Chapter 8. Circular Orbits

- *How do orbits around a black hole differ from planetary orbits around our Sun?*
- *How close to a black hole can a free stone move in a circular orbit?*
- *Can a stone reach the speed of light in a circular orbit around a black hole?*
- *Can I use a black hole circular orbit to travel forward in time? backward in time?*
- *What is the source of the energy that the so-called QUASAR radiates outward in such prodigious quantity?*

CHAPTER

8 Circular Orbits

Edmund Bertschinger & Edwin F. Taylor *

How happy is the little Stone
That orbits a Black Hole alone*
And doesn't care about Careers
And Exigencies never fears –
Whose Coat of elemental Brown
A passing Universe put on
And independent as the Sun
Associates or glows alone
Fulfilling absolute Decree
In casual simplicity –

—Emily Dickinson

*Line two in the original reads:
That rambles in the Road alone

8.1 ■ STEP OR ORBIT?

"Go straight!" shouts spacetime. The Principle of Maximal Aging interprets that command

Nature shouts at the stone "Go straight!"

A stone in orbit streaks around a black hole—or around Earth. What tells the stone how to move? Spacetime grips the stone, giving it the simplest possible command: "Go straight!" or in the more legalistic language of the Principle of Maximal Aging, "Follow the worldline of maximal aging across the next two adjoining local inertial frames." From instant to instant this directive is enough to tell the stone what to do next, the next step to take in its motion.

This chapter: circular orbits

This command for its next step is sufficient for the stone, but we want more: We seek a description of the entire orbit of the stone through spacetime—its worldline in global coordinates. The present chapter uses the global metric and the Principle of Maximal Aging to predict circular orbits of a stone around any spherically symmetric center of attraction. This prediction

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Constants of motion: map energy and map angular momentum

uses two map quantities that do not change as the motion progresses: map energy and map angular momentum. In Query 7, Section 7.5, you derived the map energy of a stone in global rain coordinates. Section 8.2 in the present chapter derives an expression for map angular momentum in global rain coordinates. Sections 8.4 shows how to use map angular momentum—together with map energy—to forecast circular orbits. We find that a *free* stone can move (a) in a *stable* circular orbit only at an r-coordinates greater than $r = 6M$, or (b) in an *unstable* circular orbit from $r = 6M$ down to $r = 3M$. No circular orbit for a free stone exists for $r < 3M$.

Comment 1. Global quantities are unicorns

Expressions for global quantities such as map energy and map angular momentum are specific to the global coordinates in which they are expressed. They are unicorns—mythical beasts—unmeasured by a local inertial observer, except by some quirk of the global coordinates (Section 6.3).

The circular orbit is a special case of an *orbit*. We have not yet carefully defined an orbit. Here is that definition.

DEFINITION 1. Orbit

An **orbit** is the path of a free stone through spacetime described by a given set of global coordinates. The path of a radially-plunging stone, with $d\phi = 0$ is a special case of the orbit.

Comment 2. Orbit vs. worldline

The *orbit* of a stone is different from its *worldline*. The worldline of a stone (Definition 9, Section 1.5) is its (free or driven) path through spacetime described by its wristwatch time. The description of a worldline does not require either coordinates or the metric.

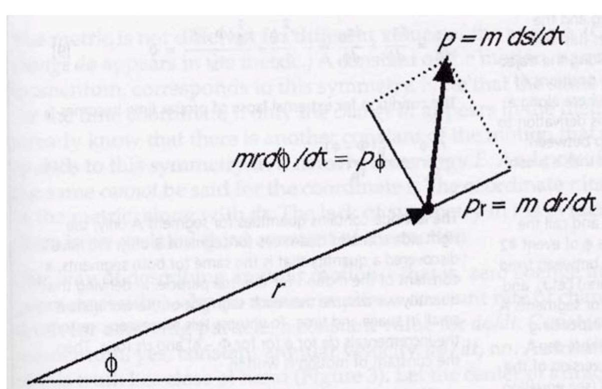

FIGURE 1 In flat spacetime, angular momentum L is the product of r and the ϕ-component of linear momentum $p_\phi = mrd\phi/d\tau$, which yields $L = mr^2 d\phi/d\tau$. Here $d\tau$ is the differential advance of wristwatch time of the stone. Box 1 shows that the same expression, written in global (either Schwarzschild or rain) coordinates, is a constant of motion around a non-spinning black hole.

8.2 ■ MAP ANGULAR MOMENTUM OF A STONE FROM MAXIMAL AGING

Vary the map angle of an intermediate event on a worldline to find map angular momentum.

