Chapter 12. Diving Panoramas

- *In which local direction (or directions) does a local inertial rain observer look to see a given star as she passes coordinate r?*

- *In which direction (or directions) does a shell observer stationary at r and φ coordinates look to see the same star?*

- *How does the panorama of the heavens change for the local rain observer as she descends?*

- *Is gravitationally blue-shifted starlight lethal for the rain observer as she approaches the singularity? Is this starlight more dangerous than killer tides?*

- *How close to the singularity will the rain observer survive?*

- *What is the last thing the local rain observer sees?*

C H A P T E R

12

Diving Panoramas

Edmund Bertschinger & Edwin F. Taylor [*]

Tell all the truth but tell it slant –
Success in Circuit lies
Too bright for our infirm Delight
The Truth's superb surprise
As Lightning to the Children eased
With explanation kind
The Truth must dazzle gradually
Or every man be blind –

—Emily Dickinson

12.1 ■ FALLING INTO THE BLACK HOLE

See the same beam in two different directions.

"What is it like to fall into a black hole?" Our book thus far can be thought of as preparation to answer this question. The simplest possible answer has two parts: "What do I *feel* as I descend?" and "What do I *see* as I descend?" You *feel* tidal accelerations that—sorry about this—"spaghettify" you before you reach the singularity (Query 23, Section 7.9). As you descend, you *see* a changing panorama of the starry heavens, developed in this chapter and narrated in Section 12.7.

"What's it like to fall into a black hole?"

The preceding Chapter 11 plotted trajectories of starlight in *global* rain coordinates and told us which beams (plural!) connect a given distant star to the *map location* of an observer. But that chapter said nothing about the direction in which that observer looks to see each beam or the beam energy she measures. These are the goals of the present chapter.

Where does a rain observer look to see a given beam?

Comment 1. The rain diver
 To dive—to be a diver—means to free-fall radially inward toward a center of attraction. A diver can drop from rest on—or be hurled radially inward from—any shell, including a shell far from the black hole. Among divers, the rain observer is a special case: a diver that drops from rest far away. In this chapter the word *diver* most often means the *rain diver*.
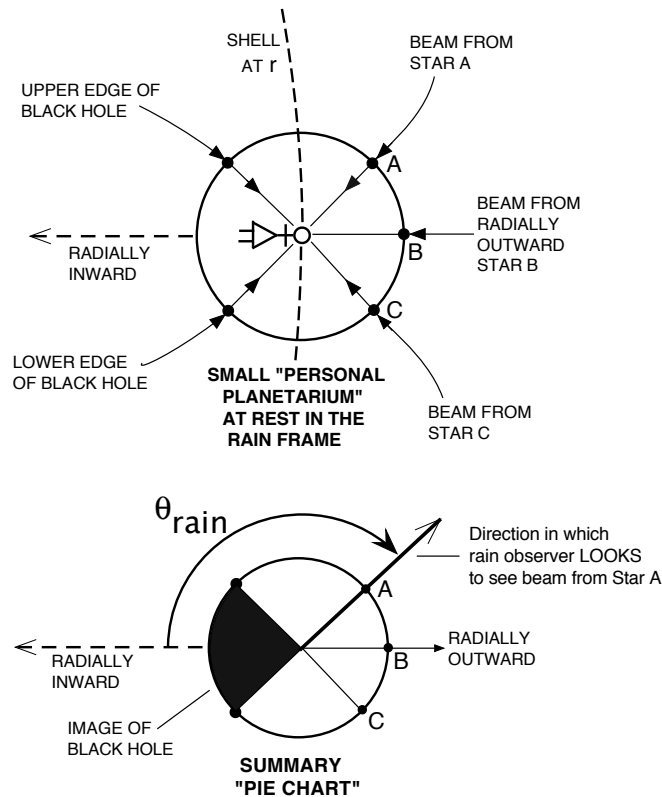
FIGURE 1   *Upper panel:* The **personal planetarium** is a small transparent sphere that encloses—and falls with—the local rain observer. The observer marks on the inside of this sphere the point-images of stars. She also draws a circle around the visual edge of the black hole. *Lower panel:* The **pie chart** summarizes rain observer markings; a black "pie slice" spans the visual image of the black hole. To see a particular beam, the rain observer *looks* in the direction $\theta_{\rm rain}$, which she measures clockwise from the radially inward direction.

12.2 ■ THE PERSONAL PLANETARIUM

⁵³ *Enjoy the view in weightless comfort.*

"Personal planetarium" to record local observations

⁵⁴ How does the local rain observer view and record starlight beams? One
⁵⁵ practical answer to this question is the **personal planetarium**: a transparent
⁵⁶ sphere at rest in the observer's local inertial frame with the observer's eye at
⁵⁷ its center (upper panel of Figure 1). Light beams run straight with respect to
⁵⁸ this local inertial frame, as shown in the figure. The observer marks on the
⁵⁹ inside of the transparent sphere the points of light she sees from stars in all
⁶⁰ directions; she also draws on the inside of the sphere a circle around the visual
⁶¹ edge of the black hole.
⁶²   We call the lower panel in figure 1 the **pie chart**. The pie chart takes its
⁶³ name from the standard graphical presentation whose black "pie slice" shows

64 the fraction of some quantity as the proportion of the whole. In our pie chart
65 the pie slice shows the range of visual angles covered by the black hole.
66   On the personal planetarium sphere the observer locates a star with the
67 angle θ_{rain} between the center of the black hole image and the dot she has
68 placed on the image of that star. For simplicity, we omit the coordinate
69 subscript "obs" for "observer" used in Chapter 11.

70   **DEFINITION 1. Angle θ_{rain}**
71   The observer in the personal planetarium looks in the direction θ_{rain} to
Definition: 72   see the to see any given star. We define angle $\theta_{\text{rain}} = 0$ to be radially
angle θ_{rain} 73   inward, from the observer's eye toward the center of the black hole and
74   the positive angle θ_{rain} to be clockwise from this direction measured in
75   her local rain frame (lower panel, Figure 1).

Full 3-dimensional   76   Can the local rain observer see a star that lies out of the plane of this
rain observer's   77 page? Of course: Every star lies on *some* slice determined by three points: the
panorama   78 star, the rain observer's eye, and the map coordinate $r = 0$. To encompass all
79 stars in the heavens, rotate each of the circles in Figure 1 around its horizontal
80 radial line. This rotation turns the pie chart into a sphere and the black hole
81 "pie slice" into a cone. From inside her planetarium, the local rain observer
82 sees the full panorama of stars in the heavens.

83 **?**   **Objection 1.** *You say, "From inside her planetarium, the local rain observer*
84   *sees the full panorama of stars in the heavens." Why isn't that the end of*
85   *the story? What more does the rain observer need to know?*

86 **!**   If she is satisfied to describe a general view of the heavens, that is
87   sufficient for her. However, she may want to know, for example, where to
88   look to see Alpha Centauri, one of Earth's nearest neighbors, as she
89   plunges past $r = 4M$. The present chapter tells her the angle θ_{rain} in
90   which she looks to see the star located at global angle ϕ_∞. Finding θ_{rain}
91   of a star is a two-step process: The present chapter says in what local
92   direction θ_{rain} the rain observer looks to see a beam with a given value of
93   b. The analysis from Chapter 11 then tells us the global angle ϕ_∞ of the
94   star that emits the beam with that value of b. Angle θ_{rain} depends on the
95   observer's instantaneous global coordinate r, so varies with r as she
96   descends. *Result:* a changing panorama, described in Section 12.8.

12.3 ■ RAIN FRAME VIEW OF LIGHT BEAMS

98 *"I spy with my little eye . . . "*

99 As she falls past r, the local rain observer sees a beam from a distant star at
From stone   100 the angle θ_{rain} with respect to the radially inward direction. We want an
to light   101 expression for this observation angle as a function of r and the b-value of that
102 light beam. Chapter 11 defines b as the ratio $b \equiv L/E$. Equation (35) in
103 Section 7.5 gives the expression for the map energy of a stone:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau} \qquad \text{(stone)} \qquad (1)$$

104   The expression for map angular momentum of a stone comes from
105   equation (10) in Section 8.2:

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} \qquad \text{(stone)} \qquad (2)$$

b in global rain   106   Divide both sides of (2) by the corresponding sides of (1) and divide
coordinates   107   numerator and denominator of the result by dT. The symbol m cancels on the
108   left side to yield the ratio $b \equiv L/E$ for light:

$$b \equiv \frac{L}{E} = \frac{r\left(\dfrac{rd\phi}{dT}\right)}{\left(1 - \dfrac{2M}{r}\right) - \left(\dfrac{2M}{r}\right)^{1/2}\dfrac{dr}{dT}} \qquad \text{(light)} \qquad (3)$$

109   Equation (3) expresses b in rain coordinates. But the local planetarium
110   observer measures visual angles in her local inertial rain frame. Box 4 in
111   Section 7.5 expressed local rain frame coordinates in global coordinates:

$$\Delta t_{\text{rain}} \equiv \Delta T \qquad (4)$$

$$\Delta y_{\text{rain}} \equiv \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2}\Delta T \qquad (5)$$

$$\Delta x_{\text{rain}} \equiv \bar{r}\Delta\phi \qquad (6)$$

112

113   Light moves with speed unity in the local inertial rain frame,
114    $\Delta s_{\text{rain}}/\Delta t_{\text{rain}} = 1$, so the Pythagorean Theorem provides labels for legs of the
115   right triangle in Figure 2:

$$\Delta x_{\text{rain}}^2 + \Delta y_{\text{rain}}^2 = \Delta s_{\text{rain}}^2 = \Delta t_{\text{rain}}^2 \qquad \text{(light, in local inertial rain frame)} (7)$$

