

# Chapter 1. Speeding

- 1.1 Special Relativity 1-1
  - 1.2 Wristwatch Time 1-3
  - 1.3 Ruler Distance 1-7
  - 1.4 Lightlike (Null) Interval 1-8
  - 1.5 Worldline of a Wandering Stone; The Light Cone 1-10
  - 1.6 The Twin “Paradox” and the Principle of Maximal Aging 1-12
  - 1.7 Energy in Special Relativity 1-16
  - 1.8 Momentum in Special Relativity 1-21
  - 1.9 Mass in Relativity 1-22
  - 1.10 The Lorentz Transformation 1-24
  - 1.11 Limits on Local Inertial Frames 1-26
  - 1.12 General Relativity: Our Current Toolkit 1-28
  - 1.13 Exercises 1-29
  - 1.14 References 1-37
- 
- *What is the key idea of relativity?*
  - *Everything is relative, right?*
  - *“Space and time form a unity called spacetime.” Huh?*
  - *Do people in relative motion age differently? Do they feel the slowing down/speeding up of their aging?*
  - *What is the farthest galaxy I can possibly visit in person?*
  - *How do relativistic expressions for energy and momentum differ from those of Newton?*
  - *When and why does special relativity break down, and what warns us that this is about to happen?*

# CHAPTER

# 1

# Speeding

Edmund Bertschinger & Edwin F. Taylor \*

28 *I've completely solved the problem. My solution was to analyze*  
29 *the concept of time. Time cannot be absolutely defined, and*  
30 *there is an inseparable relation between time and signal*  
31 *velocity.*

32 —Albert Einstein, May 1905, to his friend Michele Besso

## 1.1 ■ SPECIAL RELATIVITY

Special relativity distinguished from General relativity

34 *Special relativity and general relativity*

Begin relativity with a stone wearing a wristwatch.

35 **Special relativity** describes the very fast and reveals the unities of both  
36 space-time and mass-energy. **General relativity**, a **Theory of Gravitation**,  
37 describes spacetime and motion near a massive object, for example a star, a  
38 galaxy, or a black hole. The present chapter reviews a few key concepts of  
39 special relativity as an introduction to general relativity.

Observe two events in laboratory frame.

40 What is at the root of relativity? Is there a single, simple idea that  
41 launches us along the road to understanding? At the beginning of *Alice in*  
42 *Wonderland* a rabbit rushes past carrying a pocket watch. At the beginning of  
43 our relativity adventure a small stone wearing a wristwatch flies past us.

44 The wristwatch ticks once at Event 1, then ticks again at Event 2. At each  
45 event the stone emits a flash of light. The top panel of Figure 1 shows these  
46 events as observed in the laboratory frame. We assume that the laboratory is  
47 an **inertial reference frame**.

Definition: inertial frame

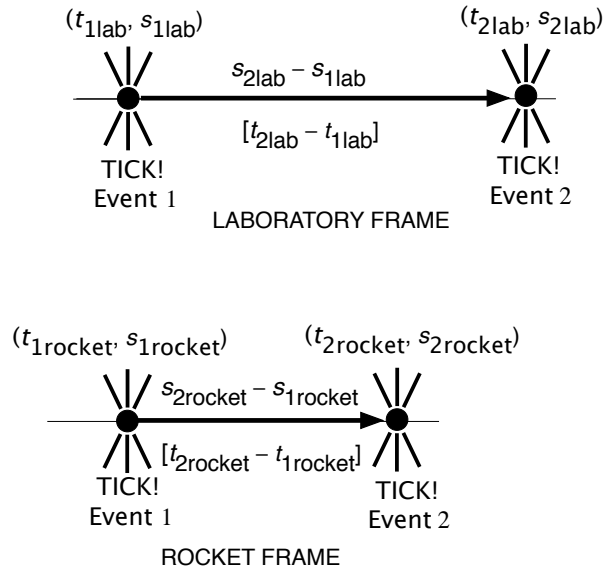
### DEFINITION 1. Inertial frame

48 An **inertial reference frame**, which we usually call an **inertial frame**, is  
49 a region of spacetime in which Newton's first law of motion holds: *A free*  
50 *stone at rest remains at rest; a free stone in motion continues that*  
51 *motion at constant speed in a straight line.*  
52

53 We are interested in the records of these two events made by someone in  
54 the laboratory. We call this someone, the **observer**:

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1-2 Chapter 1 Speeding



**FIGURE 1** A free stone moves through a laboratory at constant speed. The stone wears a wristwatch that ticks as it emits a first flash at Event 1 and a second flash at Event 2. *Top panel:* The laboratory observer records Event 1 at coordinates  $(t_{1lab}, s_{1lab})$  and Event 2 at coordinates  $(t_{2lab}, s_{2lab})$ . *Bottom panel:* An unpowered rocket ship streaks through the laboratory; the observer riding in the rocket ship records Event 1 at rocket coordinates  $(t_{1rocket}, s_{1rocket})$  and Event 2 at  $(t_{2rocket}, s_{2rocket})$ . Each observer calculates the distance and time lapse between the two events, displayed on the line between them.

Definition:  
inertial observer

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**DEFINITION 2. Observer  $\equiv$  inertial observer**

An **inertial observer** is an observer who makes measurements using the space and time coordinates of any given inertial frame. In this book we *choose* to report *every* measurement and observation using an inertial frame. Therefore in this book **observer  $\equiv$  inertial observer**.

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The top panel of Figure 1 summarizes the records of the laboratory observer, who uses the standard notation  $(t_{1lab}, s_{1lab})$  for the lab-measured time and space coordinates of Event 1 and  $(t_{2lab}, s_{2lab})$  for the coordinates of Event 2.

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The laboratory observer calculates the *difference* between the time coordinates of the two events and the *difference* between the space coordinates of the two events that she measures in her frame. The top panel of Figure 1 labels these results.

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Next an unpowered rocket moves through the laboratory along the line connecting Event 1 and Event 2. An observer who rides in the rocket measures the coordinates of the two events and constructs the bottom panel in Figure 1.

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Now the key result of special relativity: There is a surprising relation between the coordinate differences measured in laboratory and rocket frames, both of which are inertial frames. Here is that expression:

*Surprise:*  
Both observers calculate the same wristwatch time between two events.

$$\tau^2 = (t_{2\text{lab}} - t_{1\text{lab}})^2 - (s_{2\text{lab}} - s_{1\text{lab}})^2 = (t_{2\text{rocket}} - t_{1\text{rocket}})^2 - (s_{2\text{rocket}} - s_{1\text{rocket}})^2 \quad (1)$$

74 The expression on the left side of (1) is the square of the so-called **wristwatch**  
 75 **time**  $\tau$ , which we define explicitly in the following section. Special relativity  
 76 says that the wristwatch time lapse of the stone that moves directly between  
 77 events can be predicted (calculated) by both laboratory and rocket observers,  
 78 each using his or her own time and space coordinates. The middle expression  
 79 in (1) contains only laboratory coordinates of the two events. The right-hand  
 80 expression contains only rocket coordinates of the same two events. Each  
 81 observer predicts (calculates) the same value of the stone's wristwatch time  
 82 lapse as it travels between these two events.

83 **Fuller Explanation:** *Spacetime Physics*, Chapter 1. Chapter 2, Section 2.6,  
 84 shows how to synchronize the clocks in each frame with one another. Or look  
 85 up **Einstein-Poincaré synchronization**.

## 1.2.6 ■ WRISTWATCH TIME

87 *Every observer agrees on the advance of wristwatch time.*

88 Einstein said to Besso (initial quote): “Time cannot be absolutely defined . . .”  
 89 Equation (1) exhibits this ambiguity: the laboratory time lapse, rocket time  
 90 lapse, and wristwatch time lapse between two ticks of the stone's wristwatch  
 91 *can all be different from one another*. But equation (1) tells us much more: It  
 92 shows how any inertial observer whatsoever can use the space and time  
 93 coordinate separations between ticks measured in her frame to calculate the  
 94 unique **wristwatch time**  $\tau$ , the time lapse between ticks recorded on the  
 95 stone's wristwatch as it moves from Event 1 to Event 2.

### DEFINITION 3. Wristwatch time = aging

96 Equation (1) and Figure 1 show an example of the **wristwatch time**  $\tau$   
 97 between two events, in this case the time lapse recorded on a  
 98 wristwatch that is present at both events and travels uniformly between  
 99 them. Wristwatch time is sometimes called **aging**, because it is the  
 100 amount by which the wearer of the wristwatch gets older as she travels  
 101 directly between this pair of events. Another common name for  
 102 wristwatch time is **proper time**, which we do not use in this book.  
 103

104 We, the authors of this book, rate (1) as one of the greatest equations in  
 105 physics, perhaps in all of science. Even the famous equation  $E = mc^2$  is a child  
 106 of equation (1), as Section 1.7 shows.

107 Truth be told, equation (1) is not limited to events along the path of a  
 108 stone; it also applies to any pair of events in flat spacetime, no matter how  
 109 large their coordinate separations in any one frame. In the general case,  
 110 equation (1) is called the spacetime **interval** between these two events.

Example of  
**wristwatch time**  
 or **aging**

1-4 Chapter 1 Speeding

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 Definition: **interval**

**DEFINITION 4. Interval**

The spacetime **interval** is an expression whose inputs are the distance separation and time separation between a pair of events measured in an inertial frame. The term “interval” refers to the whole equation (1). There are three different possible outputs, three types of interval:

- 116 Case 1: Timelike interval,  $\tau^2 > 0$  this section
- 117 Case 2: Spacelike interval,  $\tau^2 < 0$  Section 1.3
- 118 Case 3: Lightlike interval,  $\tau^2 = 0$  Section 1.4

119 These three categories span all possible relations between a pair of  
 120 events in special relativity. When  $(t_{2lab} - t_{1lab})^2$  is greater than  
 121  $(s_{2lab} - s_{1lab})^2$ , then we have the case we analyzed for two events that  
 122 may lie along the path of a stone. We call this a **timelike interval**  
 123 because the magnitude of the time part of the interval is greater than  
 124 that of its space part.

125 What happens when  $(s_{2lab} - s_{1lab})^2$  is greater than  $(t_{2lab} - t_{1lab})^2$  in  
 126 (1), so the interval is negative? We call this a **spacelike interval**  
 127 because the magnitude of the space part of the interval is greater than  
 128 that of its time part. In this case we interchange  $(t_{2lab} - t_{1lab})^2$  and  
 129  $(s_{2lab} - s_{1lab})^2$  to yield a positive quantity we call  $\sigma^2$ , whose different  
 130 physical interpretation we explore in Section 1.3.

131 What happens when  $(s_{2lab} - s_{1lab})^2$  is equal to  $(t_{2lab} - t_{1lab})^2$  in (1),  
 132 so the interval has the value zero? We call this a **null interval** or  
 133 **lightlike interval**, as explained in Section 1.4.

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 Measure space and  
 time separations  
 in the same unit,  
 which *you* choose.

*Note:* All separations in (1) must be measured in the same unit; otherwise  
 they cannot appear as separate terms in the same equation. But we are free to  
 choose the common unit: it can be **years** (of time) and **light-years** (of  
 distance). A light-year is the distance light travels in a vacuum in one year. Or  
 we can use **meters** (of distance) along with **light-meters** (of time). A  
 light-meter of time is the time it takes light to travel one meter in a  
 vacuum—about  $3.34 \times 10^{-9}$  second. Alternative expressions for light-meter are  
**meter of light-travel time** or simply **meter of time**.

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 143  
 Speed of light  
 equals unity.

Distance and time expressed in the same unit? Then the *speed of light* has  
 the value unity, with *no* units:

$$c = \frac{1 \text{ light-year of distance}}{1 \text{ year of time}} = \frac{1 \text{ meter of distance}}{1 \text{ light-meter of time}} = 1 \quad (2)$$

144 Why the letter *c*? The Latin word *celeritas* means “swiftness” or “speed.”

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 Stone’s speed:  
 a fraction of  
 light speed

So much for the speed of *light*. How do we measure the speed of a *stone*  
 using space and time separations between ticks of its wristwatch? Typically  
 the value of the stone’s speed depends on the reference frame with respect to  
 which we measure these separations. In the top panel of Figure 1, its speed in

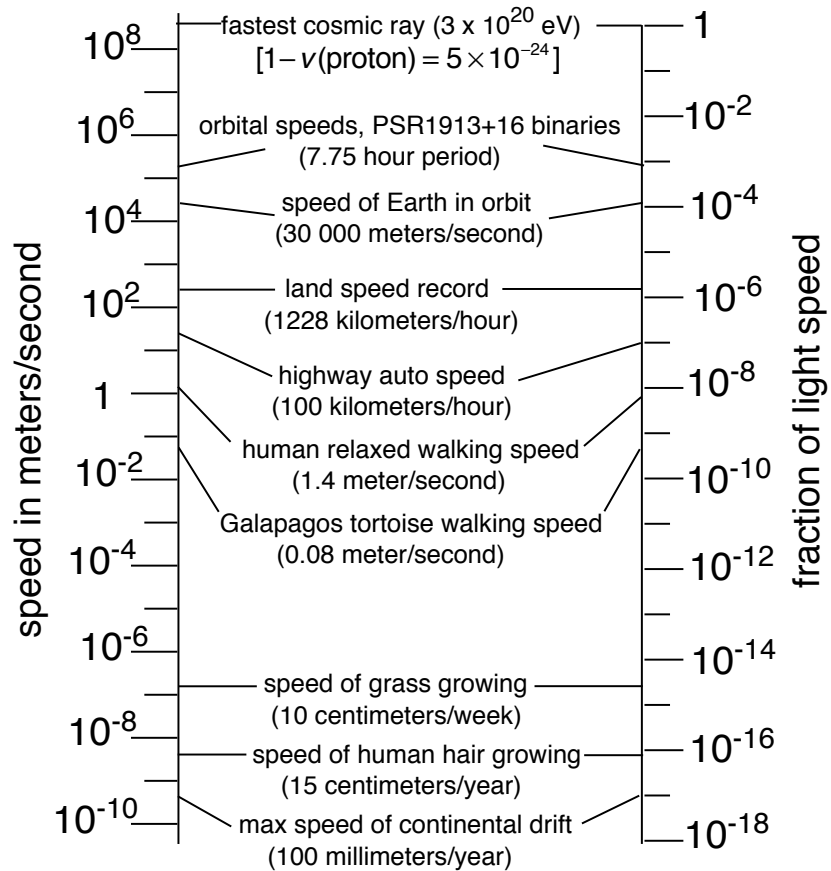


FIGURE 2 The speed ladder. Some typical speeds encountered in Nature.