Here we derive the expression for map angular momentum using global rain coordinates with its T-coordinate. The resulting expression for map angular momentum is also valid in Schwarzschild coordinates. Why? Because both global coordinate systems have the same r and ϕ coordinates, and the global t- or T-coordinate—different in the two global coordinate systems—does not appear in the expression for map angular momentum.

Start with the global rain metric, equation (15) in Section 7.4. Write down its approximation at the average r-coordinate $\bar{r}$:

$$\Delta\tau^2 \approx \left(1 - \frac{2M}{\bar{r}}\right)\Delta T^2 - 2\left(\frac{2M}{\bar{r}}\right)^{1/2}\Delta T\Delta r - \Delta r^2 - \bar{r}^2\Delta\phi^2 \qquad (9)$$

Box 1 uses the now-familiar Principle of Maximal Aging to derive the expression for map angular momentum in global rain coordinates. Box 1 tells us that $r^2 d\phi/d\tau$ is a constant of motion for a free stone moving around the non-spinning black hole. Can we recognize this constant as something familiar? Figure 1 shows that in flat spacetime the angular momentum of the stone (symbol L) has the form $L = mr^2 d\phi/d\tau$. So we identify our new constant of motion as the **map angular momentum per unit mass** of the stone: $L/m = r^2 d\phi/d\tau$.

Map angular momentum

$$\frac{L}{m} \equiv r^2\frac{d\phi}{d\tau} \qquad \text{(map angular momentum)} \qquad (10)$$

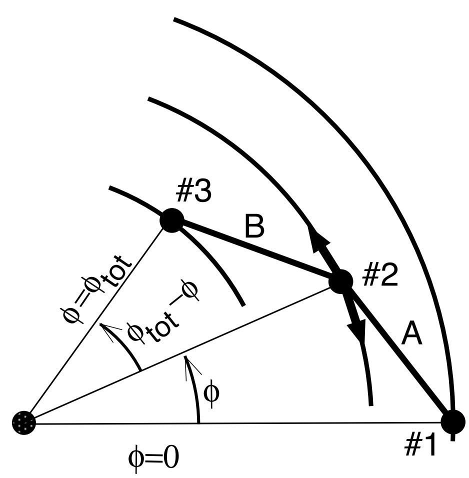

FIGURE 2 [Figure for Box 1.] Derivation of map angular momentum. Find the intermediate map angle ϕ that maximizes the stone's wristwatch time between Events #1 and #3.

Box 1. Derive the Expression for Map Angular Momentum

Strategy: Apply the Principle of Maximal Aging to maximize the wristwatch time of a free stone that moves along two adjoining worldline segments labeled A and B—for Above and Below—in Figure 2. The stone emits flashes at Events #1, #2, and #3 that mark off the segments. Fix the global rain T- and r-coordinates of all three flashes and the ϕ-coordinates of flashes #1 and #3. Vary the ϕ-coordinate of Event #2 by sliding it along a circle (double-headed arrow in Figure 2) to maximize the total wristwatch time between flashes #1 and #3. Then identify the resulting constant of motion as the map angular momentum per unit mass of the stone. Now the details.

Set the fixed ϕ-coordinate of Event #1 equal to zero and call ϕ_{tot} the fixed final ϕ-coordinate for Event #3. To change the angle ϕ of Event #2, move it in either direction along its circle (double-headed arrow in the figure). Let $\bar{r}_{\text{A}}$ and $\bar{r}_{\text{B}}$ be appropriate average values of the r-coordinate for segments A and B, respectively, and let τ_{A} and τ_{B} be the corresponding lapses of wristwatch time of the stone moving along these segments. With these substitutions, and for a small value of τ_{A}, the approximate global rain metric (9) for higher Segment A becomes:

$$\tau_{\text{A}} \approx [-\bar{r}_{\text{A}}^2\phi^2 + (\text{terms without } \phi)]^{1/2} \qquad (1)$$

To prepare for the derivative that leads to maximal aging, take the derivative of this expression with respect to ϕ:

$$\frac{d\tau_{\text{A}}}{d\phi} \approx -\frac{\bar{r}_{\text{A}}^2\phi}{\tau_{\text{A}}} \qquad (2)$$

Similarly for lower Segment B,

$$\tau_{\text{B}} \approx [-\bar{r}_{\text{B}}^2(\phi_{\text{tot}} - \phi)^2 + (\text{terms without } \phi)]^{1/2} \qquad (3)$$

$$\frac{d\tau_{\text{B}}}{d\phi} \approx \frac{\bar{r}_{\text{B}}^2(\phi_{\text{tot}} - \phi)}{\tau_{\text{B}}} \qquad (4)$$