Find rain observer   116   At what angle θ_{rain} does the rain observer look in her local frame in order
angle to see   117   to see the incoming beam with impact parameter b? The incoming beam in
light with *b*   118   Figure 2 represents any one of the multiple beams arriving at the rain observer
119   from a single star. In Figure 2 the symbols A, B, and C stand for the
120   (positive) lengths of the sides of the right triangle. In contrast, the
121   inward-moving light has negative components of motion radially and
122   tangentially in the figure. The local frame time lapse Δt_{rain} is positive along
123   the worldline. Expressing these results in local rain coordinates (4) through (6)
124   leads to the following expressions for sine and cosine of the angle θ_{rain}:
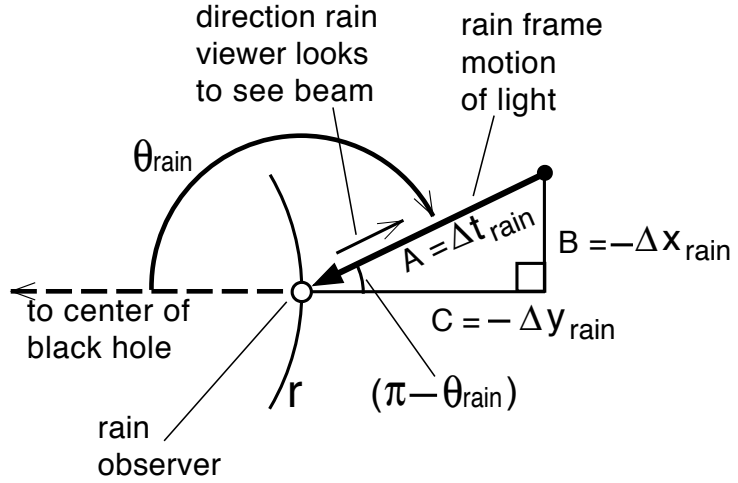
FIGURE 2   The beam from a distant star reaches the local rain observer as she dives inward past the shell at r. She measures the observation angle θ_{rain} clockwise with respect to the radially inward direction. Letters A, B, and C are the (positive) lengths of the legs of the right triangle in the local rain frame. As the light approaches the observer, it has a clockwise tangential component in the local rain frame so Δx_{rain} is negative (and $-\Delta x_{\mathrm{rain}}$ is positive, equal to side B, as shown). The light also has a radially inward component, so Δy_{rain} is also negative (and $-\Delta y_{\mathrm{rain}}$ is positive, equal to C as shown). The hypotenuse is $\Delta s_{\mathrm{rain}} = \Delta t_{\mathrm{rain}}$ from (7) is positive and lies along the worldline of the beam Δt_{rain}.

$$\sin\theta_{\mathrm{rain}} = \sin(\pi - \theta_{\mathrm{rain}}) \tag{8}$$

$$= \lim_{A\to 0}\frac{B}{A} = \lim_{\Delta t_{\mathrm{rain}}\to 0}\frac{-\Delta x_{\mathrm{rain}}}{\Delta t_{\mathrm{rain}}} = \lim_{\Delta T\to 0}\frac{-\bar{r}\Delta\phi}{\Delta T} \qquad \text{so that}$$

$$\sin\theta_{\mathrm{rain}} = -\frac{r\,d\phi}{dT} \tag{9}$$

A similar procedure leads to an expression for $\cos\theta_{\mathrm{rain}}$:

$$\cos\theta_{\mathrm{rain}} = -\cos(\pi - \theta_{\mathrm{rain}}) \tag{10}$$

$$= -\lim_{A\to 0}\frac{C}{A} = -\lim_{\Delta t_{\mathrm{rain}}\to 0}\frac{-\Delta y_{\mathrm{rain}}}{\Delta t_{\mathrm{rain}}} = \lim_{\Delta T\to 0}\frac{\Delta r}{\Delta T} + \left(\frac{2M}{r}\right)^{1/2}$$

$$= \frac{dr}{dT} + \left(\frac{2M}{r}\right)^{1/2} \qquad \text{so that}$$

$$\frac{dr}{dT} = \cos\theta_{\mathrm{rain}} - \left(\frac{2M}{r}\right)^{1/2} \tag{11}$$

Substitute expressions (9) and (11) into (3), square both sides of the result, then eliminate the remaining sine squared with the identity $\sin^2\theta = 1 - \cos^2\theta$. Rearrange the result to yield the following quadratic equation in $\cos\theta_{\mathrm{rain}}$:

$$\left(1 + \frac{b^2}{r^2}\frac{2M}{r}\right)\cos^2\theta_{\text{rain}} - 2\frac{b^2}{r^2}\left(\frac{2M}{r}\right)^{1/2}\cos\theta_{\text{rain}} - \left(1 - \frac{b^2}{r^2}\right) = 0 \qquad (12)$$

Rain observer
angle to see
light with b

129   Solve the quadratic equation (12) to find an expression for $\cos\theta_{\text{rain}}$:

$$\cos\theta_{\text{rain}} = \frac{\dfrac{b^2}{r^2}\left(\dfrac{2M}{r}\right)^{1/2} \pm F(b,r)}{1 + \dfrac{b^2}{r^2}\dfrac{2M}{r}} \qquad \text{(light)} \qquad (13)$$

Sign in numerator for global motion of light: + for $dr > 0$, − for $dr < 0$

130

131   where, from equation (17) in Section 11.2,

$$F(b,r) \equiv \left[1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{1/2} \qquad \text{(light)} \qquad (14)$$

132

133

QUERY 1. Equations for $\cos\theta_{\text{rain}}$ (*Optional*)

A. Use equations (3), (9), and (11) to derive quadratic equation (12) for $\cos\theta_{\text{rain}}$.
B. Solve quadratic equation (12) to derive (13) for $\cos\theta_{\text{rain}}$.

137

138   There is an ambiguity in (13) because $\cos\theta = \cos(-\theta)$. To remove this
139   ambiguity, substitute (9) and (11) into (3), then use (13) to substitute for
140   $\cos\theta_{\text{rain}}$. After considerable manipulation, the result is:

$$\sin\theta_{\text{rain}} = \frac{b}{r}\left[\frac{-1 \pm \left(\dfrac{2M}{r}\right)^{1/2} F(b,r)}{1 + \dfrac{b^2}{r^2}\dfrac{2M}{r}}\right] \qquad \text{(light)} \qquad (15)$$

141   which provides a stand-by correction of sign in (13). As in that equation, the
142   plus sign is for $dr > 0$ and the minus sign for $dr < 0$.

143

QUERY 2. Derivation of $\sin\theta_{\text{rain}}$ (*Optional*)

Carry out the derivation of the expression for $\sin\theta_{\text{rain}}$ in (15).
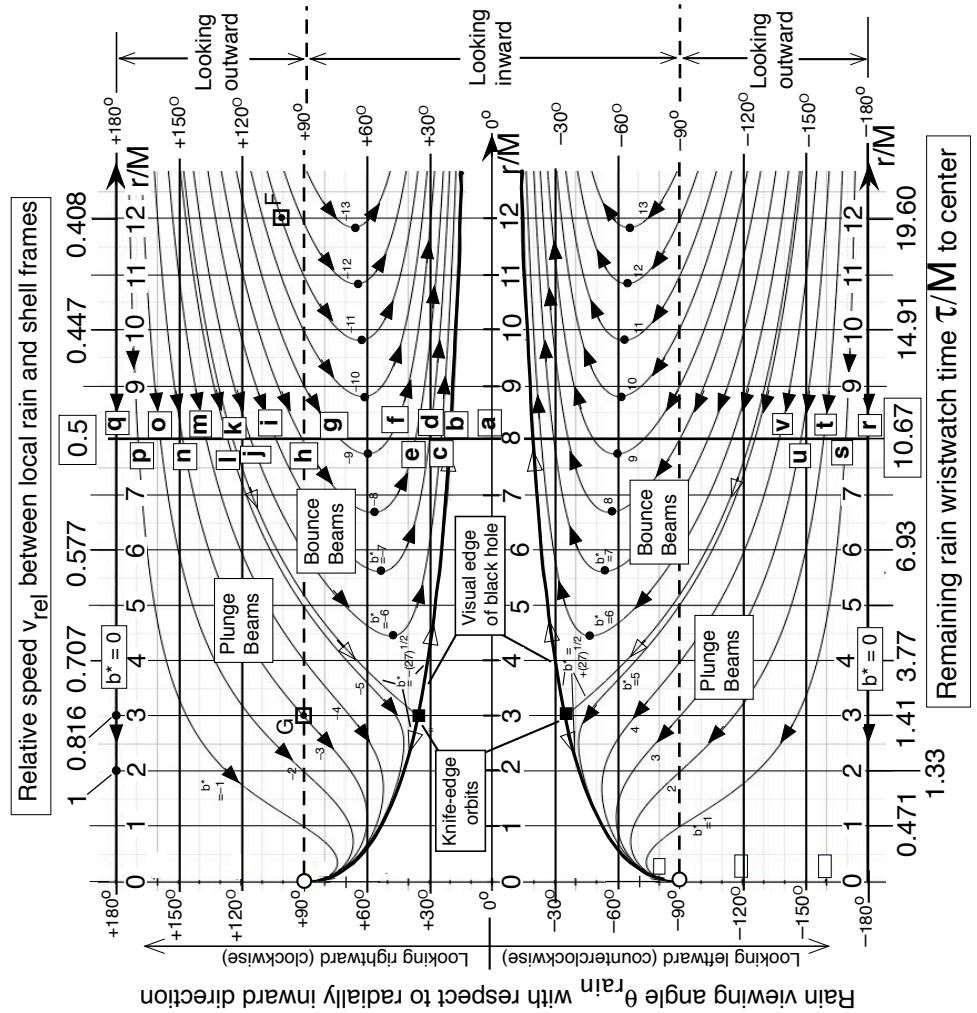
146

FIGURE 3   Angle θ_{rain} at which the local rain observer, passing the global location $(r, \phi = 0)$, looks to see starlight beams for several values of b/M, equation (13). To save space, we set $b^* \equiv b/M$. Arrows on the curve for each beam tell whether that beam is incoming or outgoing. A little black dot marks a turning point, the r-coordinate at which an incoming beam reverses dr to become an outgoing beam. Upper and lower three-branch curves with open arrowheads represent light with impact parameter $\pm b_{\text{critical}}/M = \pm (27)^{1/2}$. Two little black squares at $r = 3M$ represent circular knife-edge orbits of these critical beams on the tangential light sphere. As she descends, the rain observer sees the visual edge of the black hole at angles shown by the heavy curve. Reminder Figure 4 recalls the meaning of labels "Plunge Beams" and "Bounce Beams" in this figure. The text uses the lowercase labels **a** through **v** to explain details of this figure.