149 the laboratory frame is  $v_{\text{lab}} = (s_{2\text{lab}} - s_{1\text{lab}})/(t_{2\text{lab}} - t_{1\text{lab}})$ . In the bottom  
 150 panel, its speed in the rocket frame is  
 151  $v_{\text{rocket}} = (s_{2\text{rocket}} - s_{1\text{rocket}})/(t_{2\text{rocket}} - t_{1\text{rocket}})$ . Typically the values of these  
 152 two speeds differ from one another. However, both values are less than one.  
 153 Figure 2 samples the range of speeds encountered in Nature.

154 Equation (1) is so important that we use it to define **flat spacetime**.

155 **DEFINITION 5. Flat spacetime**

156 Definition:  
 157 **flat spacetime**

156 **Flat spacetime** is a spacetime region in which equation (1) is valid for  
 157 every pair of events.

158 The interval in equation (1) has an important property that will follow us  
 159 through special and general relativity: it has the same value when calculated  
 160 using either laboratory or rocket coordinates. We say that wristwatch time is  
 161 an **invariant quantity**.

1-6 Chapter 1 Speeding

### Sample Problems 1. Wristwatch Times

**PROBLEM 1A**

An unpowered rocket ship moves at constant speed to travel 3 light-years in 5 years, this time and distance measured in the rest frame of our Sun. What is the time lapse for this trip recorded on a clock carried with the spaceship?

**SOLUTION 1A**

The two events that start and end the spaceship's journey are separated in the Sun frame by  $s_{2\text{Sun}} - s_{1\text{Sun}} = 3$  light-years and  $t_{2\text{Sun}} - t_{1\text{Sun}} = 5$  years. Equation (1) gives the resulting wristwatch time:

$$\tau^2 = 5^2 - 3^2 = 25 - 9 = 16 \text{ years}^2 \quad (3)$$

$$\tau = 4 \text{ years}$$

which is *less* than the time lapse measured in the Sun frame.

**PROBLEM 1B**

An elementary particle created in the target of a particle accelerator arrives 5 meters of time later at a detector 4 meters from the target, as measured in the laboratory. The wristwatch of the elementary particle records what time between creation and detection?

**SOLUTION 1B**

The events of creation and detection are separated in the laboratory frame by  $s_{2\text{lab}} - s_{1\text{lab}} = 4$  meters and  $t_{2\text{lab}} - t_{1\text{lab}} = 5$  meters of time. Equation (1) tells us that

$$\tau^2 = 5^2 - 4^2 = 25 - 16 = 9 \text{ meters}^2 \quad (4)$$

$$\tau = 3 \text{ meters}$$

Again, the wristwatch time for the particle is less than the time recorded in the laboratory frame.

**PROBLEM 1C**

In Problem 1B the two events are separated by a distance of 4 meters, which means that it takes light 4 meters of light-travel time to move between them. But Solution 1B says that the particle's wristwatch records only 3 meters of time as the particle moves from the first to the second event. Does this mean that the particle travels faster than light?

**SOLUTION 1C**

This difficulty is common in relativity. The phrase "time between two events" has no unique value (initial quote of this chapter). The *time* depends on *which clock* measures the time, in this case either the laboratory clocks, which measure laboratory time separation  $t_{2\text{lab}} - t_{1\text{lab}}$ , or the particle's wristwatch, which measures lapsed wristwatch time  $\tau$ . Equation (1) already warns us that these two measures of time may not have the same value. Indeed a particle that moves faster and faster, covering a greater and greater distance  $s_{2\text{lab}} - s_{1\text{lab}}$  in the same laboratory time lapse  $t_{2\text{lab}} - t_{1\text{lab}}$ , records a wristwatch time  $\tau$  that gets smaller and smaller (Sample Problems 2), finally approaching—as a limit—the value zero, in which case a light flash has replaced the particle (Section 1.4). But for a particle with mass, the distance  $s_{2\text{lab}} - s_{1\text{lab}}$  it travels in the laboratory frame is always less than the laboratory time  $t_{2\text{lab}} - t_{1\text{lab}}$  that it takes the particle to move that distance. In other words, its laboratory speed will always be less than one, the speed of light. No particle can move faster than light moves in a vacuum. (Convince the scientific community that this statement is false, and your name will go down in history!)

**DEFINITION 6. Invariant**

Formally, a quantity is an **invariant** when it keeps the same value under some transformation. Equation (1) shows the interval between any pair of events along the path of a free stone to have the same value when calculated using coordinate separations in any inertial frame.

Transformations of coordinate separations between inertial frames are called **Lorentz transformations** (Section 1.10), so we say that the interval is a **Lorentz invariant**. However, the interval must also be an invariant under even more general transformations, not just Lorentz transformations, because all observers—not just those in inertial frames—will agree on the stone's wristwatch time lapse between any two given events. As a consequence, we most often drop the adjective *Lorentz* and use just the term **invariant**.

Definition:  
**invariant**

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### Sample Problems 2. Speeding to Andromeda

At approximately what constant speed  $v_{\text{Sun}}$  with respect to our Sun must a spaceship travel so that its occupants age only 1 year during a trip from Earth to the Andromeda galaxy? Andromeda lies 2 million light-years distant from Earth in the Sun's rest frame.

**SOLUTION** The word *approximately* in the statement of the problem tells us that we may make some assumptions. We assume that a single inertial frame can stretch all the way from Sun to Andromeda, so special relativity applies. Equation (1) leads us to predict that the speed  $v_{\text{Sun}}$  of the spaceship measured in the Sun frame is very close to unity, the speed of light. That allows us to set  $(1 + v_{\text{Sun}}) \approx 2$  in the last of the following steps:

$$\begin{aligned} \tau^2 &= (t_{2\text{Sun}} - t_{1\text{Sun}})^2 - (s_{2\text{Sun}} - s_{1\text{Sun}})^2 \quad (5) \\ &= (t_{2\text{Sun}} - t_{1\text{Sun}})^2 \left[ 1 - \left( \frac{s_{2\text{Sun}} - s_{1\text{Sun}}}{t_{2\text{Sun}} - t_{1\text{Sun}}} \right)^2 \right] \\ &= (t_{2\text{Sun}} - t_{1\text{Sun}})^2 (1 - v_{\text{Sun}}^2) \\ &= (t_{2\text{Sun}} - t_{1\text{Sun}})^2 (1 - v_{\text{Sun}})(1 + v_{\text{Sun}}) \\ &\approx 2(t_{2\text{Sun}} - t_{1\text{Sun}})^2 (1 - v_{\text{Sun}}) \end{aligned}$$

Equate the first and last expressions in (5) to obtain

$$1 - v_{\text{Sun}} \approx \frac{\tau^2}{2(t_{2\text{Sun}} - t_{1\text{Sun}})^2} \quad (6)$$

IF the spaceship speed  $v_{\text{Sun}}$  is very close to the speed of light, THEN the Sun-frame time for the trip to Andromeda is very close to the time that light takes to make the trip: 2 million years. Substitute this value for  $t_{2\text{Sun}} - t_{1\text{Sun}}$  and also demand that the wristwatch time on the spaceship (the aging of the occupants during their trip) be  $\tau = 1$  year. The result is

$$\begin{aligned} 1 - v_{\text{Sun}} &\approx \frac{1 \text{ year}^2}{2 \times 4 \times 10^{12} \text{ year}^2} \quad (7) \\ &= \frac{10^{-12}}{8} = 1.25 \times 10^{-13} \end{aligned}$$

Equation (7) expresses the result in sensible scientific notation. However, your friends may be more impressed if you report the speed as a fraction of the speed of light:  $v_{\text{Sun}} = 0.999\,999\,999\,999\,875$ . This result justifies our assumption that  $v_{\text{Sun}}$  is close to unity. *Additional question:* What is the *distance* ( $s_{2\text{rocket}} - s_{1\text{rocket}}$ ) between Earth and Andromeda measured in the rocket frame?

### 1.3.6 RULER DISTANCE

177 *Everyone agrees on the ruler distance between two events.*

178 Two firecrackers explode one meter apart and *at the same time*, as measured  
179 in a given inertial frame: in *this* frame the explosions are **simultaneous**. No  
180 stone—not even a light flash—can travel the distance between these two  
181 explosions in the zero time available in this frame. Therefore equation (1)  
182 cannot give us a value of the wristwatch time between these two events.

183 Simultaneous explosions are thus useless for measuring time. But they are  
184 perfect for measuring length. *Question:* How do you measure the length of a  
185 rod, whether it is moving or at rest in, say, the laboratory frame? *Answer:* Set  
186 off two firecrackers at opposite ends of the rod and *at the same time*  
187 ( $t_{2\text{lab}} - t_{1\text{lab}} = 0$ ) in that frame. Then *define* the rod's length in the laboratory  
188 frame as the *distance* ( $s_{2\text{lab}} - s_{1\text{lab}}$ ) between this pair of explosions  
189 simultaneous in that frame.

190 Special relativity warns us that another observer who flies through the  
191 laboratory typically does *not* agree that the two firecrackers exploded at the  
192 same time as recorded on her rocket clocks. This effect is called the **relativity**  
193 **of simultaneity**. The relativity of simultaneity is the bad news (and for many  
194 people the most difficult idea in special relativity). But here's the good news:  
195 All inertial observers, whatever their state of relative motion, can calculate the

Use simultaneous explosions to measure length of a rod.

Relativity of simultaneity



1-8 Chapter 1 Speeding

Spacelike interval  $\sigma$

196 distance  $\sigma$  between explosions as recorded in the frame in which they do occur  
 197 simultaneously. This calculation uses Case 2 of the interval (Definition 4):

$$\begin{aligned} \sigma^2 \equiv -\tau^2 &= (s_{2\text{lab}} - s_{1\text{lab}})^2 - (t_{2\text{lab}} - t_{1\text{lab}})^2 && \text{(spacelike interval)} \quad (8) \\ &= (s_{2\text{rocket}} - s_{1\text{rocket}})^2 - (t_{2\text{rocket}} - t_{1\text{rocket}})^2 \end{aligned}$$

198 The Greek letter *sigma*,  $\sigma$ , in (8)—equivalent to the Roman letter *s*—is the  
 199 length of the rod defined as the distance between explosions at its two ends  
 200 measured in a frame in which these explosions are simultaneous.

201 Equation (8) does not define a different kind of interval; it is merely  
 202 shorthand for the equation for Case 2 in Definition 4 in which  $\tau^2 < 0$ .

203 Actually, we do not need a rod or ruler to make use of this equation  
 204 (though we keep *ruler* as a label). Take any two events for which  $\tau^2 < 0$ . Then  
 205 there exists an inertial frame in which these two events occur at the same time;  
 206 we use this frame to define the **ruler distance**  $\sigma$  between these two events:

207 **DEFINITION 7. Ruler distance**

208 The **ruler distance**  $\sigma$  between two events is the distance between  
 209 these events measured by an inertial observer in whose frame the two  
 210 events occur at the same time. Another common name for ruler distance  
 211 is **proper distance**, which we do not use in this book.

Definition:  
**ruler distance**

212 Equation (8) tells us that every inertial observer can calculate the ruler  
 213 distance between two events using the space and time separations between  
 214 these events measured in his or her own frame.

215 **Fuller Explanation:** *Spacetime Physics*, Chapter 6, Regions of Spacetime

1.4 ■ LIGHTLIKE (NULL) INTERVAL

217 *Everyone agrees on the null value of the interval between two events connected*  
 218 *by a direct light flash that moves in a vacuum.*

219 Now think of the case in which the lab-frame space separation ( $s_{2\text{lab}} - s_{1\text{lab}}$ )  
 220 between two events is equal to the time separation ( $t_{2\text{lab}} - t_{1\text{lab}}$ ) between  
 221 them. In this case anything that moves uniformly between them must travel at  
 222 the speed of light  $v_{\text{lab}} = (s_{2\text{lab}} - s_{1\text{lab}})/(t_{2\text{lab}} - t_{1\text{lab}}) = 1$ . Physically, only a  
 223 direct light flash can move between this pair of events. We call the result a  
 224 **lightlike interval**:

$$\begin{aligned} \tau^2 = -\sigma^2 = 0 &= (s_{2\text{lab}} - s_{1\text{lab}})^2 - (t_{2\text{lab}} - t_{1\text{lab}})^2 && \text{(lightlike interval)} \quad (9) \\ &= (s_{2\text{rocket}} - s_{1\text{rocket}})^2 - (t_{2\text{rocket}} - t_{1\text{rocket}})^2 \end{aligned}$$

225 Because of its zero value, the lightlike interval is also called the **null interval**.

226 **DEFINITION 8. Lightlike (null) interval**

227 A **lightlike interval** is the interval between two events whose space

Definition:  
**lightlike interval**  
 or **null interval**

### Sample Problems 3. Causation

Three events have the following space and time coordinates as measured in the laboratory frame in meters of distance and meters of time. All three events lie along the  $x$ -axis in the laboratory frame. (Temporarily suppress the subscript "lab" in this Sample Problem.)

Event A:  $(t_A, x_A) = (2, 1)$

Event B:  $(t_B, x_B) = (7, 4)$

Event C:  $(t_C, x_C) = (5, 6)$

Classify the intervals between each pair of these events as timelike, lightlike, or spacelike:

- (a) between events A and B
- (b) between events A and C
- (c) between events B and C

In each case say whether or not it is possible for one of the events in the pair (which one?) to cause the other event of the pair, and if so, by what possible means.