The total wristwatch time for both segments is $\tau = \tau_{\text{A}} + \tau_{\text{B}}$. Take the derivative of this expression with respect to ϕ, substitute from (2) and (4), and set the resulting derivative equal to zero in order to apply the Principle of Maximal Aging:

$$\frac{d\tau}{d\phi} = \frac{d\tau_{\text{A}}}{d\phi} + \frac{d\tau_{\text{B}}}{d\phi} \approx -\frac{\bar{r}_{\text{A}}^2\phi}{\tau_{\text{A}}} + \frac{\bar{r}_{\text{B}}^2(\phi_{\text{tot}} - \phi)}{\tau_{\text{B}}} = 0 \qquad (5)$$

The condition for maximal lapse of wristwatch time becomes

$$\frac{\bar{r}_{\text{A}}^2\phi}{\tau_{\text{A}}} \approx \frac{\bar{r}_{\text{B}}^2(\phi_{\text{tot}} - \phi)}{\tau_{\text{B}}} \qquad (6)$$

or in our original Δ notation:

$$\frac{\bar{r}_{\text{A}}^2\Delta\phi_{\text{A}}}{\Delta\tau_{\text{A}}} \approx \frac{\bar{r}_{\text{B}}^2\Delta\phi_{\text{B}}}{\Delta\tau_{\text{B}}} \qquad (7)$$

The left side contains quantities for Segment A only; the right side quantities for Segment B only. We have discovered a quantity that has the same value for both segments, a *global constant of motion* for the free stone across *every* pair of adjoining segments along the worldline of the free stone. In deriving this quantity, we assumed that each segment of the worldline is small. To yield an equality in (7), go to the calculus limit in (7), for which $\bar{r} \to r$; the constant of motion becomes

$$\lim_{\Delta\tau\to 0}\left(\bar{r}^2\frac{\Delta\phi}{\Delta\tau}\right) = r^2\frac{d\phi}{d\tau} = \text{a constant of motion} \qquad (8)$$

where r and τ are in units of meters. The text identifies this constant of motion as *L/m*, the map angular momentum of the stone per unit mass.

Since r and τ are in units of meters, therefore L/m is also in units of meters.

8.3 ■ EQUATIONS OF MOTION FOR A STONE IN GLOBAL RAIN COORDINATES

The stone's wristwatch ticks off $d\tau$. From $d\tau$ find the resulting changes $d\phi$, dr, and dT.

Equations of motion

We now have in hand the tools needed to calculate the step-by-step advance of the free stone in global rain coordinates. Map energy and map angular momentum—global constants of motion—plus the global metric give us three equations in the three global rain unknowns dT, dr, and $d\phi$, expressed as functions of the advance $d\tau$ of the stone's wristwatch. Starting from an arbitrary initial event, the computer advances wristwatch time and calculates the consequent advance of all three map coordinates, then sums the results of these steps to plot the stone's worldline in global coordinates. We now spell out this process.

First equation of motion

The first equation of motion comes from (10):

$$\frac{d\phi}{d\tau} = \frac{L}{mr^2} \tag{11}$$

The second equation of motion comes from the expression for E/m, equation (35) in Section 7.5:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau} \qquad \text{(global rain coordinates)} \tag{12}$$

Solve (12) for $dT/d\tau$:

$$\frac{dT}{d\tau} = \left[\frac{E}{m} + \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau}\right]\left(1 - \frac{2M}{r}\right)^{-1} \tag{13}$$

Take the differential limit of (9), divide through by $d\tau^2$, and substitute into it from (11) and (13):

$$\begin{aligned} 1 = &\left[\frac{E}{m} + \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau}\right]^2\left(1 - \frac{2M}{r}\right)^{-1} \\ &- 2\left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau}\left[\frac{E}{m} + \left(\frac{2M}{r}\right)^{1/2}\frac{dr}{d\tau}\right]\left(1 - \frac{2M}{r}\right)^{-1} - \left(\frac{dr}{d\tau}\right)^2 - \left(\frac{L}{mr}\right)^2 \end{aligned} \tag{14}$$

Second equation of motiont

Multiply out and collect terms. Solve the resulting quadratic equation in $dr/d\tau$ to yield our second equation of motion for the stone:

$$\frac{dr}{d\tau} = \pm\left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2r^2}\right)\right]^{1/2} \qquad \text{(stone)} \tag{15}$$

Third equation of motion

The third equation of motion shows how dT varies with stone wristwatch time lapse $d\tau$. Substitute for $dr/d\tau$ from (15) into (13) and solve for $dT/d\tau$:

$$\frac{dT}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}\left\{\frac{E}{m} \pm \left(\frac{2M}{r}\right)^{1/2}\left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2r^2}\right)\right]^{1/2}\right\} \tag{16}$$

Comment 3. Plotting the orbit

To plot any orbit of the stone—not just a circular orbit— you (or your computer) can integrate the derivative $d\phi/dr = (d\phi/d\tau)(d\tau/dr)$ using equations (11) and (15).