Plot rain observer
angle to see light
with b

147   Figure 3 plots equation (13) for several values of $b^* \equiv b/M$; it tells us in
148   which directions θ_{rain} the local rain observer looks to see starlight beams with
149   various impact parameters b. You can think of the vertical axis as unrolling
150   the polar angle of the local rain frame.

151   **Comment 2. Mirror image of upper and lower parts of Figure 3**
152   Equation (13) is a function of b^2, so cannot distinguish between positive and
153   negative values of b, which leads to the mirror symmetry of Figure 3 above an
154   below the horizontal $\theta_{\mathrm{rain}} = 0$ axis.

155   Numbers along the top of Figure 3 show shell speeds of the rain frame at
156   various r-coordinates. You can check these numbers with equation (23) in
157   Section 6.4:

$$v_{\mathrm{rel}} = \left(\frac{2M}{r}\right)^{1/2} \qquad \text{(shell speed of rain diver, } r \geq 2M) \qquad (16)$$

158   Numbers along the bottom of Figure 3 tell the remaining wristwatch time
159   τ the rain observer has before arriving at the singularity, from equation (2) in
160   Section 7.2:

$$\tau[r \to 0] = \frac{2^{1/2}M}{3}\left(\frac{r}{M}\right)^{3/2} \qquad \text{(rain diver } \tau \text{ from } r \text{ to center)} \qquad (17)$$

161   Figure 4 reminds us of the meaning of labels in Figure 3.
162   Figure 3 carries an immense amount of information. We list here a few
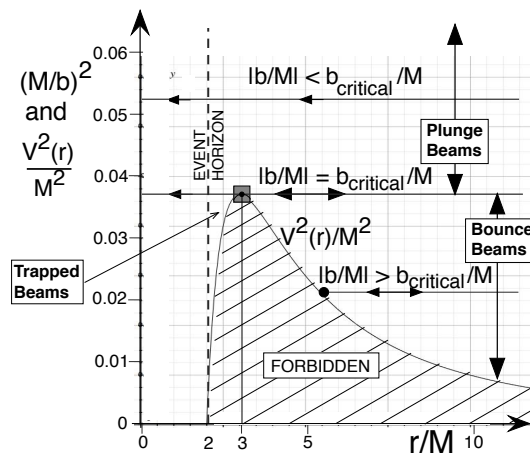163   examples:

FIGURE 4   Reminder figure of the meaning of labels in Figure 3 for starlight beams—from Figure 3 in Section 11.3. There are no trapped starlight beams.

164   **WHAT THE RAIN VIEWER SEES AS SHE PASSES** $r = 8M$

165   1. **Visual Edge of the Black Hole**

166   Two heavy curves straddle the r-axis in Figure 3; they form the outline
167   of a trumpet. As the falling rain observer passes r, she sees ahead of her
168   a black circle whose edges lie between angles $\pm\theta_{\rm rain}$ given by these two
169   heavy curves. As she descends further, the image of the black hole
170   grows until, at $r = 0$, the black hole covers the entire forward
171   hemisphere from $\theta_{\rm rain} = -90°$ to $\theta_{\rm rain} = +90°$.

Does a starlight beam escape?

172   2. **Dive or Escape?**

173   Figure 3 shows that the starlight beam with $|b| < b_{\rm critical}$ (plunge
174   beam) is incoming, with $dr < 0$ along its entire length. In contrast, the
175   starlight beam with $|b| > b_{\rm critical}$ (bounce beam) is initially incoming,
176   then reaches a turning point after which it becomes outgoing and
177   escapes. Every starlight beam that escapes has a turning point, marked
178   with the black dot in Figure 4.

179   3. **Direction in which the Rain Viewer Looks to See a Star**

180   Here is a detailed account of what the local rain observer sees as she
181   falls past $r = 8M$: In Figure 3, look at points labeled with lower-case
182   letters **a** through **v** on the vertical line at $r = 8M$. At point **a** on the
183   r-axis, the rain observer looks radially inward at the center of the black
184   hole, where she sees no starlight. When she looks somewhat to the right
185   (point **b**), she sees the visual edge of the black hole at $\theta_{\rm rain} \approx +23°$.
186   This image is brought to her by the outgoing beam with $b = -b_{\rm critical}$.
187   Farther to her right, at points **c, d, e,** and **f** she sees outgoing beams
188   with $b/M = -6, -7, -8$ and -9, respectively. She sees beams **a** through
189   **g** by looking *inward*—that is, at angles $0 \le \theta_{\rm rain} < 90°$, as labeled on
190   the right side of the figure. At point **h** she sees the beam with
191   $b/M = -8$ at $\theta_{\rm rain} = 90°$.

Where does $r = 8M$ shell observer look to see beams with different values of b?

192   **Comment 3. Look inward to see a beam coming from behind?**
193   At point **g** in Figure 3, the rain viewer sees *incoming beam* $b/M = -9$
194   ahead of her at approximate angle $70°$. *Question:* How can she look
195   *inward* to see a beam that the plot shows is coming from behind her?
196   *Answer:* Aberration (Section 12.9). In Query 6 you explain this paradox for
197   $r = 8M$.

198   To see beams **i** through **q**, the local rain observer looks *outward*, at
199   rain frame angles $90° < \theta_{\rm rain} \le 180°$, in which directions she sees
200   incoming beams with values of b from about $b/M = -9$ to $b/M = 0$.

201   Point **q** at the top of the diagram, for which $\theta_{\rm rain} = +180°$,
202   represents the radially outward direction, and is the same as point **r** at
203   the bottom of the diagram, for which $\theta_{\rm rain} = -180°$. Points **s** through **v**
204   represent directions in which the rain observer looks to the left of the
205   radially inward direction to see beams with $b/M = +1$ through
206   $b/M = +4$ as she turns her gaze back toward the center of the black
207   hole back at point **a**.

208   *Important:* The rain observer sees all of these beams
209   *simultaneously* in her local frame as she falls inward past $r = 8M$.

210   4. **Three-Dimensional Panorama**
211   Where does the local rain observer look to see beams from a star that

3-dimensional
panorama

212   does not lie in the plane of Figure 3? We know the answer to this
213   question: Rotate Figure 3 around the central r-axis until the candidate
214   star lies on the resulting global symmetry plane that contains the star,
215   the observer's eye, and $r = 0$. You can use this result to construct the
216   three-dimensional rain observer's view of every star in the heavens as
217   she passes every r-coordinate.

218   Figure 3 also reveals something complex but fascinating: Look at incoming
219   Plunge Beams at the top and bottom of Figure 3. As long as she remains
220   outside $r = 3M$, the rain observer sees Plunge Beams move steadily to smaller
221   visual angles θ_{rain}, even while the visual edges of the black hole are moving
222   steadily to larger visual angles (heavy "trumpet" lines). The rain observer sees

Star images
swing forward,
then back to
$\theta_{\mathrm{rain}} = \pm 90^{o}$.

223   *only* Plunge Beams after she passes inward through $r = 3M$; after that she
224   watches images of remote stars that emitted these Plunge Beams swing inward
225   to a minimum visual angle, then back out again to final angles $\theta_{\mathrm{rain}} = \pm 90°$.
226   In other words, after the rain observer passes inward through $r = 3M$, she sees
227   all the multiple images of stars in the heavens swing forward to meet the
228   expanding edge of the black hole, then remain at this edge as the black hole
229   continues to grow visually larger.

12.4■ CONNECT STAR MAP ANGLE TO RAIN VIEWING ANGLE.

231   *The rain observer views the heavens*

232   Can we now predict the sequence of panoramas of stars enjoyed by the rain
233   observer as she descends? Figure 3 is powerful: It tells us where each rain
234   observer looks to see a starlight beam with any given value of the impact

Want relation
between ϕ_{∞}
and θ_{rain}.

235   parameter b/M. This anchors the receiving end of the beam at the rain
236   observer. Now we need to anchor the sending end of the same beam at the
237   star—that is, to find the map angle ϕ_{∞} of the star that emits this beam.
238   To anchor both ends of each beam, we use our graphical relations among
239   values of b/M, θ_{rain}, and ϕ_{∞} for an observer located at $(r, \phi = 0)$. Figure 3

Focus on
primary beam.

240   shows the rain angle θ_{rain} at which a local rain observer looks to see a beam
241   with given b-value. Figures 8 and 10 in Section 11.8 show the relation between
242   the impact parameter b and the map angle ϕ_{∞} to the star. Taken together, the
243   figures in these two chapters (and their generating equations) solve our
244   problem: They provide the two-step procedure to go between ϕ_{∞} and θ_{rain}. To
245   begin, we focus on the **primary image**—the image due to the primary beam,
246   the most direct beam from star to observer. But the following procedure is
247   valid for any beam whose b-value connects a star to a rain observer.