**SOLUTION**

The interval between events A and B is:

$$\begin{aligned} \tau^2 &= (7 - 2)^2 - (4 - 1)^2 = 5^2 - 3^2 \\ &= 25 - 9 = +16 \end{aligned} \quad (10)$$

The time part is greater than the space part, so the interval between the events is *timelike*. Event A could have caused Event B, for example by sending a stone moving directly between them at a speed  $v_{\text{lab}} = 3/5$ . (There are other possible ways for Event A to cause Event B, for example by sending a light flash that sets off an explosion between the

two locations, with a fragment of the explosion reaching Event B at the scheduled time, and so forth. Our analysis says only that Event A *can* cause Event B, but it does not *force* Event A to cause Event B. Someone standing next to an object located at the  $x$ -coordinate of Event B could simply kick that object at the scheduled time of Event B.)

The interval between events A and C is:

$$\begin{aligned} \tau^2 &= (5 - 2)^2 - (6 - 1)^2 = 3^2 - 5^2 \\ &= 9 - 25 = -16 \end{aligned} \quad (11)$$

The space part is greater than the time part, so the interval between the events is *spacelike*. Neither event can cause the other, because to do so an object would have to travel between them at a speed greater than that of light.

The interval between events B and C is:

$$\begin{aligned} \tau^2 &= (7 - 5)^2 - (4 - 6)^2 = 2^2 - 2^2 \\ &= 4 - 4 = 0 \end{aligned} \quad (12)$$

The space part is equal to the time part, so the interval between the events is *lightlike*. Event C can cause Event B, but only by sending a direct light signal to it.

**Challenge:** How can we rule out the possibility that event B causes event A, or that event B causes event C? Would your answers to these questions be different if the same events are observed in some other frame in rapid motion with respect to the laboratory? (Answer in Exercise 1.)

228 separation and time separation are equal in every inertial frame. Only a  
 229 direct light flash can connect these two events. Because these space  
 230 and time separations are equal, the interval has the value zero, so is  
 231 also called the **null interval**.

232 **Comment 1. Einstein's derivation of special relativity**

233 Divide both sides of (9) by  $(t_{2,\text{frame}} - t_{1,\text{frame}})^2$ , where "frame" is either "lab" or  
 234 "rocket." The result tells us that the speed in any inertial frame is one,  
 235  $v_{\text{lab}} = v_{\text{rocket}} = 1$ . Einstein derived (9) starting with the *assumption* that the  
 236 speed of light is the same in all inertial frames.

237 **Fuller Explanation:** *Spacetime Physics*, Chapter 6, Regions of Spacetime.

1-10 Chapter 1 Speeding

1.5 ■ WORLDLINE OF A WANDERING STONE; THE LIGHT CONE

239 *A single curve tells all about the motion of our stone.*

240 Grasp a stone in your hand and move it alternately in one direction, then in  
 241 the opposite direction along the straight edge of your desk. Choose the  $x_{\text{lab}}$   
 242 axis along this line. Then the stone’s motion is completely described by the  
 243 function  $x_{\text{lab}}(t_{\text{lab}})$ . No matter how complicated this back-and-forth motion is,  
 244 we can view it at a glance when we plot  $x_{\text{lab}}$  along the horizontal axis of a  
 245 graph whose vertical axis represents the time  $t_{\text{lab}}$ . Figure 3 shows such a curve,  
 246 which we call a **worldline**.

Definition:  
worldline

247 **DEFINITION 9. Worldline**

248 A **worldline** is the path through spacetime taken by a stone or light  
 249 flash. By Definition 3, the total wristwatch time (aging) along the  
 250 worldline is the sum of wristwatch times between sequential events  
 251 along the worldline from a chosen initial event to a chosen final event.  
 252 The wristwatch time is an invariant; it has the same value when  
 253 calculated using either laboratory or rocket coordinates. Therefore  
 254 specification of a worldline requires neither coordinates nor the metric.

255 **Comment 2. Plotting the worldline**

256 Figure 3 shows a worldline plotted in laboratory coordinates. Typically a given  
 257 worldline will look different when plotted in rocket coordinates. We plot a  
 258 worldline in whatever coordinates we are using. Worldlines can be plotted in  
 259 spacetime diagrams for both flat and curved spacetime.

260 In the worldline of Figure 3 the stone starts at initial event O. As time  
 261 passes—as time advances upward in the diagram—the stone moves first to the  
 262 right. Then the stone slows down, that is it covers less distance to the right  
 263 per unit time, and comes to rest momentarily at event Z. (The vertical tangent  
 264 to the worldline at Z tells us that the stone covers zero laboratory distance  
 265 there: it is instantaneously at rest at Z.) Thereafter the stone accelerates to  
 266 the left in space until it arrives at event P.

Limits on  
worldline slope

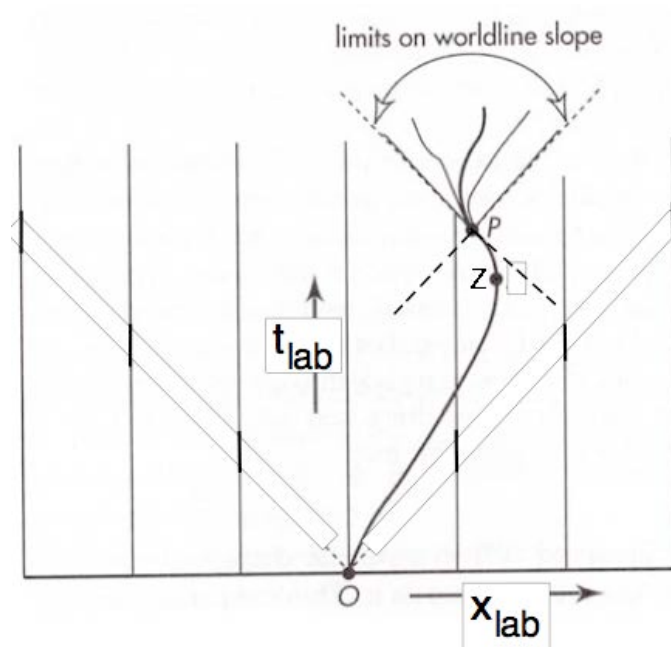
267 What possible future worldlines are available to the stone that arrives at  
 268 event P? Any material particle must move at less than the speed of light. In  
 269 other words, it travels less than one meter of distance in one meter of  
 270 light-travel time. Therefore its future worldline must make an “angle with the  
 271 vertical” somewhere between minus 45 degrees and plus 45 degrees in Figure  
 272 3, in which space and time are measured in the same units and plotted to the  
 273 same scale. These limits on the slope of the stone’s worldline—which apply to  
 274 every event on every worldline—emerge as dashed lines from event P in Figure  
 275 3. These dashed lines are worldlines of light rays that move in opposite  
 276  $x_{\text{lab}}$ -directions and cross at the event P. We call these crossed light rays a  
 277 **light cone**. Figure 4 displays the cone shape.

278 **DEFINITION 10. Light cone**

Definition:  
light cone

279 The **light cone** of an event is composed of the set of all possible  
 280 worldlines of light that intersect at that event and define its past and

Section 1.5 Worldline of a Wandering Stone; The Light Cone 1-11



**FIGURE 3** Curved **worldline** of a stone moving back and forth along a single straight spatial line in the laboratory. A point on this diagram, such as Z or P, combines  $x_{\text{lab}}$ -location (horizontal direction) with  $t_{\text{lab}}$ -location (vertical direction); in other words a point represents a spacetime *event*. The dashed lines through P are worldlines of light rays that pass through P. We call these crossed lines *the light cone of P*. For the cone shape, see Figure 4.

281 future (Figure 4). We also call it a light *cone* when it is plotted using one  
 282 space dimension plus time, as in Figure 3, and when plotted using three  
 283 space dimensions plus time—even though we cannot visualize the  
 284 resulting four-dimensional spacetime plot.

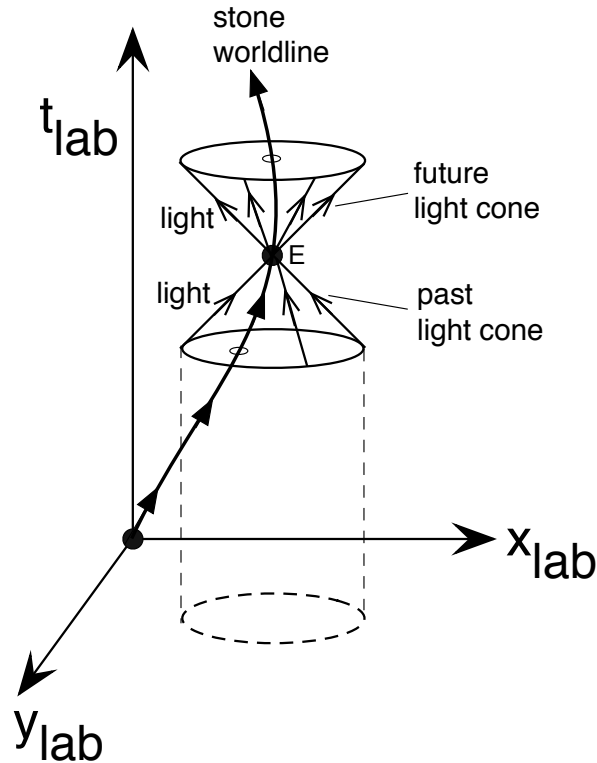
**THE LIGHT CONE AND CAUSALITY**

285 . . . *the light cone provides a mathematical tool for the analysis*  
 286 *of [general relativity] additional to the usual tools of metric*  
 287 *geometry. We believe that this tool still remains to be put to*  
 288 *full use, and that causality is the physical principle which will*  
 289 *guide this future development.*  
 290

291 —Robert W. Fuller and John Archibald Wheeler

292 **More complete explanation:** *Spacetime Physics*, Chapter 5, Trekking  
 293 Through Spacetime

1-12 Chapter 1 Speeding



**FIGURE 4** Light cone of Event E that lies on the worldline of a stone, plotted for two space dimensions plus time. The light cone consists of the upward-opening future light cone traced out by the expanding circular light flash that the stone emits at Event E, plus the downward-opening past light cone traced out by a contracting circular light flash that converges on Event E.

**1.6 ■ THE TWIN “PARADOX” AND THE PRINCIPLE OF MAXIMAL AGING**

295 *The Twin Paradox leads to a definition of natural motion.*

296 To get ready for curved spacetime (whatever that means), look more closely at  
 297 the motion of a free stone in *flat spacetime* (Definition 5), where special  
 298 relativity correctly describes motion.

Twin Paradox predicts  
 motion of a stone.

299 A deep description of motion arises from the famous **Twin Paradox**. One  
 300 twin—say a boy—relaxes on Earth while his fraternal twin sister frantically  
 301 travels to a distant star and returns. When the two meet again, the  
 302 stay-at-home brother has aged more than his traveling sister. (To predict this  
 303 outcome, extend Sample Problem 1A to include return of the traveler to the  
 304 point of origin.) Upon being reunited, the “twins” no longer look similar: the  
 305 traveling sister is *younger*: she has aged less than her stay-at-home brother.  
 306 Very strange! But (almost) no one who has studied relativity doubts the

Section 1.6 The Twin “Paradox” and The Principle of Maximal Aging 1-13

Being at rest is one *natural* motion.

307 difference in age, and every minute of every day somewhere on Earth a  
308 measurement with a fast-moving particle verifies it.

309 Which twin has the motion we can call *natural*? Isaac Newton has a  
310 definition of natural motion. He would say, “A twin at rest tends to remain at  
311 rest.” So it is the stay-at-home twin who moves in the natural way. In  
312 contrast, the out-and-back twin suffers the acceleration required to change her  
313 state of motion, from outgoing motion to incoming motion, so the twins can  
314 meet again in person. At least at her turnaround, the motion of the traveling  
315 twin is forced, *not natural*.

Moving uniformly is another *natural* motion.

316 Viewed from the second, relatively moving, inertial frame of the twin  
317 sister, the stay-at-home boy initially moves away from her with constant speed  
318 in a straight line. Again, his motion is *natural*. Newton would say, “A twin in  
319 uniform motion tends to continue this motion at constant speed in a straight  
320 line.” So the motion of the stay-on-Earth twin is also natural from the  
321 viewpoint of his sister’s frame in uniform relative motion—or from the  
322 viewpoint of any frame moving uniformly with respect to the original frame.  
323 In *any* such frame, the time lapse on the wristwatch of the stay-at-home twin  
324 can be calculated from the interval (1).

325 But there *is* a difference between the stay-at-home brother on Earth and  
326 the sister: She moves outward to a star, *then turns around* and returns to her  
327 Earthbound brother. So when her trip is over, everyone must agree: It is the  
328 brother who follows “natural” motion from parting event to reunion event.  
329 And it is the stay-at-home brother—whose wristwatch records the greater  
330 elapsed time—who **ages** the most.

Natural motion:  
Maximal  
wristwatch time.

331 The lesson we draw from the Twin Paradox in flat spacetime is that  
332 *natural* motion is the motion that maximizes the wristwatch time between *any*  
333 pair of events along its path. Now we can state the **Principle of Maximal  
334 Aging in flat spacetime**.

Definition: **Principle  
of Maximal Aging**

335 **DEFINITION 11. The Principle of Maximal Aging (flat spacetime)**  
336 The **Principle of Maximal Aging** states that the worldline a free stone  
337 follows between a pair of events in flat spacetime is the worldline for  
338 which the wristwatch time is a maximum compared with every possible  
339 alternative worldline between these events. The free stone follows the  
340 worldline of *maximal aging* between these two events.