Equations of motion in global rain coordinates

Taken together, equations (11), (15), and (16) are the *equations of motion of the stone* in global rain coordinates. Their integration yields the worldline of the stone in global rain coordinates T, r, and ϕ. Interactive software GRorbits carries out this process, plots the orbit in r and ϕ, and outputs a spreadsheet of events along the worldline of the stone.

QUERY 1. Crossing the event horizon in global rain coordinates.
A first glance at equation (16) might lead to the conclusion that $dT/d\tau$ blows up at the event horizon, so that a stone requires an unlimited lapse in the T-coordinate to cross there. Set $r = 2M(1+\epsilon)$ in this equation to show that as $\epsilon \to 0$ the right side does *not* blow up.

8.4 ■ EFFECTIVE POTENTIAL

Grasp orbit features at a single glance!

Effective potential: the r-component of motion

The orbit computation in Section 8.3 puts into our hands powerful tools to describe any motion of the free stone in the equatorial plane of a spherically symmetric center of attraction. Indeed, the wealth of possible orbits is so great that we need some classification scheme with which to sort orbits at a glance. One classification scheme uses the so-called **effective potential** that focuses on the r-component of motion. Clearer even than our computed orbits, the effective potential plot instantly shows many central features of our stone's motion.

Vicious gravitational effects close to a black hole dominate the effective potential there. In addition to the attractive potential of gravity at large r-coordinates and the effective repulsion due to map angular momentum at intermediate r-values, at still smaller r-coordinates Einstein adds a pit in the potential, shown at the left of Figures 3 and 4.

Pit in the potential

The potential? A pit in this potential? Can we get this potential from principles that are simple, clear, and solid? Yes, starting from map energy and map angular momentum, both of them global constants of motion.

To begin this process, square both sides of (15).

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right) \qquad (17)$$

Effective potential for a stone

Define a function $(V_{\rm L}(r)/m)^2$ to replace the second term on the right side of (17). Call this function the square of the **effective potential**.

$$\left(\frac{V_{\rm L}(r)}{m}\right)^2 \equiv \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right) \qquad \text{(squared effective potential)} \quad (18)$$

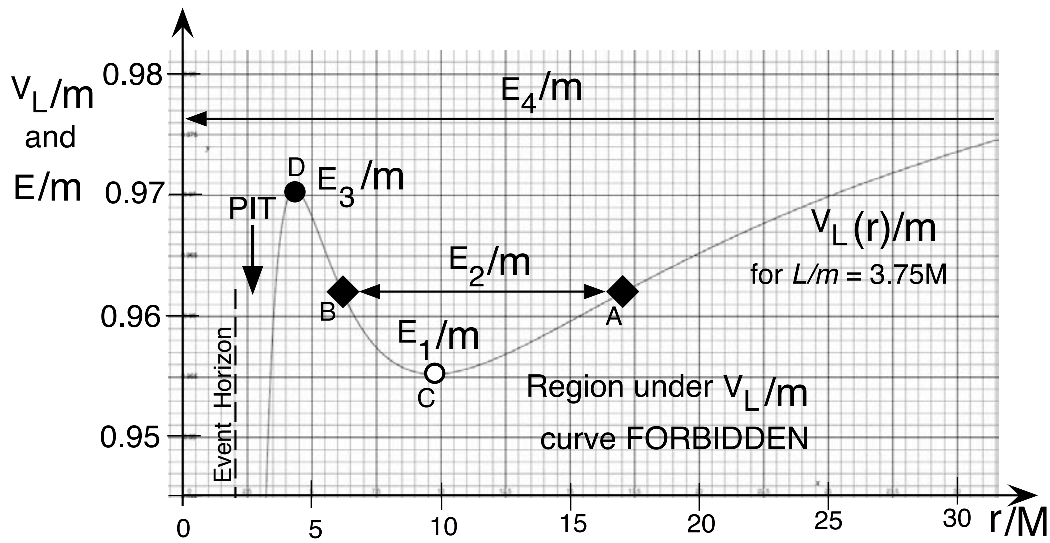

FIGURE 3 Effective potential for a stone that orbits the black hole with map angular momentum $L/m = 3.75M$. When the stone's map energy equals the minimum of the effective potential energy (little open circle at C), the stone is in a stable circular orbit. A stone with somewhat greater map energy, E_2/m, (line with double arrow) oscillates back and forth in r between turning points (little black rotated squares) labeled A and B. When the stone's map energy equals the maximum of the effective potential energy (little filled circle at D), the stone is in an unstable circular orbit. When the map energy E_4/m of an inward-moving stone is greater than the peak of the effective potential (upper horizontal line), the approaching stone crosses the event horizon and plunges to the singularity at $r \rightarrow 0$.

