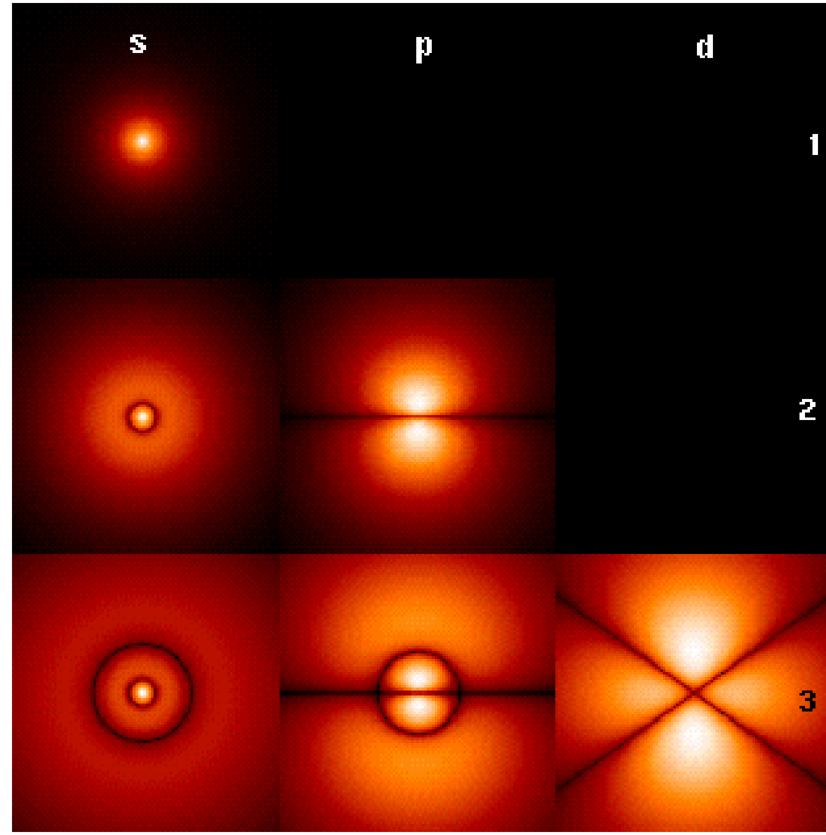
The H-atom

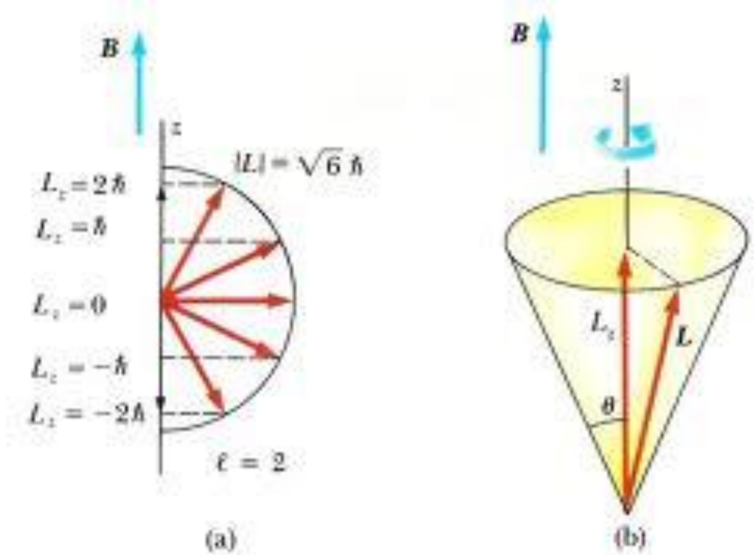
$$-\frac{\hbar^2}{2m}\Delta\psi+\left(-\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{r}\right)\cdot\psi=E\cdot\psi$$

$$\psi_{n,l,m}\rightarrow E_n=-\frac{13.6eV}{n^2}$$

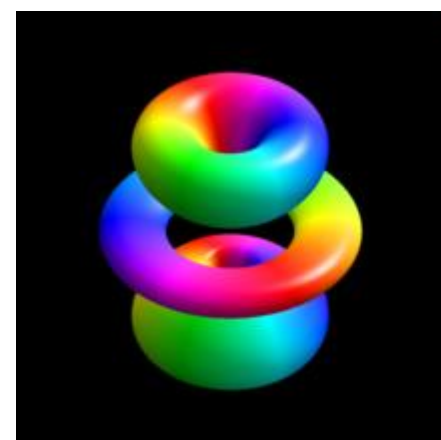
$$L=\hbar\sqrt{l(l+1)} \qquad l=0,1,2,\ldots(n-1)$$