341 **Objection 1.** *Why should I believe the Principle of Maximal Aging? Newton*  
342 *never talks about this weird idea! What does this so-called “Principle”*  
343 *mean, anyway?*



344 **Response:** For now the Principle of Maximal Aging is simply a restatement  
345 of the observation that in flat spacetime a free stone follows a straight  
346 worldline. It repeats Newton’s First Law of Motion: A free stone at rest or in  
347 motion maintains that condition. Why bother? Because general relativity  
348 revises and extends the Principle of Maximal Aging to predict the motion of  
349 a free stone in curved spacetime.

1-14 Chapter 1 Speeding



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**Objection 2.** *Wait! Have you really resolved the Twin Paradox? Both the twin sister and the twin brother sees his or her twin moving away, then moving back. Motion is relative, remember? The view of each twin is symmetrical, not only during the outward trip but also during the return trip. There is no difference between them. The experience of the two twins is identical; you cannot wriggle out of this essential symmetry! You have failed to explain why their wristwatches have different readings when they reunite.*



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Nice point. But you forget that the experience of the two twins is *not* identical. Fill in details of the story: When the twin sister arrives at the distant star and reverses her starship's direction of motion, that reversal throws her against the forward bulkhead. Ouch! She starts home with a painful lump on the right side of her forehead. Then when her ship slows down so she can stand next to her stay-at-home brother, she forgets her seat belt again. *Result:* a second painful lump, this time on the left side of her forehead. In contrast, her brother remains relaxed and uninjured during their entire separation. When the twins stand side by side, can *each* of them tell *which twin* has gone to the distant star? Of course! *More:* *Every passing observer*—whatever his or her speed or direction of motion—sees and reports the difference between the twins: “injured sister; smiling brother.” *Everyone* agrees on this difference. No contradiction and no confusion. “Paradox” resolved.

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**Comment 3. The Quintuplet “Paradox”**

In the last sentence of Definition 11, The Principle of Maximal Aging, notice the word “every” in the phrase “is a maximum compared with *every* alternative path...between the given initial and final events.” We are not just talking twins here, but triplets, quadruplets, quintuplets—indeed endless multiple births. *Example,* Figure 5: One quintuplet—**Quint #1**—follows the worldline of maximal aging between the two anchoring events by moving uniformly between them. Each of the other quints also starts from the same Initial Event A and ends at the same Final Event B, but follows a different alternative worldline—changes velocity—between initial and final events. When all the quints meet at the final event, all four traveling quints are younger than their uniformly-moving sibling, but typically by different amounts. *Every traveler, #2 through #5, who varies velocity between the two end-events is younger than its uniformly-moving sibling, Quint #1.* The Principle of Maximal Aging singles out one worldline among the limitless number of alternative worldlines between two end-events and demands that the free stone follow **this** worldline—and no other.

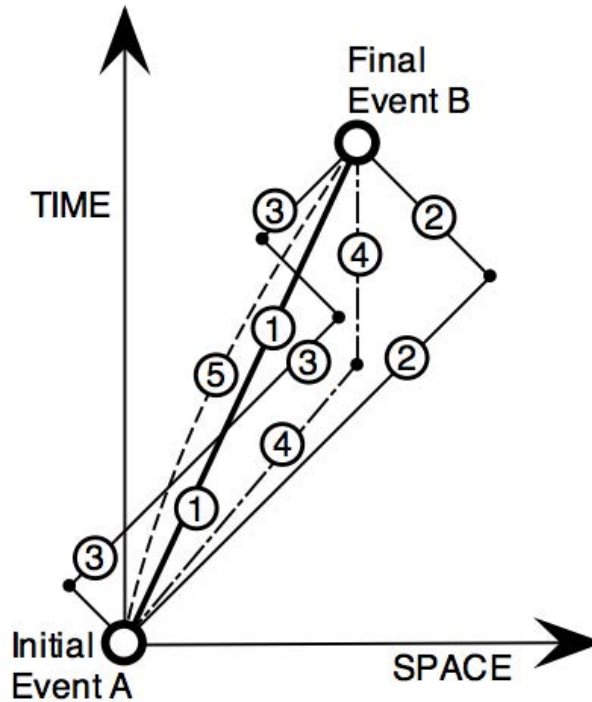
An infinite number of alternative worldlines: the free stone chooses one.

**QUERY 1. Analyze the Quintuplet Paradox**

Answer the following questions about the Quintuplet Paradox illustrated in Figure 5.

- A. Which of the five quints ages the *most* between end-events A and B? (Trick question!)
- B. Which of the five quints ages the *least* between end-events A and B?
- C. List the numbered worldlines in order, starting with the worldline along which the aging is the *least* and ending with the worldline along which the aging is the *most*.

Section 1.6 The Twin “Paradox” and The Principle of Maximal Aging 1-15



**FIGURE 5** The Quintuplet Paradox: Five alternative worldlines track the motion of five different quintuplets (**quints**) between Initial Event A and Final Event B along a spatial straight line. Quint #1 follows the (thick) worldline of maximal aging between A and B. Quint #2 moves along the (thin) worldline at 0.999 of the speed of light outward and then back again. Quint #3 follows a worldline (also a thin line) at the same *speed* as #2, but with three reversals of direction. Quint #4 shuffles (dot-dash line) to the spatial position of Final Event B, then relaxes there until her siblings join her at Event B. The (dashed) worldline of Quint #5 hugs worldline #1—the worldline of Maximal Aging—but does not quite follow it.

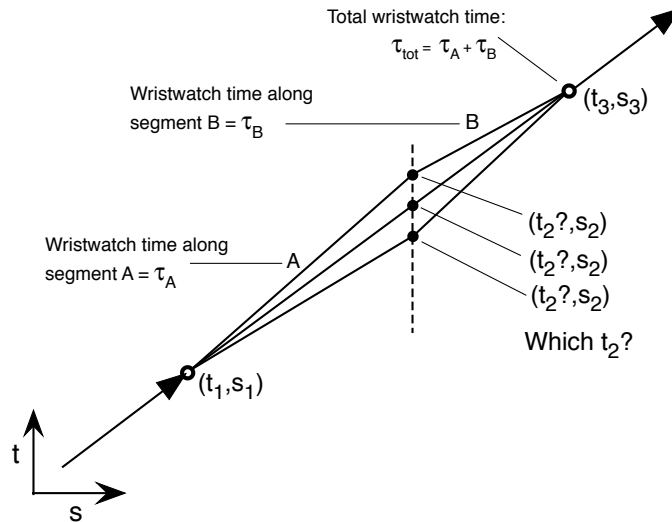
- D. True or false? If the dashed worldline of Quint #5 skims close enough to that of Quint #1—while still being separate from it—then Quint #5 will age the same as Quint #1 between end-events A and B.
- E. *Optional:* Suppose we view the worldlines of Figure 5 with respect to a frame in which Event A and Event B occur at the same spatial location. Whose *inertial* rest frame does this correspond to? Will your answers to Items A through D be different in this case?

401

402 **Fuller Explanation:** Twin “paradox:” *Spacetime Physics*, Chapter 4, Section  
 403 4.6.



1-16 Chapter 1 Speeding



**FIGURE 6** Figure for the derivation of the energy of a stone. Examine two adjacent segments, A and B, along an extended worldline plotted in, say, the laboratory frame. Choose three events at the endpoints of these two segments with coordinates  $(t_1, s_1)$ ,  $(t_2, s_2)$ , and  $(t_3, s_3)$ . All coordinates are fixed except  $t_2$ . Vary  $t_2$  to find the maximum value of the total aging  $\tau_{\text{tot}}$  (Principle of Maximal Aging). *Result:* an expression for the stone's energy  $E$ .

**1.7. ENERGY IN SPECIAL RELATIVITY**

405 *The Principle of Maximal Aging tells us the energy of a stone.*

406 Here is a modern translation (from Latin) of Isaac Newton's famous First Law  
407 of Motion:

Newton's First Law  
of motion

408 **Newton's first law of motion:** Every body perseveres in its state of  
409 being at rest or of moving uniformly straight forward except insofar as it  
410 is compelled to change its state by forces impressed.

Validity of Newton's  
First Law in special  
relativity . . .

411 In modern terminology, Newton's First Law says that, as measured in an  
412 inertial frame in flat spacetime, a free stone moves along a *straight worldline*,  
413 that is with constant speed along a straight path in space. We assumed the  
414 validity of Newton's First Law in defining the inertial frame (Definition 1,  
415 Section 1.1). In the present section the Principle of Maximal Aging again  
416 verifies this validity of the First Law. *Extra surprise!* This process will help us  
417 to derive the relativistic expression for the stone's energy  $E$ .

. . . leads to relativistic  
expression for energy.

418 Figure 6 illustrates the method: Consider two adjacent segments, A and B,  
419 of the stone's worldline with fixed events at the endpoints. Vary  $t_2$  of the  
420 middle event to find the value that gives a maximum for the total wristwatch  
421 time  $\tau_{\text{tot}}$  along the adjacent segments. Now the step-by-step derivation:

- 422 1. The wristwatch time between the first and second events along the  
423 worldline is the square root of the interval between them:

Section 1.7 Energy in Special Relativity **1-17**

$$\tau_A = \left[ (t_2 - t_1)^2 - (s_2 - s_1)^2 \right]^{1/2} \tag{13}$$

424 To prepare for the derivative that leads to maximal aging, differentiate  
 425 this expression with respect to  $t_2$ . (All other coordinates of the three  
 426 events are fixed.)

$$\frac{d\tau_A}{dt_2} = \frac{t_2 - t_1}{\left[ (t_2 - t_1)^2 - (s_2 - s_1)^2 \right]^{1/2}} = \frac{t_2 - t_1}{\tau_A} \tag{14}$$

427 2. The wristwatch time between the second and third events along the  
 428 worldline is the square root of the interval between them:

$$\tau_B = \left[ (t_3 - t_2)^2 - (s_3 - s_2)^2 \right]^{1/2} \tag{15}$$

429 Again, to prepare for the derivative that leads to extremal aging,  
 430 differentiate this expression with respect to  $t_2$ :

$$\frac{d\tau_B}{dt_2} = -\frac{t_3 - t_2}{\left[ (t_3 - t_2)^2 - (s_3 - s_2)^2 \right]^{1/2}} = -\frac{t_3 - t_2}{\tau_B} \tag{16}$$

431 3. The total wristwatch time  $\tau_{\text{tot}}$  from event #1 to event #3—the total  
 432 aging between these two events—is the sum of the wristwatch time  $\tau_A$   
 433 between the first two events plus the wristwatch time  $\tau_B$  between the  
 434 last two events:

$$\tau_{\text{tot}} = \tau_A + \tau_B \tag{17}$$

435 4. Now ask: At what intermediate  $t_2$  will a free stone pass the  
 436 intermediate point in space  $s_2$  and emit the second flash #2? Answer  
 437 by using the Principle of Maximal Aging: The time  $t_2$  will be such that  
 438 the total aging  $\tau_{\text{tot}}$  in (17) is a maximum. To find this maximum take  
 439 the derivative of  $\tau$  with respect to  $t_2$  and set the result equal to zero.  
 440 Add the final expressions (14) and (16) to obtain:

Principle of Maximal  
 Aging finds time  $t_2$   
 for middle event.

$$\frac{d\tau_{\text{tot}}}{dt_2} = \frac{t_2 - t_1}{\tau_A} - \frac{t_3 - t_2}{\tau_B} = 0 \tag{18}$$

441 6. In equation (18) the time  $(t_2 - t_1)$  is the lapse of laboratory time for  
 442 the stone to traverse segment A. Call this time  $t_A$ . The time  $(t_3 - t_2)$  is  
 443 the lapse of laboratory time for the stone to traverse segment B. Call  
 444 this time  $t_B$ . Then rewrite (18) in the simple form

Quantity whose  
 value is the  
 same for adjoining  
 segments

$$\frac{t_A}{\tau_A} = \frac{t_B}{\tau_B} \tag{19}$$

445 This result yields a maximum  $\tau_{\text{tot}}$ , *not* a minimum; see Exercise 4.

1-18 Chapter 1 Speeding

446 7. We did not say *which* pair of adjoining segments along the worline we  
 447 were talking about, so equation (19) must apply to *every* pair of  
 448 adjoining segments *anywhere* along the path. Suppose that there are  
 449 three such adjacent segments. If the value of the expression is the same  
 450 for, say, the first and second segments and also the same for the second  
 451 and third segments, then it must be the same for the first and third  
 452 segments. Continue in this way to envision a whole series of adjoining  
 453 segments, labeled A, B, C, D,..., for each of which equation (19)  
 454 applies, leading to the set of equations

$$\frac{t_A}{\tau_A} = \frac{t_B}{\tau_B} = \frac{t_C}{\tau_C} = \frac{t_D}{\tau_D} \rightarrow \frac{dt_{\text{lab}}}{d\tau} \tag{20}$$

455 where all coordinate values are given in the laboratory frame.

**Comment 4. Differences to differentials**

Differences shrink  
to differentials

456 The last step, with the arrow, in (20) is a momentous one. We take the calculus  
 457 limit by shrinking to differentials—infinitesimals—all the differences in physical  
 458 quantities. In Figure 6, for example, segments A and B shrink to infinitesimals.  
 459 Why is this step important? Because in general relativity, curvature of spacetime  
 460 means that relations between adjacent events are described accurately only  
 461 when *adjacent* events are differentially close to one another. If they are far apart,  
 462 the two events may be in regions of different spacetime curvature.  
 463

464 What does the result (20) mean? We now show that  $dt_{\text{lab}}/d\tau$  in (20) is the  
 465 expression for energy per unit mass of a free stone in the laboratory frame.  
 466 The differential form of (1) yields:

$$d\tau^2 = dt_{\text{lab}}^2 - ds_{\text{lab}}^2 = dt_{\text{lab}}^2 (1 - ds_{\text{lab}}^2/dt_{\text{lab}}^2) = dt_{\text{lab}}^2 (1 - v_{\text{lab}}^2) \tag{21}$$

467 Combine (20) with (21):

$$\frac{dt_{\text{lab}}}{d\tau} = \frac{1}{(1 - v_{\text{lab}}^2)^{1/2}} \tag{22}$$

468 Working in a single inertial frame, we have just found that  $dt/d\tau$  is  
 469 unchanging along the worldline of a free stone, which by Definition 11 is the  
 470 worldline of maximal aging. It follows that  $v_{\text{lab}}$  is constant. Hence the  
 471 Principle of Maximal Aging leads to the result that in flat spacetime the free  
 472 stone moves at constant speed. (The derivation of relativistic momentum in  
 473 Section 1.8 shows that the free stone's *velocity* is also constant, so that it  
 474 moves along a straight worldline in every inertial frame.)

