Quantum Information Processing, BME 2019 Spring
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Exercises

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I. EXERCISES

Keywords: density matrix, thermal state, composite system, reduced density matrix, entanglement, von Neumann entropy, relaxation, dephasing, Bloch-Redfield equations.

1. *Single-qubit density matrix.*
   Calculate the density matrices of these single-qubit states in the computational basis, plot their real-part and imaginary-part bar charts (i.e., the 'city plots'), and their Bloch vectors:

   (a) $|x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, (b) $|\bar{x}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

   (c) a statistical (or 'incoherent') mixture of $|x\rangle$ with probability ('weight') 3/4, and $|\bar{x}\rangle$ with probability 1/4,

   (d) a statistical mixture of $|x\rangle$ with probability 1/2 and $|y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ with probability 1/2.

2. *Thermal single-qubit density matrix.*
   (a) Calculate the single-qubit thermal density matrix for a Larmor frequency of 5 GHz and temperature of 50 mK, and plot its real-part and imaginary-part bar charts.

   (b) Plot the thermal occupation of the ground state and the excited state for a Larmor frequency of 5 GHz, as the function of the temperature in the window between 0 K and 4.2 K.

3. *Two-qubit density matrices.*
   Calculate the density matrix of each of the four Bell states. Plot their real-part and imaginary-part bar charts. Calculate the reduced density matrix of the first qubit for all four Bell states. Plot their real-part bar charts.

4. *Thermal two-qubit density matrix.* Take two uncoupled qubits at 50 mK, with Larmor frequencies of 5 GHz and 6 GHz, respectively. Calculate the thermal density matrix in the product basis and plot the bar charts.

5. *Entanglement due to Heisenberg exchange interaction and Dzyaloshinskii-Moriya interaction*
   (a) Consider two localized spin-1/2 particles that are interacting via isotropic antiferromagnetic Heisenberg interaction, described by the Hamiltonian $H = J\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$, with interaction strength $J > 0$. Note that each spin is described by 1/2 Pauli matrices in this Hamiltonian, e.g., $\boldsymbol{S}_1 = \boldsymbol{\sigma}_1/2$. Assume $J/h = 5$ GHz and calculate the ground state of this Hamiltonian and the corresponding density matrix.

   (b) Calculate the reduced density matrix of the first spin, and the von Neumann entropy of this reduced state.

   (c) (homework) Repeat (b) for the second spin, to demonstrate that the von Neumann entanglement entropy is unique.

   (d) (homework) Add the Dzyaloshinskii-Moriya term to the Hamiltonian, $H' = H + D(S_{1x}S_{2y} - S_{1y}S_{2x})$, with $D/h = 2$ GHz. Calculate and plot the von Neumann entanglement entropy for a thermal state of this system, as the function of temperature in the window between 0 K and 4.2 K.