248   **FROM STAR MAP ANGLE ϕ_∞ TO RAIN VIEWING ANGLE $\theta_{\rm rain}$**

From ϕ_∞,
find $\theta_{\rm rain}$

249   At what angle $\theta_{\rm rain}$ does a local rain observer located at map
250   coordinates $(r, \phi = 0)$ look to see the primary image of a star at
251   given map angle ϕ_∞? Here is the two-step procedure:

252   Step A.   Figures 8 and 10 in Section 11.8 tell us the b-value of the
253   primary beam that connects the star at map angle ϕ_∞ to this
254   observer's location and whether that beam is incoming or
255   outgoing.

256   Step B.   For that b-value—and knowledge of whether the beam is
257   incoming or outgoing—Figure 3 gives the local viewing angle
258    $\theta_{\rm rain}$ in which the rain observer looks to see his primary image
259   of that star.

260   **Comment 4. Reminder: Two angles, ϕ_∞ and $\theta_{\rm rain}$**
261   We measure the map angle ϕ_∞ to a star—as we measure all map
262   angles—*counterclockwise* with respect to the radially *outward* direction. In
263   contrast—and for our own convenience—we measure the rain observing angle
264    $\theta_{\rm rain}$ *clockwise* from the radially *inward* direction.

265   We can also run this process backward, from rain observation angle $\theta_{\rm rain}$

From $\theta_{\rm rain}$,
find ϕ_∞.

266   to map angle ϕ_∞. The rain observer sees a star at rain angle $\theta_{\rm rain}$. Find the
267   map angle ϕ_∞ to that star using Step B above, followed by Step A: From the
268   value of $\theta_{\rm rain}$, Figure 3 tells us the b-value of the beam and whether it is
269   incoming or outgoing. From this information, Figures 8 and 10 in Section 11.8
270   give us the map angle ϕ_∞ of the star from which this beam comes.

271   **Comment 5. Automate rain panorama plots.**
272   To plot panoramas of the rain viewer, we read numbers from curves in figures,
273   which yield only approximate values. Nothing stops us from converting the data

Automate rain
panorama plots.

274   in these figures (and the equations from which they come) into look-up tables or
275   mathematical functions directly used by a computer. Then from the map angle
276   ϕ_∞ to every star in the heavens, the computer automatically projects onto the
277   inside of the personal planetarium (Figure 1) the visual panorama seen by the
278   rain observer.

279   The following Queries and Sample Problems provide examples and
280   practice connecting star map angle ϕ_∞ with the angle $\theta_{\rm rain}$ in which a local
281   rain observer looks to see that star.

282

QUERY 3. Sequential changes in rain viewing angles of different stars
A rain viewer first looks at a given star when she is far from the black hole; later she looks at the same
star as she hurtles inward past each of the r-values $r/M = 12, 5, 2, 1$, and just before she reaches the
singularity. Do the following Items twice: once using plots in the figures, and second *optional* using
equations (13) and (14).

Find the viewing angle $\theta_{\rm rain}$ at each r-coordinate for the star at each of the following map angles.
At each r, find the value b/M of the beam that the rain observer sees from that star.

Sample Problems 1. Find local rain angles

Find the rain angles θ_{rain} at which the rain observer hurtling past map location $(r = 8M, \phi = 0)$ looks to see Beams A through D in Figure 10, Section 11.7.

SOLUTION

A. Beam A comes from a star at map angle $\phi_\infty = 70°$. In Figure 10, Section 11.7, the arrowhead and label nearest the little open square labeled A tell us that the beam that connects that star with this observer is incoming and has $b/M = -8$. In Figure 3, the incoming $b/M = -8$ beam crosses the vertical line $r = 8M$ at $\theta_{\text{rain}} \approx +90°$. So to see the star at $\phi_\infty = 70°$, the rain observer at $(r = 8M, \phi = 0)$ looks 90° to the right of the radially inward direction.

B. Beam B in Figure 10, Section 11.7 comes from a star at map angle $\phi_\infty = 158°$. The little open square labeled B in that figure shows that this beam also has the value $b/M = -8$ but the arrowheads now show that it is outgoing when it reaches our observer. In Figure 3 the outgoing $b/M = -8$ beam crosses the vertical line $r = 8M$ at $\theta_{\text{rain}} \approx +40°$. So to see this star the rain observer looks 40° to the right of the radially inward direction.

C. Beam C in Figure 10, Section 11.7 is incoming from a star at map angle $\phi_\infty = -70°$ with $b/M = +8$. To see this star our rain observer looks at the angle $\theta_{\text{rain}} \approx -90°$ to the *left* of the radially inward direction.

D. By symmetry with Beam B, to see Beam D in Figure 10, Section 11.7 our observer looks at the angle $\theta_{\text{rain}} \approx -40°$ to the *left* of the radially inward direction—that is, looking inward.

A. The star at $\phi_\infty = 30°$

B. The star at $\phi_\infty = 90°$

C. The star at $\phi_\infty = 150°$

QUERY 4. Given ϕ_∞, find θ_{rain}.
Each of the following items lists the map angle ϕ_∞ to a star and the r-coordinate of a rain observer at map coordinates $(r, \phi = 0)$ who looks at that star. In each case find the rain angle θ_{rain} (with respect to the radially inward direction) at which the local rain observer looks to see that star.

A. $\phi_\infty = +30°$, $r = 6M$

B. $\phi_\infty = -120°$, $r = 10M$

C. $\phi_\infty = +90°$, $r = 2.5M$

D. $\phi_\infty = -180°$, $r = 12M$

E. The rain observer is at the turning point of the beam that has $b/M = -7$. In what rain direction θ_{rain} does she look to see that star? At what map angle ϕ_∞ is the star that she sees?

QUERY 5. Given θ_{rain}, find ϕ_∞.
Each of the following items lists the r-coordinate of a rain observer at map coordinates $(r, \phi = 0)$ and the rain angle θ_{rain} at which she looks to see a given star. In each case find the map angle ϕ_∞ of that star.

A. $r = 4M$, $\theta_{\text{rain}} = -115°$

Sample Problems 2. What can the local rain observer at $r = 6M$ see?

In the following cases the local rain observer is passing one of our standard locations, $(r, \phi = 0)$.

A. What is the range of b-values of starlight beams that the rain observer at $r = 6M$ can see? When this observer looks inward, $0 \leq |\theta_{\rm rain}| < 90°$, what is the range of b-values of beams that she can see? **SOLUTION:** In Figure 3, look at the vertical line at $r = 6M$. Beams with b-values in the range $0 \leq |b/M| \leq\approx 7.3$, either incoming or outgoing, cross that vertical line. These are the beams that she can see. Beams that the rain observer can see looking inward, $-90° < \theta_{\rm rain} < +90°$, have b-values in the range $b_{\rm critical}/M \leq |b/M| \leq\approx 7.3$.

B. In Part A, the largest value of $|b|$ for light seen by a rain observer at r occurs for a beam whose turning point is at that r-coordinate. What is that maximum value of $|b|$ for $r = 6M$? **SOLUTION:** Use equation (36) in Section 11.4. For $r_{\rm tp} = 6$, the answer is $b/M = \pm 7.348$, the correct value compared with the approximate value we read off the plot in Figure 3.

C. At what rain angle $\theta_{\rm rain}$ will the rain viewer passing $r = 6M$ look to see a star at map angle $\phi_\infty = 180°$, exactly on the opposite side of the black hole from her? **SOLUTION:** Begin with Figure 10, Section 11.7. Look at

the intersection of the vertical line at $r = 6M$ and top and bottom horizontal lines at $\phi_\infty = \pm 180°$. The b values of these beams is $b/M \approx \pm 6.6$; the figure tells us that these are outgoing beams. The plus or minus refers to beams that come around opposite sides of the black hole. Now return to Figure 3 and find the intersection of vertical line $r = 6M$ with outgoing beams whose impact parameters are $b/M = \pm 6.6$. These intersections correspond to $\theta_{\rm rain} \approx \pm 110°$. These angles are greater than $\pm 90°$, so the rain observer looks somewhat behind her to see the star on the opposite side of the black hole.

D. The local rain observer passing $r = 6M$ sees a star at angle $\theta_{\rm rain} = +74°$. What is the map angle ϕ_∞ to that star? Is this beam incoming or outgoing? **SOLUTION:** In Figure 3 the point $(r = 6M, \theta_{\rm rain} = +74°)$ lies on the curve for the incoming beam with $b/M = -7$. Now go to Figure 10, Section 11.7 to find the intersection of $r = 6M$ with the incoming beam with $b/M = -7$. This occurs at the map angle $\phi_\infty \approx +90°$.

E. Can the rain observer at $r = 6M$ see a star that lies outside of the plane of Figure 3, for example? **SOLUTION:** Sure: just rotate every relevant figure around its horizontal axis until the desired star lies in the resulting plane, then carry out the analysis as before.

B. $r = 10M$, $\theta_{\rm rain} = +80°$

C. $r = 2.5M$, $\theta_{\rm rain} = -145°$

D. $r = 6M$, $\theta_{\rm rain} = +155°$

E. The rain observer sees the visual edge of the black hole at $\theta_{\rm rain} = +70°$. What is the map angle of the star that he sees at this visual edge?

12.5 ■ ABERRATION

Rain on the windshield

Aberration:
spectacular
consequences

Aberration is the difference in direction in which light moves as observed in overlapping inertial frames in relative motion. Aberration has spectacular consequences for what the local rain observer sees as she approaches and crosses the black hole's event horizon. Figure 5 shows an analogy: Rain that falls on a stationary and on a fast-moving car comes from different directions as viewed by a rider in the car. Light moves differently than rain, but the general idea is the same.