$$L_z=\hbar m_L \qquad m_l=0,\pm1,\pm2,\ldots\pm l$$



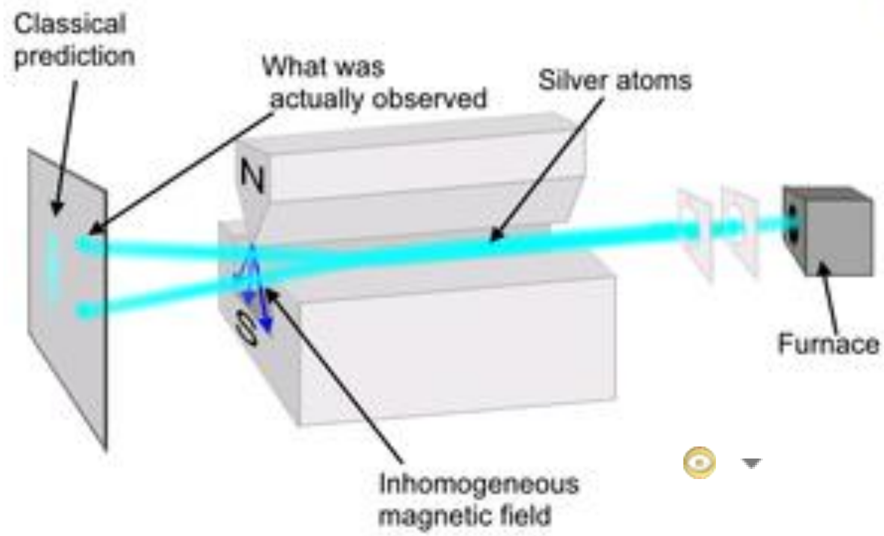


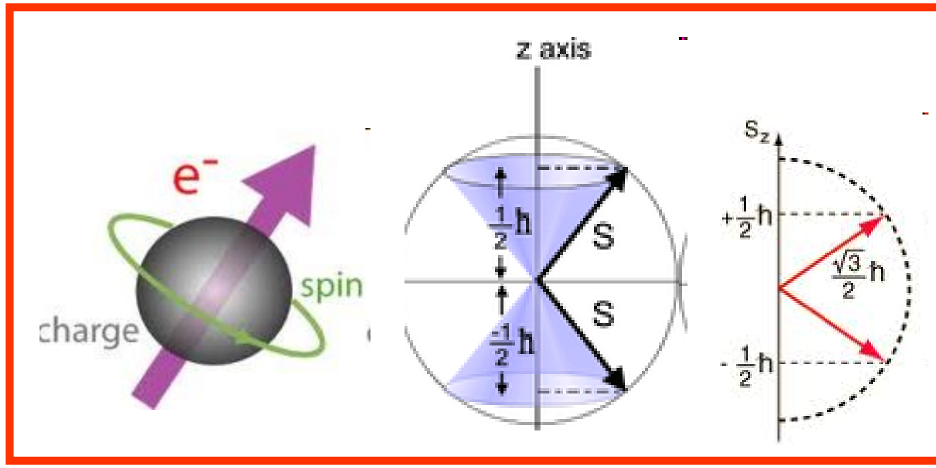
$$P(\vec{r})=\left|\psi_{4,3,1}\right|^2$$