475 We show below that at low speeds (22) reduces to Newton's expression for  
 476 kinetic energy plus rest energy, all divided by the stone's mass  $m$ . This  
 477 supports our decision to call the expression in (22) the energy per unit mass of  
 478 the stone:

Section 1.7 Energy in Special Relativity **1-19**

$$\frac{E_{\text{lab}}}{m} = \frac{dt_{\text{lab}}}{d\tau} = \frac{1}{(1 - v_{\text{lab}}^2)^{1/2}} = \gamma_{\text{lab}} \quad (23)$$

479

480 The last expression in (23) introduces a symbol—Greek lower case  
481 gamma—that we use to simplify later equations.

$$\gamma_{\text{lab}} \equiv \frac{1}{(1 - v_{\text{lab}}^2)^{1/2}} \quad (24)$$

482

483 We call  $E_{\text{lab}}/m$  a **constant of motion** because the free stone’s energy  
484 does not change as it moves in the laboratory frame. This may seem trivial for  
485 a stone that moves with constant speed in a straight line. In general relativity,  
486 however, we will find an “energy” that is a constant of motion for a free stone  
487 in orbit around a center of gravitational attraction.

488 We applied the Principle of Maximal Aging to motion in the laboratory  
489 frame. An almost identical derivation applies in the rocket frame. Coordinates  
490 of the initial and final events will differ from those in Figure 6, but the result  
491 will still be that  $dt_{\text{rocket}}/d\tau$  is constant along the free stone’s worldline:

$$\frac{E_{\text{rocket}}}{m} = \frac{dt_{\text{rocket}}}{d\tau} = \frac{1}{(1 - v_{\text{rocket}}^2)^{1/2}} = \gamma_{\text{rocket}} \quad (25)$$

492

493 Typically the value of the energy will be different in different inertial  
494 frames. We expect this, because the speed of a stone is not necessarily the  
495 same in different frames.

496 Equations (23) and (25) tell us that the energy of a stone in a given  
497 inertial frame increases without limit when the stone’s speed approaches the  
498 value one, the speed of light, in that frame. Therefore the speed of light is the  
499 limit of the speed of a stone—or of any particle with mass—measured in any  
500 inertial frame. The other limit of (23) is a stone at rest in the laboratory. In  
501 this case, equation (23) reduces to

$$E_{\text{lab}} = m \quad (\text{when speed of stone } v_{\text{lab}} = 0) \quad (26)$$

502 We express  $m$ , the mass of the stone, in units of energy. If you insist on using  
503 conventional units, such as joules for energy and kilograms for mass, then a  
504 conversion factor  $c^2$  intrudes into our simple expression. The result is the most  
505 famous equation in all of physics:

$$E_{\text{lab,conv}} = m_{\text{conv}}c^2 \quad (\text{when speed of stone } v_{\text{lab}} = 0) \quad (27)$$

506 Here the intentionally-awkward subscript “conv” means “conventional units.”  
507 Equations (26) and (27) both quantify the *rest energy* of a stone; both tell us

Rest energy

1-20 Chapter 1 Speeding

**Sample Problems 4. Energy Magnitudes**

**PROBLEM 4A**

The “speed ladder” in Figure 2 shows that the fastest wheeled vehicle moves on land at a speed approximately  $v \approx 10^{-6}$ . The kinetic energy of this vehicle is what fraction of its rest energy?

**SOLUTION 4A**

For such an “everyday” speed, the approximation on the right side of equation (28) should be sufficiently accurate. Then  $v^2 \approx 10^{-12}$  and approximate equation (28) tells us that:

$$\frac{\text{kinetic energy}}{\text{rest energy}} = \frac{mv^2}{2m} = \frac{v^2}{2} \approx 5 \times 10^{-13} \quad (29)$$

**PROBLEM 4B**

With what speed  $v$  must a stone move so that its kinetic energy equals its rest energy?

**SOLUTION 4B**

This problem requires relativistic analysis. Equation (23) gives total energy and (26) gives rest energy. Kinetic energy is the difference between the two:

$$\frac{E_{\text{lab}} - m}{m} = \frac{1}{(1 - v^2)^{1/2}} - 1 = 1 \quad (30)$$

from which

$$1 - v^2 = \frac{1}{2^2} = \frac{1}{4} \quad (31)$$

so that

$$v = \left(\frac{3}{4}\right)^{1/2} = 0.866 \quad (32)$$

This speed is a fraction of the speed of light, which means that  $v_{\text{conv}} = 0.866 \times 3.00 \times 10^8$  meters/second =  $2.60 \times 10^8$  meters/second.

**PROBLEM 4C**

Our Sun radiates  $3.86 \times 10^{26}$  watts of light. How much mass does it convert to radiation every second?

**SOLUTION 4C**

This problem provides exercise in converting units. One watt is one joule/second. The units of energy are the units of (force  $\times$  distance) or (mass  $\times$  acceleration  $\times$  distance). Therefore the units of joule are kilogram-meter<sup>2</sup>/second<sup>2</sup>. From (27):

$$\begin{aligned} m &= \frac{E_{\text{conv}}}{c^2} \quad (33) \\ &= \frac{3.86 \times 10^{26} \text{ kilogram-meters}^2/\text{second}^2}{(3.00 \times 10^8 \text{ meters/second})^2} \\ &\approx 4.3 \times 10^9 \text{ kilograms} \\ &\approx 4.3 \times 10^6 \text{ metric tons} \end{aligned}$$

This is the mass—a few million metric tons—that our Sun, a typical star, converts into radiation every second.

508 that mass itself is a treasure trove of energy. On Earth, nuclear reactions  
509 release less than one percent of this available energy. In contrast, a  
510 particle-antiparticle annihilation can release *all* of the mass of the combining  
511 particles in the form of radiant energy (gamma rays).

512 At everyday speeds, the expression for  $E_{\text{lab}}$  in (23) reduces to an  
513 expression that contains Newton’s kinetic energy. How do we get to Newton’s  
514 case? Simply ask: How fast do things move around us in our everyday lives?  
515 At this writing, the fastest speed achieved by a wheeled vehicle on land is 1228  
516 kilometers per hour (Figure 2), which is 763 miles per hour or 280 meters per  
517 second. As a fraction of light speed, this vehicle moves at  $v = 9.3 \times 10^{-7}$  (no  
518 units). For such a small fraction, we can use a familiar approximation (inside  
519 the front cover):

$$\begin{aligned} E_{\text{lab}} &= \frac{m}{(1 - v_{\text{lab}}^2)^{1/2}} = m(1 - v_{\text{lab}}^2)^{-1/2} \approx m\left(1 + \frac{v_{\text{lab}}^2}{2}\right) \quad (28) \\ &\approx m + \frac{1}{2}mv_{\text{lab}}^2 = m + (KE)_{\text{Newton}} \quad (v_{\text{lab}} \ll 1) \end{aligned}$$

520 You can verify that the approximation is highly accurate when  $v_{\text{lab}}$  has the  
521 value of the land speed record—and is an even better approximation for the

Section 1.8 Momentum in Special Relativity 1-21

522 everyday speeds of a bicycle or football. The final term in (28) is Newton's  
 523 (low speed) expression for the kinetic energy of the stone. The first term is the  
 524 rest energy of the stone, equation (26).

525 We can also separate the relativistic expression for energy into rest energy  
 526 and kinetic energy. Define the relativistic kinetic energy of a stone in any  
 527 frame with the equation

$$KE \equiv E - m = m(\gamma - 1) \quad (\text{any frame, any speed}) \quad (34)$$

528

**Comment 5. Deeper than Newton?**

529 Newton's First Law of Motion, quoted at the beginning of this section, was his  
 530 brilliant assumption. In the present section we have derived this result using the  
 531 Principle of Maximal Aging. Is our result deeper than Newton's? We think so,  
 532 because the Principle of Maximal Aging has wider application than special  
 533 relativity. It informs our predictions for the motion of a stone around both the  
 534 non-spinning and the spinning black hole. Deep indeed!  
 535

536 **Fuller Explanation:** Energy in flat spacetime: *Spacetime Physics*, Chapter 7,  
 537 Momenergy.

**1.8.3 ■ MOMENTUM IN SPECIAL RELATIVITY**

539 *The interval plus the Principle of Maximal Aging give us an expression for the*  
 540 *linear momentum of a stone.*

541 To derive the relativistic expression for the momentum of a stone, we use a  
 542 method similar to that for the derivation of energy in Section 1.7. Figure 7  
 543 corresponds to Figure 6, which we used to derive the stone's energy.  
 544 Momentum has components in all three space directions; first we derive its  
 545  $x_{\text{lab}}$  component, which we write as  $p_{x,\text{lab}}$ . In the momentum case the time  $t_2$   
 546 for the intermediate flash emission is *fixed*, while we vary the space coordinate  
 547  $s_2$  of this intermediate event to find the location that yields maximum  
 548 wristwatch time between initial and final events. We ask you to carry out this  
 549 derivation in the exercises. The result is a second expression whose value is  
 550 constant for a free stone in either the laboratory frame or the rocket frame:

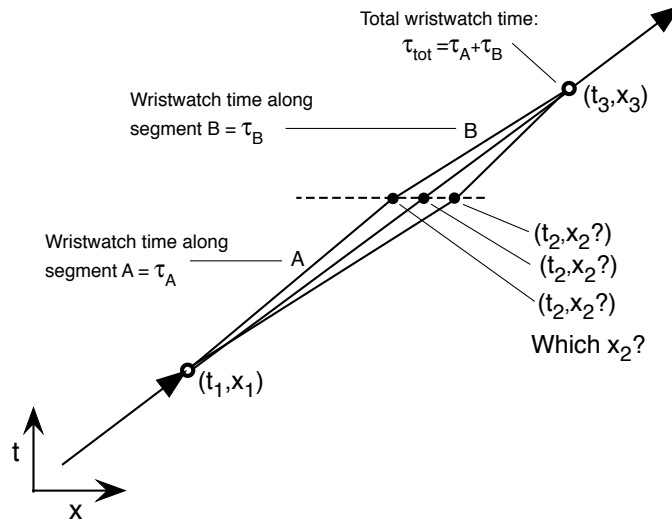
$$\frac{p_{x,\text{lab}}}{m} = \frac{dx_{\text{lab}}}{d\tau} = \frac{v_{x,\text{lab}}}{(1 - v_{\text{lab}}^2)^{1/2}} = \gamma_{\text{lab}} v_{x,\text{lab}} \quad (35)$$

$$\frac{p_{x,\text{rocket}}}{m} = \frac{dx_{\text{rocket}}}{d\tau} = \frac{v_{x,\text{rocket}}}{(1 - v_{\text{rocket}}^2)^{1/2}} = \gamma_{\text{rocket}} v_{x,\text{rocket}} \quad (36)$$

551

552 where  $v_{\text{lab}}$  and  $v_{\text{rocket}}$  are each constant in the respective frame, and  $\gamma$  was  
 553 defined in (24). Expressions for the  $y_{\text{lab}}$  and  $z_{\text{lab}}$  components of momentum

1-22 Chapter 1 Speeding



**FIGURE 7** Figure for the derivation of the  $x$ -component of momentum of a stone. You will carry out this derivation in the exercises.

$p_{x,\text{lab}}/m = dx_{\text{lab}}/d\tau$  is a constant of motion.

are similar to (35) and (36). The result for each component of momentum reminds us that the free stone moves with constant speed in a straight line in every inertial frame.

Each component of the free stone's momentum in the laboratory frame is a *constant of motion*, like its energy  $E_{\text{lab}}/m$  in the laboratory frame, because each component of momentum does not change as the free stone moves in the laboratory frame. Momentum components of the stone in the rocket frame are also constants of motion, though equations (35) and (36) show that corresponding components in the two frames are not equal, because the stone's velocity is not the same in the two frames.

At slow speed,  $v \ll 1$ , we recover Newton's components of momentum in both frames. This justifies our calling components in (35) and (36) *momentum*.

**Fuller Explanation:** Momentum in flat spacetime: *Spacetime Physics*, Chapter 7, Momenergy.

**1.9 ■ MASS IN RELATIVITY**

*The mass  $m$  of a stone is an invariant!*

Find mass from energy and momentum.

An important relation among mass, energy, and momentum follows from the timelike interval and our relativistic expressions for energy and momentum. Suppose a moving stone emits two flashes differentially close together in distance  $ds_{\text{lab}}$  and in time  $dt_{\text{lab}}$ , with similar differentials in the rocket frame. Then (1) gives the lapse of wristwatch time  $d\tau$ :

$$d\tau^2 = dt_{\text{lab}}^2 - ds_{\text{lab}}^2 = dt_{\text{rocket}}^2 - ds_{\text{rocket}}^2 \quad (37)$$

### Box 1. No Mass Change with Speed!

The fact that no stone moves faster than the speed of light is sometimes “explained” by saying that “the mass of a stone increases with speed,” leading to what is called “relativistic mass” whose increase prevents acceleration to a speed greater than that of light. This interpretation can be applied consistently, but what could it mean in practice? Someone riding along with the faster-moving stone detects no change in the number of atoms in the stone, nor any change whatever in the individual atoms, nor in the binding energy between atoms. Where’s the “change” in what is claimed to be a “changing mass”? We observe no change in the stone that can possibly account for the varying value of its “relativistic mass.”