We deal here with different viewing directions in overlapping local inertial frames, so special relativity suffices for this analysis. Exercise 18 in Chapter 1
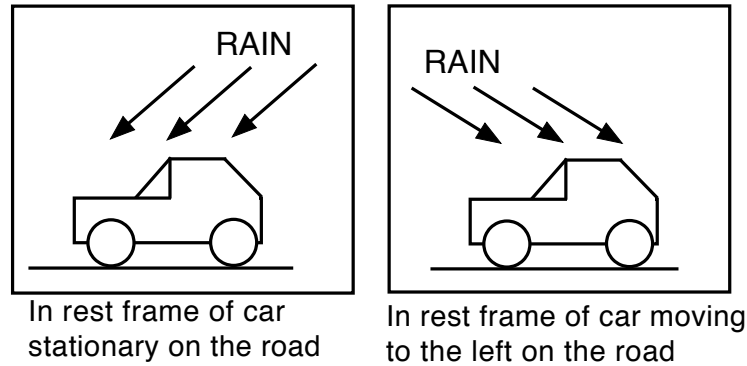
FIGURE 5   Aberration of rain as an analogy of the aberration of light. In the left panel no rain falls directly on the windshield; in the right panel the driver sees the rain coming from a forward direction.

derived expressions for aberration between laboratory and rocket frames in special relativity. We need to modify these equations in four ways:

Modify special relativity aberration equation for local rain and shell frames.

Modify Chapter 1 aberration equations.

1. Choose the shell frame (outside the event horizon) to be the laboratory frame and the local rain frame to be the rocket frame.

2. The direction of relative motion is along the common Δy_{frame} line instead of along the common Δx_{frame} line in Chapter 1.

3. The local rain frame moves in the negative y_{shell} direction, so v_{rel} in the aberration equations must be replaced by $-v_{\text{rel}}$. Equation (16) gives $v_{\text{rel}} = (2M/r)^{1/2}$.

4. The original special relativity aberration equations describe the direction (angle ψ) in which light *moves*. In contrast, our angles θ_{shell} and θ_{rain} refer to the direction in which the observer *looks* to see the beam, which is the opposite direction. Because of this, $\cos \psi$ in Chapter 1 becomes $-\cos \theta$ in the present chapter.

When all these changes are made, the aberration equation (54) in exercise 18 of Chapter 1 becomes:

$$\cos \theta_{\text{shell}} = \frac{\cos \theta_{\text{rain}} + \left(\dfrac{2M}{r}\right)^{1/2}}{1 + \left(\dfrac{2M}{r}\right)^{1/2} \cos \theta_{\text{rain}}} \qquad \text{(light)} \qquad (18)$$

QUERY 6. Resolve the paradox in Comment 3 First of all, note that the question posed in Comment 3 in Section 12.3 is bogus. The beam "comes from behind" only in global coordinates; but we must not trust global coordinates to tell us about measurements or observations. Instead, you can use
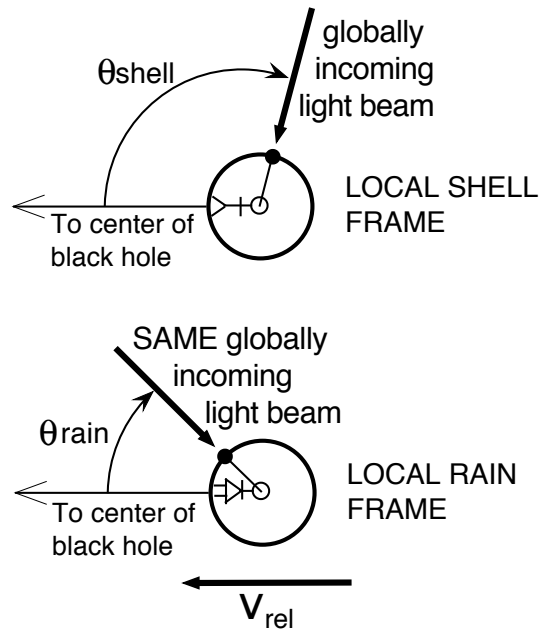
FIGURE 6   Example of light aberration for shell and local rain observers from equation (18). The shell observer at $r = 3M$ looks at the angle $\theta_{\text{shell}} = 105°$ to see the beam from a star. The rain observer who passes this shell sees the beam at $\theta_{\text{rain}} = 45°$.

our equations to show that the descending local rain observer at $r = 8M$ sees the incoming beam with $b/M = -9$ at the inward angle $\theta_{\text{rain}} \approx 72°$, as follows:

A. Substitute the data from point **g** into equation (14) to show that
$F(b, r) = F(-9M, 8M) = 0.225$.

B. Plug the results of Item A plus $r = 8M$ and $b/M = -9$ into equation (13) to show that
$\cos \theta_{\text{rain}} = 0.310$.

C. From Item B, show that $\theta_{\text{rain}} = 72°$. Does this result match the vertical location of point **g** in Figure 3?

QUERY 7. Rain view at the event horizon.

A. When the rain observer passes through $r = 2M$, at what angle θ_{rain} does she see the edge of the black hole?

B. Can the rain observer use her panorama of stars to detect the moment when she crosses the event horizon?

C. Does your answer to Item B violate our iron rule that a diver cannot detect when she crosses the horizon?

12.6 ■ RAIN FRAME ENERGY OF STARLIGHT

₃₆₉ *Falling through gravitationally blue-shifted light.*

₃₇₀ Section 7.8 predicted that as she approaches the singularity, tidal forces will
₃₇₁ end the experience of the rain diver during a fraction of a second, as measured
₃₇₂ on her wristwatch—independent of black hole mass. But starlight increases its
₃₇₃ frequency, and hence its locally-measured energy, as it falls toward the black
₃₇₄ hole. Will the rain observer receive a lethal dose of high-energy starlight before
₃₇₅ she reaches the singularity? To engage this question, we analyze the energy of
₃₇₆ light $E_{\rm rain}$ measured in the local rain frame compared to its map energy E.

Rain observer in danger from starlight?

₃₇₇ Equation (1) gives the map energy E of a stone in global rain coordinates. The
₃₇₈ special relativity expression for rain frame energy with the substitution
₃₇₉ $\Delta t_{\rm rain} \equiv \Delta T$ from (4) yields

$$\frac{E_{\rm rain}}{m} = \lim_{\Delta\tau\to 0}\left(\frac{\Delta t_{\rm rain}}{\Delta\tau}\right) = \lim_{\Delta\tau\to 0}\left(\frac{\Delta T}{\Delta\tau}\right) = \frac{dT}{d\tau} \qquad \text{(stone)} \qquad (19)$$

₃₈₀ Modify this expression to describe light. Substitute $dT/d\tau$ from (19) into (1)
₃₈₁ and solve for $E_{\rm rain}$:

$$E_{\rm rain} = \left(1 - \frac{2M}{r}\right)^{-1} E\left[1 + \left(\frac{2M}{r}\right)^{1/2}\frac{m}{E}\frac{dr}{d\tau}\right] \qquad \text{(stone)} \qquad (20)$$

₃₈₂ Recall equation (2) in Section 11.2:

$$\frac{dr}{d\tau} = \pm\left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2r^2}\right)\right]^{1/2} \qquad \text{(stone)} \qquad (21)$$

₃₈₃ Use equation (21) to replace the expression $(m/E)(dr/d\tau)$ just inside the
₃₈₄ right-hand square bracket in (20):

$$\frac{m}{E}\frac{dr}{d\tau} = \pm\left[1 - \left(1 - \frac{2M}{r}\right)\left(\frac{m^2}{E^2} + \frac{L^2}{E^2r^2}\right)\right]^{1/2} \qquad \text{(stone)} \qquad (22)$$

₃₈₅ This equation is for a stone. Turn it into an equation for light by going to the
₃₈₆ limit of small mass and high speed: $m \to 0$ and $L/E \to b$. Plug the result into
₃₈₇ (20) and divide through by E, which then becomes:

$$\frac{E_{\rm rain}}{E} = \frac{1 \pm \left(\dfrac{2M}{r}\right)^{1/2}\left[1 - \dfrac{b^2}{r^2}\left(1 - \dfrac{2M}{r}\right)\right]^{1/2}}{1 - \dfrac{2M}{r}} \qquad \text{(light)} \qquad (23)$$

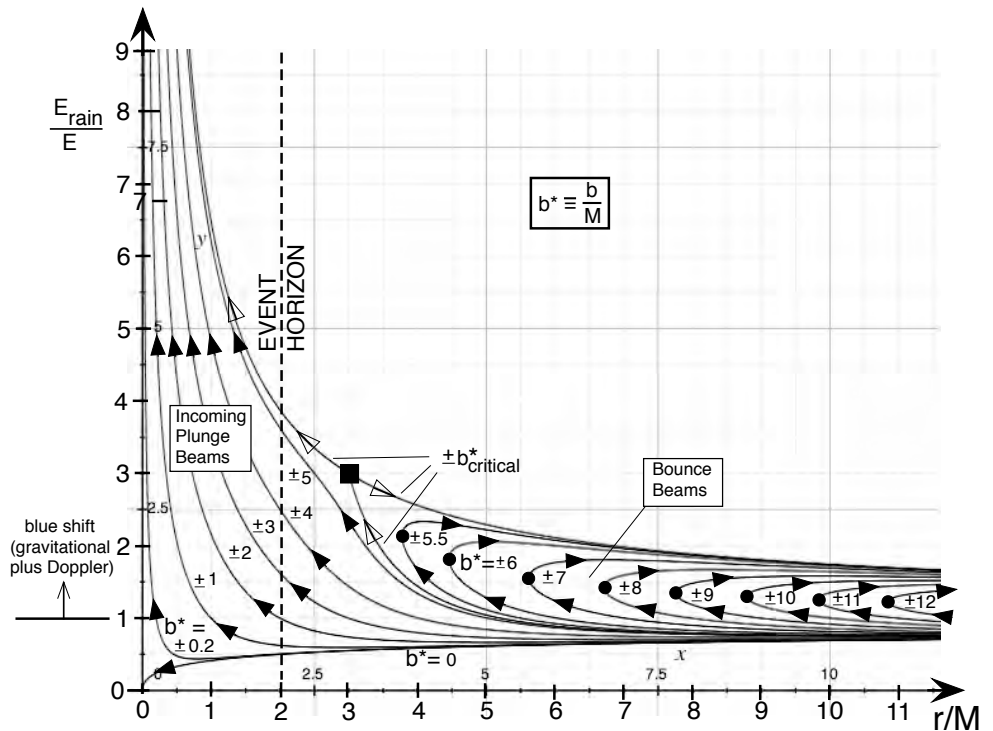₃₈₈ Use (14) to write this as:

FIGURE 7   Ratio E_{rain}/E of starlight measured by the local rain observer, from equation (24). The curve rising out of each turning point describes an outgoing beam. Beams with $0 \leq |b/M| \leq 5$ are incoming plunge beams.

$$\frac{E_{\text{rain}}}{E} = \frac{1 \pm \left(\dfrac{2M}{r}\right)^{1/2} F(b,r)}{1 - \dfrac{2M}{r}} \qquad \text{(light)} \qquad (24)$$

Numerator: minus sign for incoming starlight, plus sign for outgoing light

389

390   where equation (14) defines $F(b,r)$. In (23)—and therefore in (24)—the impact
391   parameter b is squared; therefore the beam has the same map energy whether
392   it moves clockwise or counterclockwise around the black hole, as we expect.
393   In the exercises you derive an expression E_{rain}/E for the stone.
394   Figure 7 plots results of equation (24). We expect the rain frame energy of
395   an *incoming* beam to depend on two competing effects: the gravitational blue
396   shift (increase in local frame energy) of the falling light, reduced for the diving
397   observer by her inward motion. The result is the Doppler downshift in energy
398   of the light viewed by this local rain observer—compared to the same light
399   beam viewed by the local shell observer. In contrast, starlight that has passed

Rain-frame
energies
of beams

400 its turning point and heads outward again in global coordinates moves opposite
401 to the incoming rain observer, so she will measure its energy to be Doppler
402 up-shifted. Figure 7 shows that these two effects yield a net blue shift for some
403 beams and parts of other beams, and a net red shift for still other beams.

QUERY 8. Rain energy of light at large r.

What happens to the value of E_{rain} as $r \to \infty$? Show that

$$\lim_{r\to\infty} E_{\text{rain}} = E \qquad \text{(light)} \tag{25}$$

QUERY 9. Rain energies of radially-moving starlight.

Find an expression for the rain energy of light with $b = 0$ that moves radially inward (for any value of r) or outward (for $r > 2M$). Show that in this case equation (24) becomes

$$\frac{E_{\text{rain}}}{E} = \frac{1 \pm \left(\dfrac{2M}{r}\right)^{1/2}}{1 - \dfrac{2M}{r}} = \frac{1 \pm \left(\dfrac{2M}{r}\right)^{1/2}}{\left[1 + \left(\dfrac{2M}{r}\right)^{1/2}\right]\left[1 - \left(\dfrac{2M}{r}\right)^{1/2}\right]} = \frac{1}{1 \mp \left(\dfrac{2M}{r}\right)^{1/2}} \quad \text{(light, } b=0\text{)} \tag{26}$$

where the upper sign is for outgoing beams. But starlight with $b = 0$ cannot be outgoing, so:

$$\frac{E_{\text{rain}}}{E} = \frac{1}{1 + \left(\dfrac{2M}{r}\right)^{1/2}} \qquad \text{(starlight, } b=0\text{)} \tag{27}$$

This is the curve displayed at the bottom of Figure 7.

A. Show that E_{rain}/E has the value 1/2 at $r = 2M$, and that this result is consistent with the $b = 0$ curve in Figure 7.

B. A shell observer remote from a black hole shines radially inward a laser of map energy E_{laser}, measured in his local frame, which is also global E in flat spacetime. A local rain observer moving inward along the same radial line looks radially outward at this laser beam as she descends. Write a short account about the ratio $E_{\text{rain}}/E_{\text{laser}}$ of this laser light that she measures outside the event horizon, when she is at the event horizon, and as she approaches the singularity. If she is given the value of E_{laser}, can she detect when she crosses the event horizon? Could you design an "event horizon alarm" for our black hole explorations? Does your design violate our iron rule that a diver cannot detect when she crosses the event horizon?

QUERY 10. Details of Figure 7
Without equations, provide qualitative explanations of rain frame beam energies in Figure 7.

A. Show that E_{rain} approaches the value E at large r, as demonstrated in (25).

B. Why do beam energies not depend on the sign of b?

C. Why do the outgoing Bounce Beams at any given r have greater rain frame energy than incoming Bounce Beams?

D. For the Plunge Beams, $0 \leq |b| \leq b_{\text{critical}}$ at any given r, why do beams with larger values of $|b|$ have greater rain frame energy than beams with smaller values of $|b|$?

QUERY 11. *Optional:* **Trouble at the event horizon?**

The denominator $1 - 2M/r$ in (24) goes to zero at the event horizon. Does this mean that at $r = 2M$ starlight has infinite rain frame energy for every value of b? To answer, use our standard approximation (inside the front cover); set $r = 2M(1 + \epsilon)$, where $0 < \epsilon \ll 1$ and verify that E_{rain}/E is finite at the event horizon, provided that the beam is not an outgoing Plunge Beam. Show that your approximation at $r/M = 2$ correctly predicts values of E_{rain}/E for two or three of the curves in Figure 7.

QUERY 12. Killer starlight?

What is the energy of starlight measured by the local rain observer as she approaches the singularity? Let $r/M = \epsilon$, where $0 < \epsilon \ll 1$ and show that for starlight (incoming Plunge Beams: minus sign in (24)):

$$\lim_{r \to 0} \frac{E_{\text{rain}}}{E} = \frac{|b|}{r} \qquad \text{(incoming Plunge Beams)} \tag{28}$$

Lethal starlight is bad, but tides are worse.

In Query 12 you show that close to the singularity the energy of starlight measured by the plunging local rain observer increases as the *inverse first power* of the decreasing r. The other mortal danger to the rain observer comes from tidal accelerations. Section 7.9 showed that the rain observer "ouch time" from first discomfort to arrival at the singularity is two-ninths of a second, independent of the mass of the black hole. Equations (38) through (40) in Section 9.7 tell us that tidal accelerations increase as the *inverse third power* of the decreasing r-coordinate, which is proportionally faster than the inverse first power increase in the rain frame energy of incoming starlight.

Which will finally be lethal for the rain observer: killer starlight or killer tides? Inverse third power tidal acceleration appears to be the winning candidate. Analyzing tidal acceleration is straightforward: its effects are simply mechanical. In contrast, we have trouble predicting results for light: they depend not only on the rain frame energy of the light but also on its intensity and the rain observer's wristwatch exposure time. This book says nothing about the focusing properties of curved spacetime near the black hole—an advanced topic—so we lack the tools to predict the (short-term!) consequences of the rain observer's accumulated exposure to starlight as she descends.

12-20 Chapter 12 Diving Panoramas

Assume tides
are lethal

466   We have a lot of experience protecting humans against radiation of
467 different wavelengths. Perhaps a specially-designed personal planetarium
468 (Section 12.2) will allow the rain observer to survive killer starlight all the way
469 down to her tidal limit. In contrast, we know nothing that can shield us from
470 tidal effects. In the description of the final fall in Section 12.7 we assume that
471 it is killer tides that prove lethal for the rain diver.

12.7 ■ THE FINAL FALL
Free-fall to the center

474   We celebrate with the final parade of an all-star cast. Let's follow general
475 relativists Richard Matzner, Tony Rothman, and Bill Unruh (see the
476 references) looking at the starry heavens as we free-fall straight down into a
477 non-spinning black hole so massive, so large that even after crossing the event
478 horizon we have nearly two hours of existence ahead of us—roughly the length
479 of a movie—to behold the whole marvelous ever-changing spectacle. Almost
480 everything we have learned about relativity—both special and
481 general—contributes to our appreciation of this mighty sequence of panoramas.

Time of a movie
inside the
event horizon

Panoramas Seen by the Rain Frame Observer

483 *—Adapted from Matzner, Rothman, and Unruh. Some numerical values*
484 *calculated by Luc Longtin.*

485 Imagine a free-fall journey into a billion-solar-mass black hole
486 ($M = 10^9 M_{\rm Sun} = 1.477 \times 10^9$ kilometers $= 1.6 \times 10^{-4}$ light-years—about
487 one-third of the estimated mass of the black hole at the center of galaxy M87).
488 The map r-coordinate of the event horizon—double the above figure—is about
489 the size of our solar system. We adjust our launch velocity to match the
490 velocity which a rain frame, falling from far away, would have at our shell
491 launch point at $r = 5\,000M$. Our resulting inward shell launch velocity,
492 $v = -(2M/r)^{1/2}$ with respect to a local shell observer, is equal to two percent
493 of the speed of light. We record each stage in the journey by giving both the
494 time-to-crunch on our wristwatch and our current map r.

Fall into a one-
billion-solar-mass
black hole.

26 years to the end

495   *The beginning of the journey, 26 years before the end.* At this point the
496 black hole is rather unimpressive. There is a small region (about 1 degree
497 across—i.e., twice the size of the Moon seen from Earth) in which the star
498 pattern looks slightly distorted and within it (covering about one-tenth of a
499 degree) a disk of total blackout. Careful examination shows that a few stars
500 nearest the rim of the blacked-out region have second images on the opposite
501 side of the rim. Had these images not been pointed out to us, we probably
502 would have missed the black hole entirely.