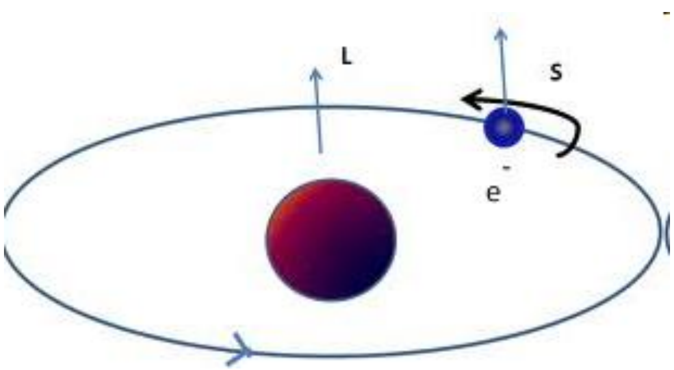


The spin of electron

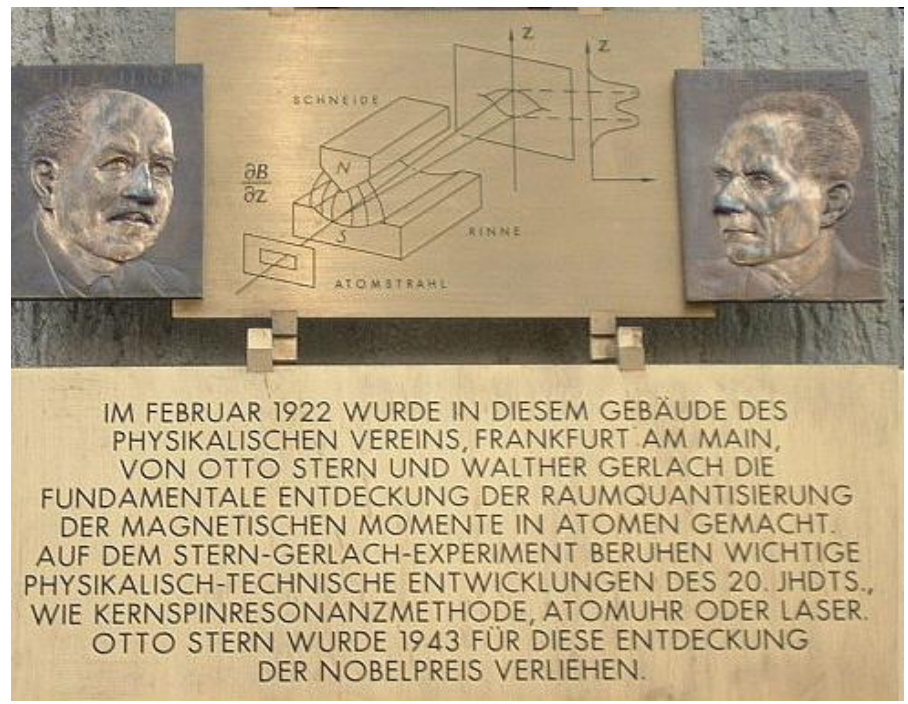
Stern–Gerlach-experiment





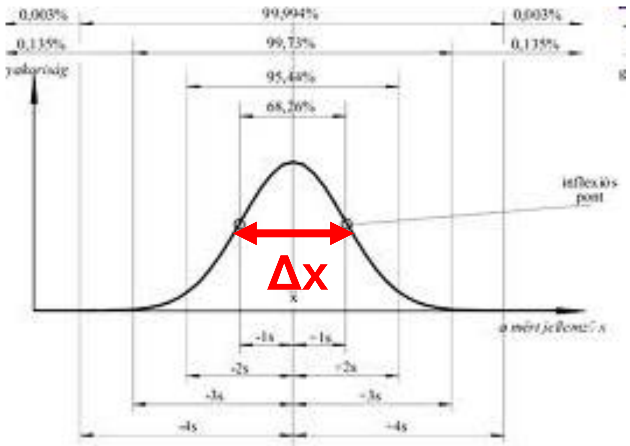


$$\psi_{n,l,m} \rightarrow \psi_{n,l,m,s}$$



Heisenberg's uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

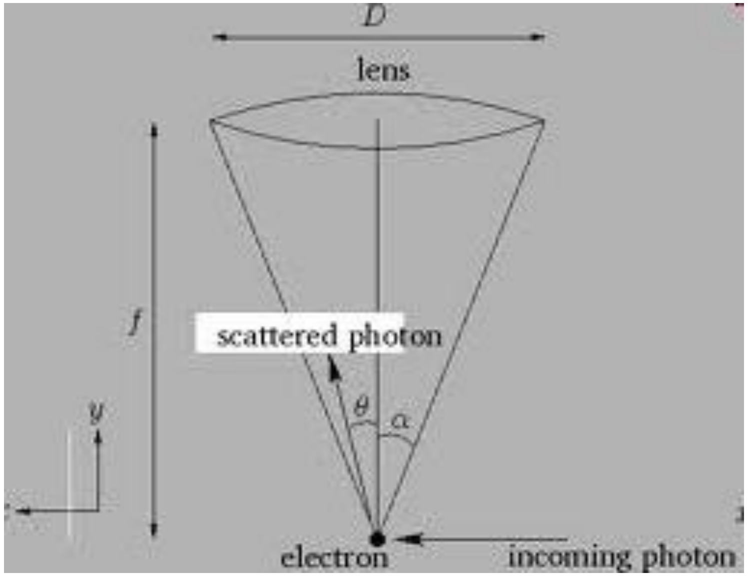


$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Lifetime of excitation → spectral width

Einstein:
God does not play dice…

Quantum physics:
Yes, he does…!!!



Resolution of microscope:

$$\Delta x = \frac{0.61\lambda}{\sin\alpha}$$

The electron is tossed by
a photon: the uncertainty of ***p***:

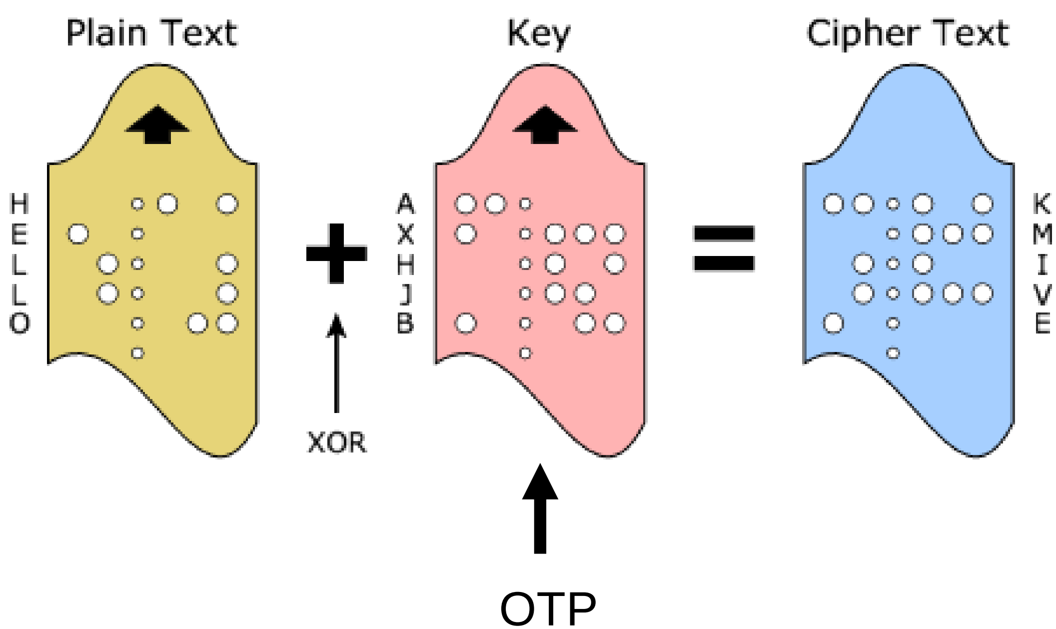
$$\Delta p = p\sin\alpha = \frac{h}{\lambda}\sin\alpha$$

This is not a proof!!!

Quantum Key Distribution I.

*In cryptography, the one-time pad (**OTP**) is a type of encryption that is impossible to crack if used correctly. Each bit or character from the plaintext is encrypted by a modular addition with a bit or character from a secret random key (or pad) of the same length as the plaintext, resulting in a **ciphertext**. If the key is truly random, as large as or greater than the plaintext, never reused in whole or part, and kept secret, the ciphertext will be impossible to decrypt or break without knowing the key.* (Wikipedia)

Vernam code (Vernam cipher):



Quantum Key Distribution II. (How to send a secret message?)