Our viewpoint in this book is that mass is a *Lorentz invariant*, something whose value is the same for all inertial observers when they use (39) or (40) to reckon the mass. In relativity, every invariant is a diamond. Do not throw away a diamond!

To preserve the diamond of invariant mass, we will never—outside the confines of this box—use the phrase “rest mass.” (Horrors!). Why not? Because “rest mass” (Ouch!) implies that there is such a thing as “non-rest mass”—mass that changes with speed. Oops, there goes your precious diamond down the drain.

In contrast, the phrase *rest energy* is fine; it *is true* that energy changes with speed; the energy of a stone *does* have different values as measured by inertial observers in uniform relative motion. In the special case of a stone at rest in any inertial frame, however, the value of its rest energy *in that frame* is equal to the value of its mass—equation (26)—provided you use the same units for mass as for energy.

“Rest mass”? NO!  
Rest energy? YES!

For more on this subject see *Spacetime Physics*, **Dialog: Use and Abuse of the Concept of Mass**, pages 246–251.

575 Divide equation (37) through by the invariant  $d\tau^2$  and multiply through by  
576 the invariant  $m^2$  to obtain

$$m^2 = \left( m \frac{dt_{\text{lab}}}{d\tau} \right)^2 - \left( m \frac{ds_{\text{lab}}}{d\tau} \right)^2 = \left( m \frac{dt_{\text{rocket}}}{d\tau} \right)^2 - \left( m \frac{ds_{\text{rocket}}}{d\tau} \right)^2 \quad (38)$$

577 Substitute expressions (23) and (35) for energy and momentum to obtain:

$$m^2 = E_{\text{lab}}^2 - p_{\text{lab}}^2 = E_{\text{rocket}}^2 - p_{\text{rocket}}^2 \quad (39)$$

579 In (39) mass, energy, and momentum are all expressed in the same units, such  
580 as kilograms or electron-volts. In conventional units (subscript “conv”), the  
581 equation has a more complicated form. In either frame:

$$(m_{\text{conv}}c^2)^2 = E_{\text{conv}}^2 - p_{\text{conv}}^2c^2 \quad (40)$$

Stone’s energy  
(also momentum)  
may be different  
for different  
observers...

... but its mass  
has the same  
(invariant!) value  
in all frames.

582 Equations (39) and (40) are central to special relativity. There is nothing like  
583 them in Newton’s mechanics. The stone’s energy  $E$  typically has different  
584 values when measured in different inertial frames that are in uniform relative  
585 motion. Also the stone’s momentum  $p$  typically has different values when  
586 measured in different frames. However, the values of these two quantities in  
587 *any* given inertial frame can be used to determine the value of the stone’s mass  
588  $m$ , which is independent of the inertial frame. *The stone’s mass  $m$  is a Lorentz*  
589 *invariant* (Definition 6 and Box 1).



1-24 Chapter 1 Speeding

590 **Fuller Explanation:** Mass and momentum-energy in flat spacetime:  
 591 *Spacetime Physics*, Chapter 7, Momenergy.

1.10 ■ THE LORENTZ TRANSFORMATION

593 *Relative motion; relative observations*

594 To develop special relativity, Einstein assumed that the laws of physics are the  
 595 same in every inertial frame, an assertion called **The Principle of**  
 596 **Relativity**. Let two different inertial frames, such as those of a laboratory and  
 597 an unpowered rocket ship, be in uniform relative motion with respect to one  
 598 another. Special relativity is valid in each of these frames. *More:* Special  
 599 relativity links the coordinates of an event in one frame with the coordinates  
 600 of the same event in the other frame; it also relates the energy and momentum  
 601 components of a stone measured in one frame to the corresponding quantities  
 602 measured in the other frame. Let an inertial (unpowered) rocket frame pass  
 603 with relative velocity  $v_{\text{rel}}$  in the  $x$ -direction through an overlapping laboratory  
 604 frame. Call the laboratory coordinate separations between two events  
 605  $(\Delta t_{\text{lab}}, \Delta x_{\text{lab}}, \Delta y_{\text{lab}}, \Delta z_{\text{lab}})$  and the rocket coordinate separations between the  
 606 same events  $(\Delta t_{\text{rocket}}, \Delta x_{\text{rocket}}, \Delta y_{\text{rocket}}, \Delta z_{\text{rocket}})$ . From now on we use the  
 607 Greek letter capital delta,  $\Delta$ , as a shorthand for separation, to avoid lengthy  
 608 expressions, for example  $\Delta t_{\text{lab}} = t_{2,\text{lab}} - t_{1,\text{lab}}$ . These separations are related  
 609 by the **Lorentz transformation equations**:

Lorentz transform  
from lab to rocket

$$\begin{aligned} \Delta t_{\text{rocket}} &= \gamma_{\text{rel}} (\Delta t_{\text{lab}} - v_{\text{rel}} \Delta x_{\text{lab}}) & (41) \\ \Delta x_{\text{rocket}} &= \gamma_{\text{rel}} (\Delta x_{\text{lab}} - v_{\text{rel}} \Delta t_{\text{lab}}) \\ \Delta y_{\text{rocket}} &= \Delta y_{\text{lab}} \quad \text{and} \quad \Delta z_{\text{rocket}} = \Delta z_{\text{lab}} \end{aligned}$$

610 where equation (24) defines  $\gamma_{\text{rel}}$ . We do not derive these equations here; see  
 611 Fuller Explanation at the end of this section. The reverse transformation, from  
 612 rocket to laboratory coordinates, follows from symmetry: replace  $v_{\text{rel}}$  by  $-v_{\text{rel}}$   
 613 and interchange rocket and lab labels in (41) to obtain

Lorentz transform  
from rocket to lab

$$\begin{aligned} \Delta t_{\text{lab}} &= \gamma_{\text{rel}} (\Delta t_{\text{rocket}} + v_{\text{rel}} \Delta x_{\text{rocket}}) & (42) \\ \Delta x_{\text{lab}} &= \gamma_{\text{rel}} (\Delta x_{\text{rocket}} + v_{\text{rel}} \Delta t_{\text{rocket}}) \\ \Delta y_{\text{lab}} &= \Delta y_{\text{rocket}} \quad \text{and} \quad \Delta z_{\text{lab}} = \Delta z_{\text{rocket}} \end{aligned}$$

614 For a pair of events infinitesimally close to one another, we can reduce  
 615 differences in (42) and (41) to coordinate differentials. Further: It is also valid  
 616 to divide the resulting equations through by the Lorentz invariant differential  
 617  $d\tau$  and multiply through by the invariant mass  $m$ . Then substitute from  
 618 equations (23) and (35). *Result:* Two sets of equations that transform the  
 619 energy  $E$  and the components  $(p_x, p_y, p_z)$  of the momentum of a stone between  
 620 these two frames:

Transform energy  
and momentum from  
lab to rocket

$$E_{\text{rocket}} = \gamma_{\text{rel}} (E_{\text{lab}} - v_{\text{rel}} p_{x,\text{lab}}) \tag{43}$$

$$p_{x,\text{rocket}} = \gamma_{\text{rel}} (p_{x,\text{lab}} - v_{\text{rel}} E_{\text{lab}})$$

$$p_{y,\text{rocket}} = p_{y,\text{lab}} \quad \text{and} \quad p_{z,\text{rocket}} = p_{z,\text{lab}}$$

Transform energy  
and momentum from  
rocket to lab

621 Here  $p_{x,\text{rocket}}$  is the  $x$ -component of momentum in the rocket frame, and so  
622 forth. The reverse transformation, again by symmetry:

$$E_{\text{lab}} = \gamma_{\text{rel}} (E_{\text{rocket}} + v_{\text{rel}} p_{x,\text{rocket}}) \tag{44}$$

$$p_{x,\text{lab}} = \gamma_{\text{rel}} (p_{x,\text{rocket}} + v_{\text{rel}} E_{\text{rocket}})$$

$$p_{y,\text{lab}} = p_{y,\text{rocket}} \quad \text{and} \quad p_{z,\text{lab}} = p_{z,\text{rocket}}$$

623 We can now predict and compare measurements in inertial frames in  
624 relative motion. And remember, special relativity assumes that every inertial  
625 frame extends without limit in every direction and for all time.

Lorentz boost

626 **Comment 6. Nomenclature: Lorentz boost**

627 Often a Lorentz transformation is called a **Lorentz boost**. The word *boost* does  
628 not mean sudden change, but rather a change in the frame from which we make  
629 measurements and observations.

630 **Comment 7. Constant of motion vs. invariant**

631 An *invariant* is not the same as a *constant of motion*. Here is the difference:

632 An invariant is a quantity that has the same value *in all inertial frames*. Two  
633 sample invariants: (a) the wristwatch time between any two events, (b) the mass  
634 of a stone. The term *invariant* must always tell or imply what the change is that  
635 leads to the same result. Carefully stated, we would say: "The wristwatch time  
636 between two events and the mass of a stone are each invariant with respect to a  
637 Lorentz transformation between the laboratory and the rocket frame."

638 By contrast, a *constant of motion* is a quantity that stays unchanged along the  
639 worldline of a free stone *as calculated in a given inertial frame*. Two sample  
640 constants of motion: (a) the energy and (b) the momentum of a free stone as  
641 observed or measured in, say, the laboratory frame. In other inertial frames  
642 moving relatively to the lab frame, the energy and momentum of the stone are  
643 also constants of motion; however, these quantities typically have *different*  
644 *values in different inertial frames*.

645 *Conclusion:* Invariants (diamonds) and constants of motion (rubies) are both  
646 truly precious.

647 **Fuller Explanation:** *Spacetime Physics*, Special Topic: Lorentz  
648 Transformation.

1-26 Chapter 1 Speeding

1.14 ■ LIMITS ON LOCAL INERTIAL FRAMES

650 *Limits on the extent of an inertial frame in curved spacetime*

651 Flat spacetime is the arena in which special relativity describes Nature. The  
 652 power of special relativity applies strictly only in an inertial frame—or in each  
 653 one of a collection of overlapping inertial frames in uniform relative motion. In  
 654 every inertial frame, by definition, a free stone released from rest remains at  
 655 rest and a free stone launched with a given velocity maintains the magnitude  
 656 and direction of that velocity.

Limits on size of  
 local inertial  
 frames? We need  
 general relativity.

657 If it were possible to embrace the Universe with a single inertial frame,  
 658 then special relativity would describe our Universe, and we would not need  
 659 general relativity. But we *do* need general relativity, precisely because typically  
 660 an inertial frame is inertial in only a limited region of space and time. Near a  
 661 center of attraction, every inertial frame must be **local**. An inertial frame can  
 662 be set up, for example, inside a sufficiently small “container,” such as (a) an  
 663 unpowered rocket ship in orbit around Earth or Sun, or (b) an elevator on  
 664 Earth whose cables have been cut, or (c) an unpowered rocket ship in  
 665 interstellar space. In each such inertial frame, for a limited extent of space and  
 666 time, we find no evidence of gravity.

Inertial frame  
 cannot be too  
 large, because . . .

667 Well, *almost* no evidence. Every inertial enclosure in which we ride near  
 668 Earth cannot be too large or fall for too long a frame time without some  
 669 unavoidable change in relative motion between a pair of free stones in the  
 670 enclosure. Why? Because each one of a pair of widely separated stones within a  
 671 large enclosed space is affected differently by the nonuniform gravitational field  
 672 of Earth—as Newton would say. For example, two stones released from rest  
 673 side by side are both attracted toward the center of Earth, so they move closer  
 674 together as measured inside a falling long narrow horizontal railway coach  
 675 (Figure 8, left panel). Their motion toward one another has nothing to do with  
 676 gravitational attraction between these stones, which is entirely negligible.

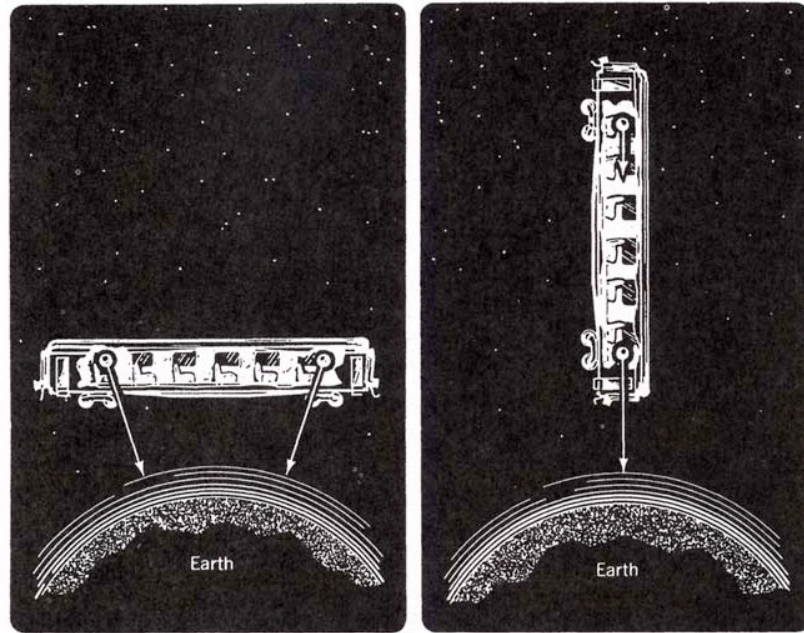
. . . tidal accelerations  
 occur in large frames.

677 As another example, think of two stones released from rest far apart  
 678 vertically, one directly above the other in a long narrow vertical falling railway  
 679 coach (Figure 8, right panel). For vertical separation, their gravitational  
 680 accelerations toward Earth are both in the same direction. However, the stone  
 681 nearer Earth is more strongly attracted to Earth, so gradually leaves the other  
 682 stone behind, according to Newton’s analysis. As a result, viewed from inside  
 683 the coach the two stones move farther apart. *Conclusion:* The large enclosure  
 684 is not an inertial frame.