300 days

503   *Three hundred days before the end, at $r = 500M$.* Some noticeable change
504 has occurred. The dark circular portion of the sky has now grown to one full
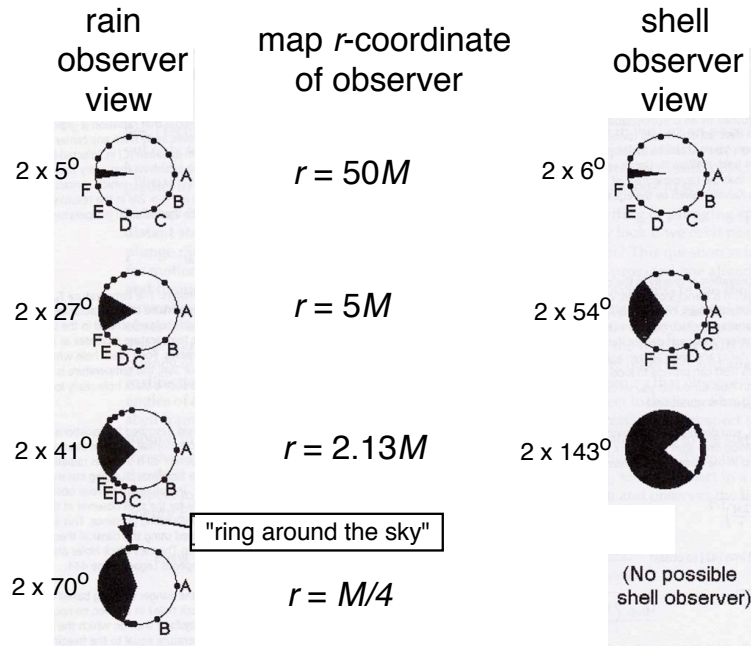505 degree in width.

rain observer view	map *r*-coordinate of observer	shell observer view
2 x 5°	$r = 50M$	2 x 6°
2 x 27°	$r = 5M$	2 x 54°
2 x 41°	$r = 2.13M$	2 x 143°
2 x 70°	"ring around the sky" $r = M/4$	(No possible shell observer)

FIGURE 8   Pie charts showing rain viewing directions of stars and the visual edge of the black hole seen in sequence by a rain frame viewer (left-hand column) and by a set of stationary shell observers at different radii (right-hand column). Dots labeled A through F represent directions of stars plotted by rain and shell observers on their personal planetariums. In the final instants of her journey (at smaller radii than shown here), the sky behind the rain observer is black, nearly empty of stars, and the black hole covers the sky ahead of her. Cleaving the forward half of the firmament from the backward half is a bright ring around the sky. This figure does not show multiple images of stars due to one or several orbits of starlight around the black hole. (Figure based on the work of M. Sikora, courtesy of M. Abramowicz.)

One week

506   *One week before the end, at $r = 41M$.* The image has grown immensely.
507   There is now a pure dark patch ahead with a diameter of about 22 degrees
508   (approximately the size of a dinner plate held at arm's length). The original
509   star images that lay near the direction of the black hole have been pushed
510   away from their original positions by about 15 degrees. Further, between the
511   dark patch itself and these images lies a band of second images of each of these
512   stars. Looking at the edge of this darkness with the aid of a telescope, we can
513   even see faint second images of stars that lie behind us! This light has looped
514   around the black hole on its way to our eye (Figure 11, Section 11.7). From
515   this point on, Doppler shift and gravitational blue shift radically change the
516   observed frequencies of light that originate from different stars.

12-22 Chapter 12 Diving Panoramas

12 hours

517   *Twelve hours before the end, at $r = 7M$.* A sizeable portion of the sky
518   ahead of us is now black; the diameter of the black hole image covers a
519   44-degree angle, over 10 percent of the entire visual sphere.

3.3 hours

520   *3.3 hours before the end.* As we pass inward through $r = 3M$, we see all
521   the stars in the heavens swing forward to meet the expanding edge of the
522   black hole, then remain at this edge as the black hole continues to grow
523   visually larger.

2 hours

524   *Two hours before the end.* We are now at $r = 2.13M$, just outside the
525   event horizon and our speed is 97 percent that of light as measured in the local
526   shell frame that we are passing. Changes in viewing angle (aberrations) are
527   now extremely important. Anything we see after an instant from now will be a
528   secret taken to our grave, because we will no longer be able to send any
529   information out to our surviving colleagues. Although we will be "inside" the
530   black hole, not all of the sky in front of us appears entirely dark. Our high
531   speed causes light beams to arrive at our eyes at extreme forward angles. Even
532   so, a disk subtending a total angle of 82 degrees in front of us is fully black—a
533   substantial fraction of the forward sky.

534   Behind us we see the stars grow dim and spread out; for us their images
535   are not at rest, but continue to move forward in angle to meet the advancing
536   edge of the black hole. This apparent star motion is again a forward-shift due

Secondary
images

537   to our increasing speed. But there is a more noticeable feature of the sky: We
538   can now see second images of all the stars in the sky surrounding the black
539   hole. These images are squeezed into a band about 5 degrees wide around the
540   image of the black hole. These second images are now brighter than were the
541   original stars. Surrounding the ring of second images are the still brighter
542   primary images of stars that lie ahead of us, behind the black hole. The band
543   of light caused by both the primary and secondary images now shines with a
544   brightness ten times that of Earth's normal night sky.

2 minutes

545   *Approximately two minutes before oblivion: $r = M/7$.* The black hole now
546   subtends a total angle of 150 degrees from the forward direction—almost the
547   entire forward sky. Behind us star images are getting farther apart and rushing
548   forward in angle. Only 20 percent of star images are left in the sky behind us.
549   In a 10-degree-wide band surrounding the outer edges of the black hole, not
550   only second but also third and some fourth images of the stars are now visible.
551   This band running around the sky now glows 1000 times brighter than the
552   night sky viewed from Earth.

Final seconds

553   *The final seconds.* The sky is dark everywhere except in that rapidly
554   thinning band around the black disk. This luminous band—glowing ever
555   brighter—runs completely around the sky perpendicular to our direction of
556   motion. At 3 seconds before oblivion it shines brighter than Earth's Moon.
557   New star images rapidly appear along the inner edge of the shrinking band as
558   higher and higher-order star images become visible from light wrapped many
559   times around the black hole. The stars of the visible Universe seem to brighten

560   and multiply as they compress into a thinner and thinner ring transverse to
561   our direction of motion.

Awesome ring bisects the sky.

562   Only in the last 2/9 of a second on our wristwatch do tidal forces become
563   strong enough to end our journey and our view of that awesome ring bisecting
564   the sky.

12.8 ■ EXERCISES
565

566   **1. Impact parameter at a turning point**

567   From equation (29) in Section 11.5, show that $b/M \to \infty$ not only as $r_{\rm tp} \to \infty$
568   but also as $r_{\rm tp}/M \to 2^+$, where the subscript tp means turning point. Since
569   b/M is finite for values between these two limits, therefore there must be at
570   least one minimum in the b vs. $r_{\rm tp}$ curve. Verify the map location and value of
571   this minimum, shown in Figure 9. Remember that beams for which $r_{\rm tp} < 3M$
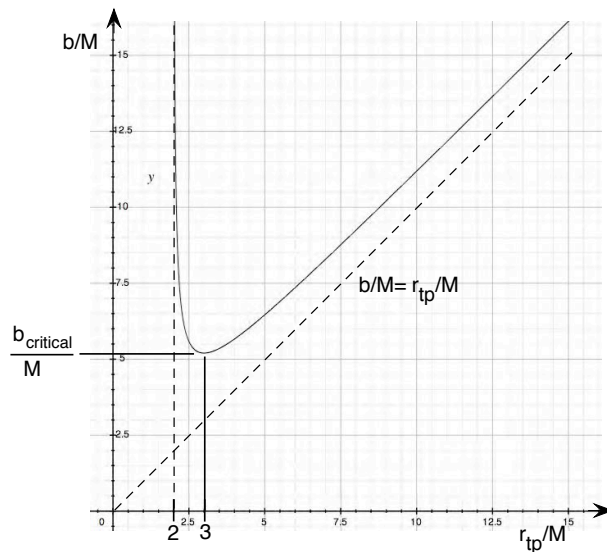572   cannot represent starlight.

FIGURE 9   Plot of b/M vs. $r_{\rm tp}/M$ from equation (29) in Section 11.5.