James Bond

?? message ??

e: 01000101

message = **M** = ete

t: 01010100

M = 01000101 01010100 01000101

Key : **K** = 11011000 00010100 11010111

Addition law:

$$0 \oplus 0 = 0$$

$$0 \oplus 1 = 1$$

$$1 \oplus 0 = 1$$

$$1 \oplus 1 = 0$$

Ciphertext = **C** = M⊕K

M	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**1**
K	1	1	0	1	1	0	0	0
C	1	0	0	1	1	1	0	1

Quantum Key Distribution III.

To recover (unwrap) the message from the ciphertext:

$$M^* = K \oplus C$$

J.B.: $C = M \oplus K$

$$M^* = K \oplus (M \oplus K) = M \oplus (K \oplus K) = M$$

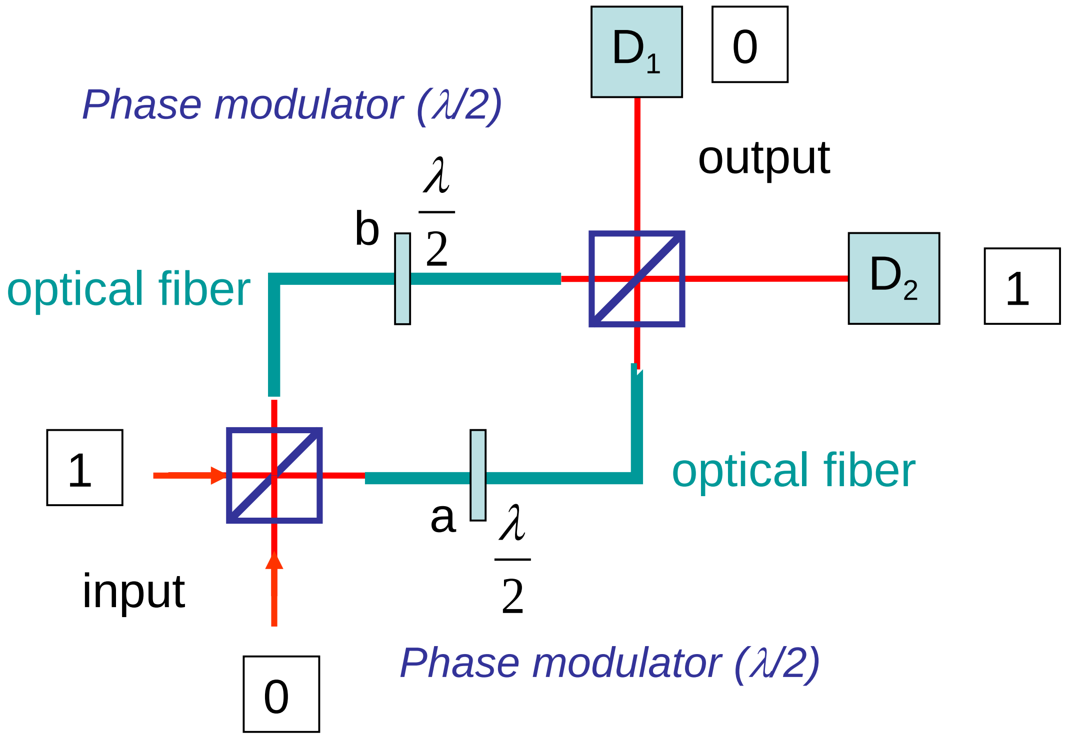
C	1	0	0	1	1	1	0	1
K	1	1	0	1	1	0	0	0
M	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**1**

But how the key can be distributed???

Quantum cryptography I.

Quantum cryptography. The idea: **Bennett** & **Brassard** (**1984**).
(Quantum Key Distribution scheme).
The name of protocol : **BB84**.