685 A rider in either railway car such as those shown in Figure 8 sees the pair  
 686 of horizontally-separated stones accelerate *toward* one another and a pair of  
 687 vertically-separated stones accelerate *away* from one another. These relative  
 688 motions earn the name **tidal accelerations**, because they arise from the same  
 689 kind of nonuniform gravitational field that accounts for ocean tides on  
 690 Earth—tides due to the field of the Moon, which is stronger on the side of  
 691 Earth nearer the Moon.

Unavoidable tidal  
 accelerations?  
 Then unavoidable  
 spacetime curvature!

692 As we fall toward the center of attraction, there is no way to avoid the  
 693 relative—*tidal*—accelerations at different locations in the long railway car. We



**FIGURE 8** Einstein's old-fashioned railway coach in free fall, showing relative accelerations of a pair of free stones, as described by Newton (not to scale). *Left panel:* Two horizontally separated free stones are both attracted toward the center of Earth, so as viewed by someone who rides in the falling horizontal railway car, this pair of stones accelerate *toward* one another. *Right panel:* A free stone nearer Earth has a greater acceleration than that of a free stone farther from Earth. As viewed by someone who rides in the falling vertical railway car, this pair of free stones accelerate *away* from one another. We call these relative accelerations **tidal accelerations**.

694 can do nothing to eliminate tidal accelerations completely. These relative  
 695 accelerations are central indicators of the **curvature of spacetime**.

696 Even though we cannot completely eliminate tidal accelerations near a  
 697 center of gravitational attraction, we can often reduce them sufficiently so that  
 698 they do not affect the results of a local measurement that takes place entirely  
 699 in that frame.

Make every  
 measurement  
 in a local  
 inertial frame.

700 *Conclusion:* Almost everywhere in the Universe we can set up a *local*  
 701 inertial frame in which to carry out a measurement. Throughout this book we  
 702 *choose* to make every observation and measurement and carry out every  
 703 experiment in a local inertial frame (Section 1.13). This leads to one of the key  
 704 ideas in this book (see back cover):

705 **We choose to report every measurement and observation using an**  
 706 **inertial frame—a local inertial frame in curved spacetime.**

707 But the local inertial frame tells only part of the story. How can we  
 708 analyze a pair of events widely separated near the Earth, near the Sun, or near

1-28 Chapter 1 Speeding

General relativity:  
patchwork quilt  
of inertial frames.

709 a neutron star—events too far apart to be enclosed in a single inertial frame?  
710 For example, how do we describe the motion of a comet whose orbit  
711 completely encircles the Sun, with an orbital period of many years? The comet  
712 passes through a whole series of local inertial frames, but cannot be tracked  
713 using a single global inertial frame—which does not exist. Special relativity  
714 has reached its limit! To describe motion that oversteps a single local inertial  
715 frame, we must turn to a theory of curved spacetime such as Einstein’s general  
716 relativity—his **Theory of Gravitation**—that we start in Chapter 3, Curving.

**Comment 8. Which way does wristwatch time flow?**

717  
718 In your everyday life, time flows out of what you call your past, into what you call  
719 your future. We label this direction **the arrow of time**. But equation (37) contains  
720 only squared differentials, which allows wristwatch time lapse to be negative—to  
721 run backward—instead of forward along your worldline. So why does your life  
722 flow in only one direction—from past to future on your wristwatch? A subtle  
723 question! We do not answer it here. In this book we simply assume one-way flow  
724 of wristwatch time along any worldline. This assumption will lead us on an  
725 exciting journey!

726 **Fuller Explanation:** *Spacetime Physics*, Chapter 2, Falling Free, and  
727 Chapter 9, Gravity: Curved Spacetime in Action.

**1.12 ■ GENERAL RELATIVITY: OUR CURRENT TOOLKIT**

729 *Ready for a theory of curved spacetime.*

General relativity:  
amazing predictive  
power

730 The remainder of this book introduces Einstein’s general theory of relativity,  
731 currently our most powerful toolkit for understanding gravitational effects.  
732 You will be astonished at the range of observations that general relativity  
733 describes and correctly predicts, among them gravitational waves, space  
734 dragging, the power of quasars, deflection and time delay of light passing a  
735 center of attraction, the tiny precession of the orbit of planet Mercury, the  
736 focusing of light by astronomical objects, and the existence of gravitational  
737 waves. It even makes some predictions about the fate of the Universe.

General relativity  
faces extension  
or revision.

738 In spite of its immense power, Einstein’s general relativity has some  
739 inadequacies. General relativity is incompatible with quantum mechanics that  
740 describes the structure of atoms. Sooner or later a more fundamental theory is  
741 sure to replace general relativity and surmount its limits.

What makes up 96%  
of the Universe?

742 We now have strong evidence that so-called “baryonic  
743 matter”—everything we can see and touch on Earth (including ourselves) and  
744 everything we currently see in the heavens—constitutes only about four  
745 percent of the *stuff* that affects the expansion of the Universe. What makes up  
746 the remaining 96 percent? Current theories of **cosmology**—the study of the  
747 history and evolution of the Universe (Chapter 15)—examine this question  
748 using general relativity. But an alternative possibility is that general relativity  
749 itself requires modification at these huge scales of distance and time.

750 Theoretical research into quantum gravity is active; so are experimental  
751 tests looking for violations of general relativity, experiments whose outcomes

In the meantime,  
general relativity  
is a powerful toolkit.

752 might guide a new synthesis. Meanwhile, Einstein’s general relativity is highly  
753 successful and increasingly important as an everyday toolkit. The conceptual  
754 issues it raises (and often satisfies) are profound and are likely to be part of  
755 any future modification. Welcome to this deep, powerful, and intellectually  
756 delicious subject!

757 **Comment 9. Truth in labeling: “Newton” and “Einstein”**

758 Throughout this book we talk about Newton and Einstein as if each were  
759 responsible for the current form of his ideas. This is false: Newton published  
760 nothing about kinetic energy; Einstein did not believe in the existence of black  
761 holes. Hundreds of people have contributed—and continue to contribute—to the  
762 ongoing evolution and refinement of ideas created by these giants. We do not  
763 intend to slight past or living workers in the field. Rather, we use “Newton” and  
764 “Einstein” as labels to indicate which of their worlds we are discussing at any  
765 point in the text.

?

766 **Objection 3.** *You have told me a lot of weird stuff in this chapter, but I am*  
767 *interested in truth and reality. Do moving clocks **really** run slow? Are*  
768 *clocks synchronized in one frame **really** unsynchronized in a*  
769 *relatively-moving frame? Give me the truth about **reality**!*

!

770 *Truth and reality are mighty words indeed, but in both special and general*  
771 *relativity they are distractions; we strongly suggest that you avoid them as*  
772 *you study these subjects. Why? Because they direct your attention away*  
773 *from the key question that relativity is designed to answer: *What does this**  
774 *inertial observer measure and report?* Ask THAT question and you are  
775 ready for general relativity!

776 **Fuller Explanation:** *Spacetime Physics*, Chapter 9, Gravity: Curved  
777 Spacetime in Action

778 *Now Besso has departed from this strange world a little ahead*  
779 *of me. That means nothing. We who believe in physics, know*  
780 *that the distinction between past, present and future is only a*  
781 *stubbornly persistent illusion.*

782 —Albert Einstein, 21 March 1955, in a letter to Michele  
783 Besso’s family; Einstein died 18 April 1955.

784 **Comment 10. Chapter preview and summary**

785 This book does not provide formal chapter previews or summaries. To preview  
786 the material, read the section titles and questions on the left hand initial page of  
787 each chapter, then skim through the marginal comments. Do the same to  
788 summarize material and to recall it at a later date.

## 1-30 Chapter 1 Speeding

## 1.13 ■ EXERCISES

790 **1. Answer to challenge problem in Sample Problem 3:**

791 Event B cannot cause either Event A or Event C because it occurs *after* those  
792 events in the given frame. The temporal order of events with a timelike  
793 relation will not change, no matter from what frame they are observed: See  
794 Section 2.6, entitled “The Difference between Space and Spacetime.”

795 **2. Spatial Separation I**

796 Two firecrackers explode at the same place in the laboratory and are separated  
797 by a time of 3 seconds as measured on a laboratory clock.

- 798     **A.** What is the spatial distance between these two events in a rocket in  
799         which the events are separated in time by 5 seconds as measured on  
800         rocket clocks?  
801     **B.** What is the relative speed  $v_{\text{rel}}$  between rocket and laboratory frames?

802 **3. Spatial Separation II**

803 Two firecrackers explode in a laboratory with a time difference of 4 seconds  
804 and a space separation of 5 light-seconds, both space and time measured with  
805 equipment at rest in the laboratory. What is the distance between these two  
806 events in a rocket in which they occur at the same time?

807 **4. Maximum wristwatch time**

808 Show that equation (18) corresponds to a maximum, not a minimum, of total  
809 wristwatch time of the stone, equation (17), as it travels across two adjacent  
810 segments of its worldline.

811 **5. Super Cosmic Rays**

812 The Pierre Auger Observatory is an array of cosmic ray detectors lying on the  
813 vast plain *Pampa Amarilla* (yellow prairie) in western Argentina, just east of  
814 the Andes Mountains. The purpose of the observatory is to study cosmic rays  
815 of the highest energies. The highest energy cosmic ray detected had an energy  
816 of  $3 \times 10^{20}$  electron-volts.

- 817     **A.** A regulation tennis ball has a mass of 57 grams. If this tennis ball is  
818         given a kinetic energy of  $3 \times 10^{20}$  electron volts, how fast will it move,  
819         in meters per second? (*Hint:* Try Newton’s mechanics.)  
820     **B.** Suppose a proton has the energy  $3 \times 10^{20}$  electron-volts. How long  
821         would it take this proton to cross our galaxy (take the galaxy diameter  
822         to be  $10^5$  light-years) as measured on the proton’s wristwatch? Give  
823         your answer in seconds.

- 824 C. What is the diameter of the galaxy measured in the rest frame of the  
825 proton?

## 826 6. Mass-Energy Conversion

- 827 A. How much mass does a 100-watt bulb dissipate (in heat and light) in  
828 one year?
- 829 B. Pedaling a bicycle at full throttle, you generate approximately one-half  
830 horsepower of *useful* power. (1 horsepower = 746 watts). The human  
831 body is about 25 percent efficient; that is, 25 percent of the food  
832 burned can be converted to useful work. How long a time will you have  
833 to ride your bicycle in order to lose 1 kilogram by direct conversion of  
834 mass to energy? Express your answer in years. (One year =  $3.16 \times 10^7$   
835 seconds.) How can weight-reducing gymnasiums stay in business?  
836 What is misleading about the way this exercise is phrased?
- 837 C. One kilogram of hydrogen combines chemically with 8 kilograms of  
838 oxygen to form water; about  $10^8$  joules of energy is released. A very  
839 good chemical balance is able to detect a fractional change in mass of 1  
840 part in  $10^8$ . By what factor is this sensitivity more than enough—or  
841 insufficient—to detect the fractional change of mass in this reaction?

## 842 7. Departure from Newton

843 Use equations (33) and (34) to check the limits of Newtonian's expression for  
844 kinetic energy:

- 845 A. An asteroid that falls from rest at a great distance reaches Earth's  
846 surface with a speed of 10 kilometers/second (if we neglect atmospheric  
847 resistance). By what percent is Newton's prediction for kinetic energy  
848 in error for this asteroid?
- 849 B. At what speed does the all-speed expression for kinetic energy (34)  
850 yield a kinetic energy that differs from Newton's prediction—embodied  
851 in equation (33)—by one percent? ten percent? fifty percent?  
852 seventy-five percent? one hundred percent? Use the percentage  
853 expression  $100 \times [KE - (KE)_{\text{Newton}}]/KE$ , where  $KE$  is the relativistic  
854 expression for kinetic energy.

## 855 8. Units and Conversions

- 856 A. Show that the speed of a stone in an inertial frame (as a fraction of the  
857 speed of light) is given by the expression

$$v_{\text{inertial}} = \left( \frac{ds}{dt} \right)_{\text{inertial}} = \left( \frac{p}{E} \right)_{\text{inertial}} \quad (45)$$



**1-32 Chapter 1 Speeding**

- 858 **B.** What speed  $v$  does (45) predict when the mass of the particle is zero,  
859 as is the case for a flash of light? Is this result the one you expect?
- 860 **C.** The mass and energy of particles in beams from accelerators is often  
861 expressed in GeV, that is billions of electron-volts. Journal articles  
862 describing these measurements refer to particle momentum in units of  
863 GeV/c. Explain.

**864 9. The Pressure of Light**

865 A flash of light has zero mass. Use equation (40), in conventional units, to  
866 answer the following questions.

- 867 **A.** You can feel on your hand an object with the weight of 1 gram mass.  
868 Shine a laser beam downward on a black block of wood that you hold  
869 in your hand. You detect an increased force as if the block of wood had  
870 increased its mass by one gram. What power does the laser beam  
871 deliver, in watts?
- 872 **B.** The block of wood described in part A absorbs the energy of the laser  
873 beam. Will the block burst into flame?

**874 10. Derivation of the Expression for Momentum**

- 875 **A.** Carry out the derivation of the relativistic expression for momentum  
876 described in Section 1.8. Lay out this derivation in a series of numbered  
877 steps that parallel those for the derivation of the energy in Section 1.7.
- 878 **B.** Write an expression for  $p$  in conventional units.

**879 11. Verifying energy-momentum transformation equations**

880 Derive transformation equations (43) and (44) using the procedure outlined  
881 just before these equations.

**882 12. Newtonian transformation**

883 Show that for Newton, where all velocities are small compared to the speed of  
884 light, the Lorentz transformation equations (41) reduce to the familiar  
885 Galilean transformation equations and lead to the universality of time.

**886 13. The Photon**

887 NOTE: Exercises 13 through 18 are related to one another.

- 888 **A.** A photon is a quantum of light, a particle with zero mass. Apply  
889 equation (39) for a photon moving only in the  $\pm x$ -direction. Show that  
890 in this conversion to light,  $p_x \rightarrow \pm E$ .