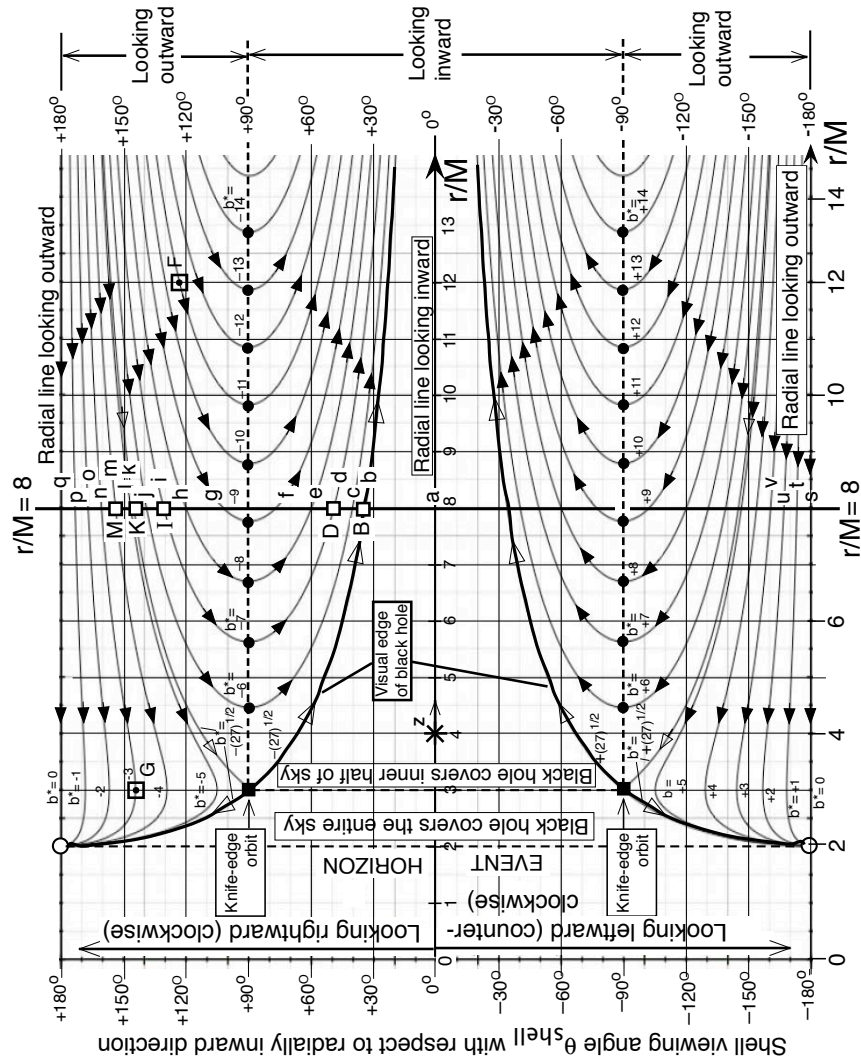**FIGURE 10**   Angle θ_{shell} at which the shell observer located at global coordinates $(r, \phi = 0)$ looks to see starlight beam with values of $b^* \equiv b/M$. Arrows on each curve tell us whether that beam is incoming or outgoing. A black dot marks a turning point, the r-coordinate at which an incoming beam reverses its dr to become an outgoing beam. Upper and lower three-branch curves with open arrowheads represent light with impact parameter $b/M = \pm b_{\text{critical}}/M = \pm(27)^{1/2}$. Two little black squares at $r = 3M$ represent circular knife-edge orbits of these critical beams on the tangential light sphere. When viewed by starlight, the shell observer sees the black hole at shell angles inside the heavy curve, which represents its visual edge. The shell observer near the event horizon looks radially outward to see the entire heavens contracted to a narrow cone (Figure 8).

573   **2. Direction of a star seen by the local shell observer.**

574   Exercise 18 in Section 1.13 shows the relation between the directions in which
575   light moves in inertial laboratory and rocket frames. Replace laboratory with

local shell frame and rocket with local rain frame. The direction of relative
motion is along the local y-axes, and the rain frame moves moves in the
negative $\Delta y_{\mathrm{shell}}$ direction, so the sign of the relative velocity v_{rel} must be
reversed in the special relativity formulas. Then equation (54) in Section 1.13
becomes

$$\cos \psi_{\mathrm{shell}} = \frac{\cos \psi_{\mathrm{rain}} - v_{\mathrm{rel}}}{1 - v_{\mathrm{rel}} \cos \psi_{\mathrm{rain}}} \qquad (\psi = \text{direction of light motion}) \qquad (29)$$

where ψ_{shell} is the direction the light moves in the shell frame and ψ_{rain} its
direction of motion in the rain frame.

A. In the notation of Chapter 1, angles ψ are the directions in which the
   light *moves*; in the notation of the present chapter angles θ are the
   angles in which an observer *looks* to see the beam. The cosines of two
   angles that differ by $180°$ are the negatives of one another. Show that
   equation (18) becomes, in the notation of our present chapter:

$$\cos \theta_{\mathrm{shell}} = \frac{\cos \theta_{\mathrm{rain}} + v_{\mathrm{rel}}}{1 + v_{\mathrm{rel}} \cos \theta_{\mathrm{rain}}} \qquad (\theta = \text{direction viewer looks}) \qquad (30)$$

B. Why can't equations (18) and (30) be used inside the event horizon?

C. A shell observer at a given r and a local rain observer who passes
   through that map location both view the same beam. Items (a)
   through (c) below give the value of b and r in each case, and whether
   the beam is incoming or outgoing. For each case, find θ_{rain} from Figure
   3; use (30) with v_{rel} from (16) to convert to shell angle θ_{shell}; then check
   your result in Figure 10.

   (a) Outgoing beam with $b/M = -12$ observed at $r = 12M$.
   (b) Incoming beam with $b/M = -7$ observed at $r = 6M$.
   (c) Incoming beam with $b/M = -4$ observed at $r = 3M$.

D. Look at the list "What the Rain Viewer Sees as She Passes $r = 8M$" in
   Section 12.3. Use the lowercase bold letters on the $r = 8M$ vertical line
   in Figure 10 to write a similar analysis of what the local shell observer
   at $r = 8M$ sees.

3. Expression E_{rain}/E for a stone.

Section 12.6 derives the expression E_{rain}/E for *light.* Derive the same
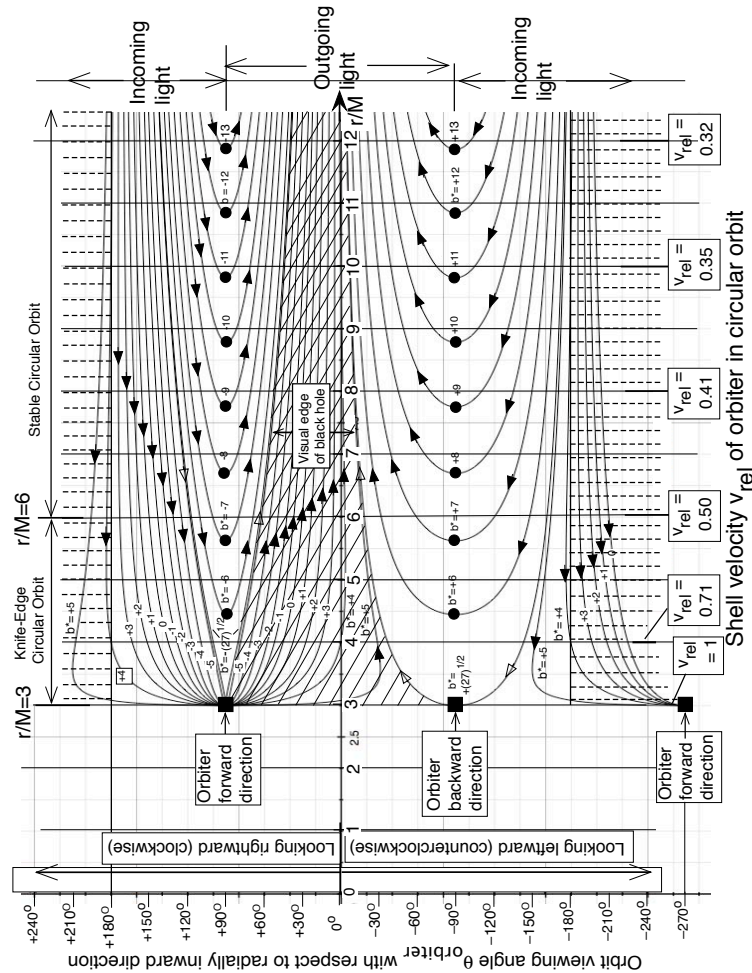expression for a *stone.*

FIGURE 11   Observation angles θ_{orbiter} at which an orbiter looks to see beams with different impact parameters $b^* = b/M$. The edges of the diagonally shaded region are the orbiter's visual edges of the black hole. Values of v_{rel} along the bottom are the relative velocities of the orbiter with respect to the local shell frame. In the limiting case of the orbit at $r = 3M$ (moving at the speed of light in the shell frame), the inner half of the sky is black for the orbiter. Upper and lower regions shaded by vertical dashed lines include some of the curves between $\theta_{\text{orbiter}} = -180°$ and $\theta_{\text{orbiter}} = +180°$.

4. Direction of a star seen by an orbiting observer

In what direction does the observer in circular orbit look to see the same beam? Special relativity can answer this question, because it requires a simple aberration transformation—similar to (30)—first from θ_{rain} to θ_{shell} and then from θ_{shell} to θ_{orbiter}. Equation (16) gives the relative speed in the radial direction between the rain diver and the shell observer, while equation (33) in Section 8.5 gives the relative speed in the tangential direction between shell and orbiting observers: $v_{\text{rel}} = v_{\text{shell}} = (r/M - 2)^{-1/2}$. The resulting

613 transformations, although messy, use nothing but algebra and trigonometry.
614 The results are plotted in Figure 11, which is similar to Figure 3.
615   WE HAVE NOT FINISHED THIS EXERCISE. PLEASE ASK
616 SEVERAL NON-TRIVIAL QUESTIONS THAT CAN BE ANSWERED
617 USING FIGURE 11. THEN ANSWER YOUR OWN QUESTIONS.

12.9 ■ REFERENCES

619 Initial quote: Emily Dickinson, poem number 143, version A, about 1860, *The*
620   *Poems of Emily Dickinson, Variorum Edition*, Edited by R. W. Franklin,
621   Cambridge Massachusetts, The Belknap Press of Harvard University, 1998,
622   Volume I, page 183.
623 Description of final dive (Section 12.7) and Figure 8 are adapted from Richard
624   Matzner, Tony Rothman, and Bill Unruh, "Grand Illusions: Further
625   Conversations on the Edge of Spacetime," in *Frontiers of Modern Physics:*
626   *New Perspectives on Cosmology, Relativity, Black Holes and Extraterrestrial*
627   *Intelligence*, edited by Tony Rothman, Dover Publications, Inc., New York,
628   1985, pages 69–73. Luc Longtin provided corrections for "times before
629   oblivion" in in the Section The Final Fall and calculated numbers for Figure
630   8.