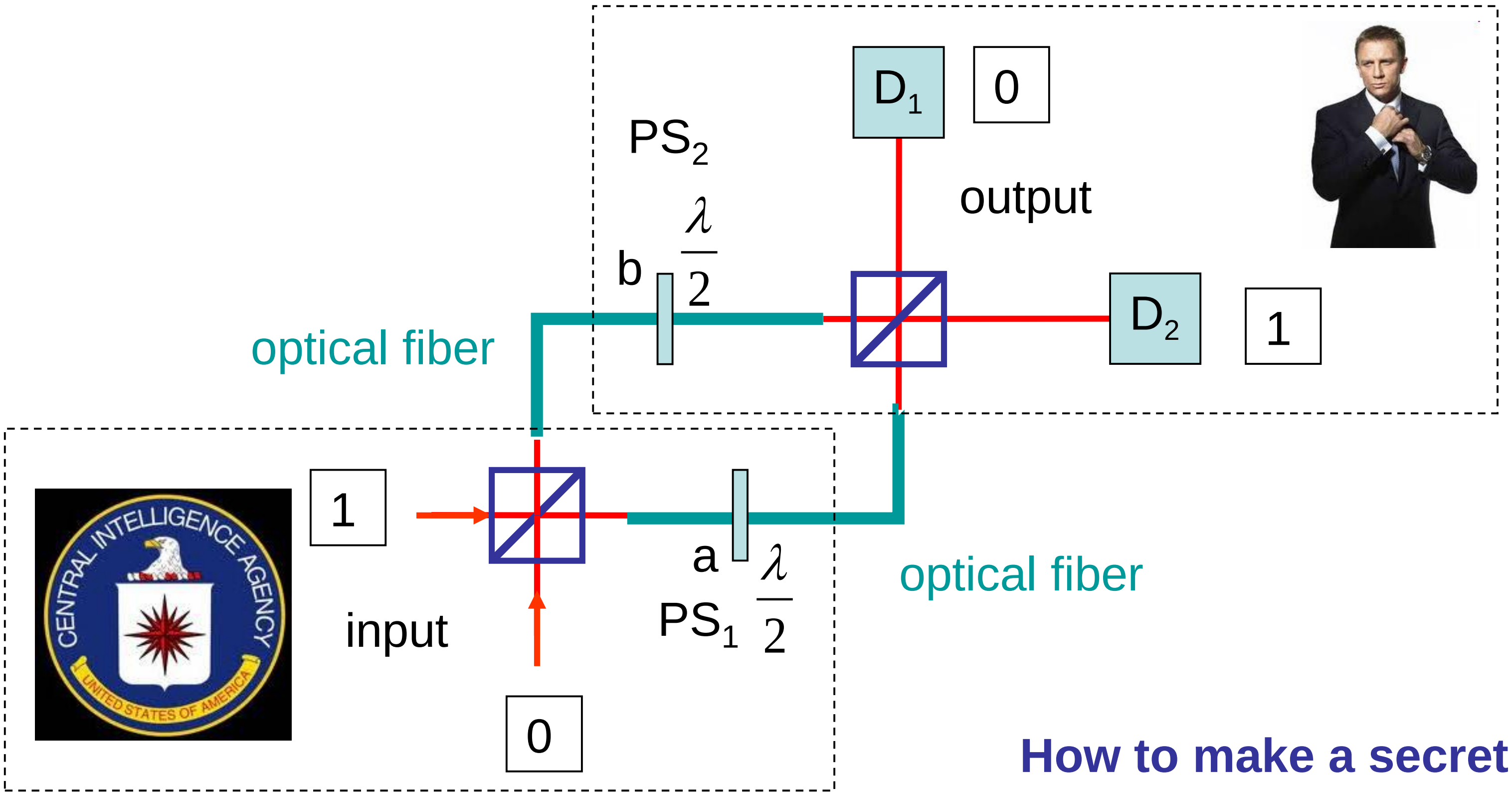
Non polarizing beam splitter:
$\Delta\varphi=\pi/2$ in reflection.



M-Z interferometer
Input - output

	a	b	c	d
Symbol	**BS_1 input**	**PS_1**	**PS_2**	**BS_2 output**
1	0	0	0	0
2	1	0	0	1
3	0	$\lambda/2$	$\lambda/2$	0
4	1	$\lambda/2$	$\lambda/2$	1
5	1	$\lambda/2$	0	0
6	1	0	$\lambda/2$	0
7	0	$\lambda/2$	0	1
8	0	0	$\lambda/2$	1

Quantum cryptography II. (Quantum communication)



How to make a secret key :

1. The CIA agent uses a random number generator to decide which input has to be used.
2. The CIA agent uses a random number generator to decide switch on or not PS_1.
3. Mr. Bond uses a random number generator to decide switch on or not PS_2.
4. They compare – for example - every odd results by phone (public channel).
5. If the input and output result agree with the M-Z protocol, the symbols of even numbers give the secret key.

Random generator simply: flip a coin (heads or tails?)