Section 1.13 Exercises **1-33**

- 891 B. Write down the Lorentz transformation equations (43) and (44) for a  
892 photon moving in the positive  $x$ -direction.
- 893 C. Write down the Lorentz transformation equations (43) and (44) for a  
894 photon moving in the negative  $x$ -direction.
- 895 D. Show that *it does not matter* what units you use for  $E$  in your photon  
896 Lorentz transformation equations, as long as the units for each  
897 occurrence of  $E$  are the same.

898 **14. One-Dimensional Doppler Equations**

899 A mongrel equation (neither classical nor quantum-mechanical) connects the  
900 quantum energy  $E$  of a single photon with the frequency  $f$  of a classical  
901 electromagnetic wave. In conventional units, this equation is:

$$E_{\text{conv}} = hf_{\text{conv}} \quad (\text{photon, conventional units}) \quad (46)$$

902 where  $f_{\text{conv}}$  is the frequency in oscillations per second and  $h$  is Planck's  
903 constant. In mks units,  $E_{\text{conv}}$  has the units joules, and **Planck's constant**  $h$   
904 has the numerical value  $h = 6.63 \times 10^{-34}$  joule-second.

- 905 A. Substitute (46) into your transformation equations for the photon, and  
906 replace  $\gamma_{\text{rel}}$  in those equations with its definition  $(1 - v_{\text{rel}}^2)^{-1/2}$ . Planck's  
907 constant disappears from the resulting equations between frequency  
908  $f_{\text{lab}}$  in the laboratory frame and frequency  $f_{\text{rocket}}$  in the rocket frame:

$$f_{\text{lab}} = \left[ \frac{1 \pm v_{\text{rel}}}{1 \mp v_{\text{rel}}} \right]^{1/2} f_{\text{rocket}} \quad (\pm x, \text{ light}) \quad (47)$$

$$f_{\text{rocket}} = \left[ \frac{1 \mp v_{\text{rel}}}{1 \pm v_{\text{rel}}} \right]^{1/2} f_{\text{lab}} \quad (\pm x, \text{ light}) \quad (48)$$

909 These are the **one-dimensional Doppler equations** for light moving  
910 in either direction along the  $x$ -axis.

- 911 B. The relation between frequency  $f_{\text{conv}}$  and wavelength  $\lambda_{\text{conv}}$  for a  
912 classical plane wave in an inertial frame, in conventional units

$$f_{\text{conv}} \lambda_{\text{conv}} = c \quad (\text{classical plane wave}) \quad (49)$$

913 Rewrite equations (47) and (48) for the relation between laboratory  
914 wavelength  $\lambda_{\text{lab}}$  and rocket wavelength  $\lambda_{\text{rocket}}$ .

915 **15. Speed-Control Beacon**

916 An advanced civilization sets up a beacon on a planet near the crowded center  
917 of our galaxy and asks travelers approaching directly or receding directly from  
918 the beacon to use the Doppler shift to measure their speed relative to the

**1-34** Chapter 1 Speeding

919 beacon, with a speed limit at  $v = 0.2$  relative to that beacon. The beacon  
 920 emits light of a single *proper* wavelength  $\lambda_0$ , that is, the wavelength measured  
 921 in the rest frame of the beacon. Four index colors are:

$$\lambda_{\text{red}} = 680 \times 10^{-9} \text{meter} = 680 \text{ nanometers} \quad (50)$$

$$\lambda_{\text{yellow}} = 580 \times 10^{-9} \text{meter} = 580 \text{ nanometers}$$

$$\lambda_{\text{green}} = 525 \times 10^{-9} \text{meter} = 525 \text{ nanometers}$$

$$\lambda_{\text{blue}} = 475 \times 10^{-9} \text{meter} = 475 \text{ nanometers}$$

- 922 A. Choose the beacon proper wavelength  $\lambda_0$  so that a ship approaching at  
 923 half the speed limit,  $v = 0.1$ , sees green light. What is the proper  
 924 wavelength  $\lambda_0$  of the beacon beam? What color do you see when you  
 925 stand next to the beacon?
- 926 B. As your spaceship moves directly toward the beacon described in Part  
 927 A, you see the beacon light to be blue. What is your speed relative to  
 928 the beacon? Is this below the speed limit?
- 929 C. In which direction, toward or away from the beacon, are you traveling  
 930 when you see the beacon to be red? What is your speed relative to the  
 931 beacon? Is this below the speed limit?

**16. Radar**

932 An advanced civilization uses radar to help enforce the speed limit in the  
 933 crowded center of our galaxy. Radar relies on the fact that with respect to its  
 934 rest frame a spaceship reflects a signal back with a frequency equal to the  
 935 incoming frequency measured in its frame.  
 936

- 937 A. Show that a radar signal of frequency  $f_0$  at the source is received back  
 938 from a directly approaching ship with the reflected frequency  $f_{\text{reflect}}$   
 939 given by the expression:

$$f_{\text{reflect}} = \frac{1+v}{1-v} f_0 \quad (\text{radar}) \quad (51)$$

940 where  $v$  is the speed of the spaceship with respect to the signal source.

- 941 B. What is the wavelength  $\lambda_{\text{reflect}}$  of the signal reflected back from a  
 942 spaceship approaching at the speed limit of  $v = 0.2$ ?
- 943 C. The highway speed of a car is very much less than the speed of light.  
 944 Use the approximation formula inside the front cover to find the  
 945 following approximate expression for  $f_{\text{reflect}} - f_0$ :

$$f_{\text{reflect}} - f_0 \approx 2v f_0 \quad (\text{highway radar}) \quad (52)$$

946 The US state of Massachusetts Highway Patrol uses radar with  
 947 microwave frequency  $f_0 = 10.525 \times 10^9$  cycles/second. By how many

948 cycles/second is the reflected beam shifted in frequency when reflected  
949 from a car approaching at 100 kilometers/hour = 27.8 meters/second?

### 950 17. Two-dimensional Velocity Transformations

951 An electron moves in the laboratory frame with components of velocity  
952  $(v_{x,\text{lab}}, v_{y,\text{lab}})$  and in the rocket frame with components of velocity  
953  $(v_{x,\text{rocket}}, v_{y,\text{rocket}})$ .

954 A. Use the differential form of the Lorentz transformation equations (42)  
955 to relate the velocity components of the electron in laboratory and  
956 rocket frames:

$$v_{x,\text{lab}} = \frac{v_{x,\text{rocket}} + v_{\text{rel}}}{1 + v_{\text{rel}}v_{x,\text{rocket}}} \quad v_{y,\text{lab}} = \frac{v_{y,\text{rocket}}}{\gamma_{\text{rel}}(1 + v_{\text{rel}}v_{x,\text{rocket}})} \quad (53)$$

957 This is called the **Law of Transformation of Velocities**.

958 B. With a glance at the Lorentz transformation (42) and its inverse (41),  
959 make an argument that to derive the inverse of (53), one simply replaces  
960  $v_{\text{rel}}$  with  $-v_{\text{rel}}$  and interchanges lab and rocket labels, leading to:

$$v_{x,\text{rocket}} = \frac{v_{x,\text{lab}} - v_{\text{rel}}}{1 - v_{\text{rel}}v_{x,\text{lab}}} \quad v_{y,\text{rocket}} = \frac{v_{y,\text{lab}}}{\gamma_{\text{rel}}(1 - v_{\text{rel}}v_{x,\text{lab}})} \quad (54)$$

961 C. Does the law of transformation of velocities allow the electron to move  
962 faster than light when observed in the laboratory frame? For example,  
963 suppose that in the rocket frame the electron moves in the positive  
964  $x_{\text{rocket}}$ -direction with velocity  $v_{x,\text{rocket}} = 0.75$  and the rocket frame also  
965 moves in the same direction with the same relative speed  $v_{\text{rel}} = 0.75$ .  
966 What is the value of the velocity  $v_{x,\text{lab}}$  of the electron in the laboratory  
967 frame?

968 D. Suppose two light flashes move with opposite velocities  $v_{x,\text{rocket}} = \pm 1$  in  
969 the rocket frame. What are the corresponding velocities  $v_{x,\text{lab}}$  of the  
970 two light flashes in the laboratory frame?

971 E. Light moves with velocity components  
972  $(v_{x,\text{rocket}}, v_{y,\text{rocket}}, v_{z,\text{rocket}}) = (0, -1, 0)$  in the rocket frame. *Predict* the  
973 magnitude  $|v_{\text{lab}}|$  of its velocity measured in the laboratory frame. Does  
974 a calculation verify your prediction?

### 975 18. Aberration of light

976 Light that travels in one direction in the laboratory travels in another direction  
977 in the rocket frame unless the light moves along the line of relative motion of  
978 the two frames. This difference in light travel direction is called **aberration**.

979 A. Transform the angle of light propagation in two spatial dimensions.  
980 Recall that laboratory and rocket  $x$ -coordinates lie along the same line,

## 1-36 Chapter 1 Speeding

981 and in each frame measure the angle  $\psi$  of light motion with respect to  
 982 this common forward  $x$ -direction. Make the following argument: Light  
 983 travels with the speed one, which is the hypotenuse of the velocity  
 984 component triangle. Therefore for light  $v_{x,\text{inertial}} \equiv v_{x,\text{inertial}}/1 = \cos \psi$ .  
 985 Show that this argument converts the first of equations (53) to:

$$\cos \psi_{\text{lab}} = \frac{\cos \psi_{\text{rocket}} + v_{\text{rel}}}{1 + v_{\text{rel}} \cos \psi_{\text{rocket}}} \quad (\text{light}) \quad (55)$$

986 B. From equation (39) show that for light tracked in any inertial frame  
 987  $|p_{\text{inertial}}| = E_{\text{inertial}}$ . Hence  $p_{x,\text{inertial}}/E_{\text{inertial}} = \cos \psi$  and the first of  
 988 equations (44) becomes, for light

$$E_{\text{lab}} = E_{\text{rocket}} \gamma_{\text{rel}} (1 + v_{\text{rel}} \cos \psi_{\text{rocket}}) \quad (\text{light}) \quad (56)$$

989 C. Make an argument that to derive the inverses of (55) and (56), you  
 990 simply replace  $v_{\text{rel}}$  with  $-v_{\text{rel}}$  and interchange laboratory and rocket  
 991 labels, to obtain the aberration equations:

$$\cos \psi_{\text{rocket}} = \frac{\cos \psi_{\text{lab}} - v_{\text{rel}}}{1 - v_{\text{rel}} \cos \psi_{\text{lab}}} \quad (\text{light}) \quad (57)$$

$$E_{\text{rocket}} = E_{\text{lab}} \gamma_{\text{rel}} (1 - v_{\text{rel}} \cos \psi_{\text{lab}}) \quad (\text{light}) \quad (58)$$

992 D. A source at rest in the rocket frame emits light uniformly in all  
 993 directions in that frame. Consider the 50 percent of this light that goes  
 994 into the forward hemisphere in the rocket frame. Show that in the  
 995 laboratory frame this light is concentrated in a narrow forward cone of  
 996 half-angle  $\psi_{\text{headlight,lab}}$  given by the following equation:

$$\cos \psi_{\text{headlight,lab}} = v_{\text{rel}} \quad (\text{headlight effect}) \quad (59)$$

997 The transformation that leads to concentration of light in the forward  
 998 direction is called the **headlight effect**.

999 **19. Cherenkov Radiation**

1000 Can an electron move faster than light? No and yes. No, an electron cannot  
 1001 move faster than light *in a vacuum*; yes, it can move faster than light in a  
 1002 medium in which light moves more slowly than its standard speed in a  
 1003 vacuum. P. A. Cherenkov shared the 1958 Nobel Prize for this discovery that  
 1004 an electron emits coherent radiation when it moves faster than light moves in  
 1005 any medium.

1006 What is the minimum kinetic energy that an electron must have to emit  
 1007 Cherenkov radiation while traveling through water, where the speed of light is  
 1008  $v_{\text{light}} \approx 0.75c$ ? Express this kinetic energy as both the fraction (kinetic  
 1009 energy)/ $m$  of its mass  $m$  and in electron-volts (eV). Type “Cherenkov

1010 radiation” into a computer search engine to see images of the blue light due to  
1011 Cherenkov radiation emitted by a radioactive source in water.

## 1012 20. Live Forever?

1013 Luc Longtin shouts, “I can live forever! Here is a variation of equation (1):  
1014  $\Delta\tau^2 = \Delta t_{\text{Earth}}^2 - \Delta s_{\text{Earth}}^2$ . Relativity allows the possibility that  $\Delta\tau \ll \Delta t_{\text{Earth}}$ .  
1015 In the limit,  $\Delta\tau \rightarrow 0$ , so the hour hand on my wristwatch does not move.  
1016 Eternal life!

1017 “I have decided to ride a 100 kilometer/hour train back and forth my  
1018 whole life. THEN I will age much more slowly.” Comment on Luc’s ecstatic  
1019 claim without criticizing him.

- 1020 A. When he carries out his travel program, how much younger will  
1021 100-year-old Luc be than his stay-at-home twin brother Guy?
- 1022 B. Suppose Luc rides a spacecraft in orbit around Earth (speed given in  
1023 Figure 2). In this case, how much younger will 100-year-old Luc be  
1024 than brother Guy?
- 1025 C. Suppose Luc manages to extend his life measured in Earth-time by  
1026 riding on a fast cosmic ray (speed given in Figure 2). When Luc returns  
1027 to Earth in his old age, it is clear that his brother Guy will no longer be  
1028 among the living. However, would Luc *experience* his life as much  
1029 longer than he would have experienced it if he remained on Earth?  
1030 That is, would he “enjoy a longer life” in some significant sense, for  
1031 example counting many times the total number of heartbeats  
1032 experienced by Guy?

## 1.14 ■ REFERENCES

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**1-38** Chapter 1 Speeding

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