Subscript L on $V_{\rm L}(r)$ reminds us that this effective potential is different for different values of the map angular momentum L. Substitute (18) into (17) and take the square root of both sides:

$$\frac{dr}{d\tau} = \pm\left[\left(\frac{E}{m}\right)^2 - \left(\frac{V_{\rm L}(r)}{m}\right)^2\right]^{1/2} \qquad (19)$$

The squared effective potential $(V_{\rm L}(r)/m)^2$ is what we subtract from the squared map energy term $(E/m)^2$ to obtain $(dr/d\tau)^2$. The plus sign in (19) describes increase in r-coordinate, the minus sign describes decreasing r.

Figure 3 plots effective potential $V_{\rm L}(r)/m$ from (18) and shows the r-range for motion of stones with three different map energies.

Forbidden map energy region; turning points

Note that $dr/d\tau$ in equation (19) is real only where $(E/m)^2$ has a value greater than $(V_{\rm L}(r)/m)^2$. This has important consequences: The stone cannot exist with a map energy in the region under the effective potential curve: that is the **forbidden map energy region**. As a result, the horizontal map energy line labeled E_2/m in Figure 3 terminates wherever it meets the $V_L(r)/m$ curve. At these points, called **turning points** in r, the map energy and the effective potential are equal: $E/m = V_{\rm L}/m$, so that $dr/d\tau = 0$ in (19). At a turning point the r-component of map motion goes to zero (while the stone continues to sweep around in the ϕ-direction). In Figure 3 the stone's r map position oscillates back and forth between turning points in r labeled A and B. Earth and each solar planet oscillates back and forth with an r-component of motion similar to that labeled E_2/m in Figure 3, each around a minimum of its own solar effective potential that depends on its map angular momentum.

Definition: **Forbidden map energy region**

DEFINITION 2. Forbidden map energy region

The **forbidden map energy region** is a region in a $V_\mathrm{L}(r)/m$ vs. r/M plot in which equations of motion of the stone (Section 8.3) become imaginary or complex. Hence the stone cannot move—or even exist—with map energy in the forbidden map energy region.

QUERY 2. Demonstrate forbidden map energy regions

Verify the statement in Definition 2 that "In the forbidden map energy region, the equations of motion of a stone (Section 8.3) become imaginary or complex." for *each* equation of motion in Section 8.3.

DEFINITION 3. Turning point, circle point, and bounce point

Figures 3 and 4 show little filled circles, little open circles, and little rotated filled squares, each one located on the effective potential curve. These points are called *turning points*. (Section 8.5 defines the meaning of the "half-black" circle numbered one in Figure 4.)

Definition: **Turning point**

A **turning point** is a value of r for which $E = V_\mathrm{L}(r)$. At a turning point, $dr/d\tau = 0$. Examples of turning points: points A through D in Figure 3 and points 1 through 5 in Figure 4. We distinguish two kinds of turning points: circle point and bounce point:

Definition: **Circle point**

A **circle point** is a turning point at a maximum or minimum of the effective potential. At a circle point $dr/d\tau$ equals zero and remains zero, at least temporarily, so a stone at a circle point is in either an unstable or a stable circular orbit. We plot a circle point as either a little filled circle (at an unstable circular orbit r-value) or a little open circle (at a stable circular orbit r-value). Examples of bounce points: C and D in Figure 3 and points labeled 1 through 5 in Figure 4.

Definition: **Bounce point**

A **bounce point** is a turning point that is *not* at a maximum or minimum of the effective potential. At a bounce point, $dr/d\tau$ for a free stone reverses sign. We plot a bounce point as a little filled rotated square. Examples of bounce points: A, and B in Figure 3. A stone that moves between bounce points—such as the stone with map energy E_2/m in Figure 3, is in a bound orbit that is *not* circular (Chapter 9).

Three payoffs of effective potential

Here are four important payoffs of the effective potential. First, it gives $dr/d\tau$ in terms of E, L, and r. Second, at every r it shows us the map energy region that is forbidden to the stone. Third, it fixes r-values of the turning points for given E and L. Fourth, and most important, it helps us to categorize—at a glance—different kinds of orbits, including circular orbits.

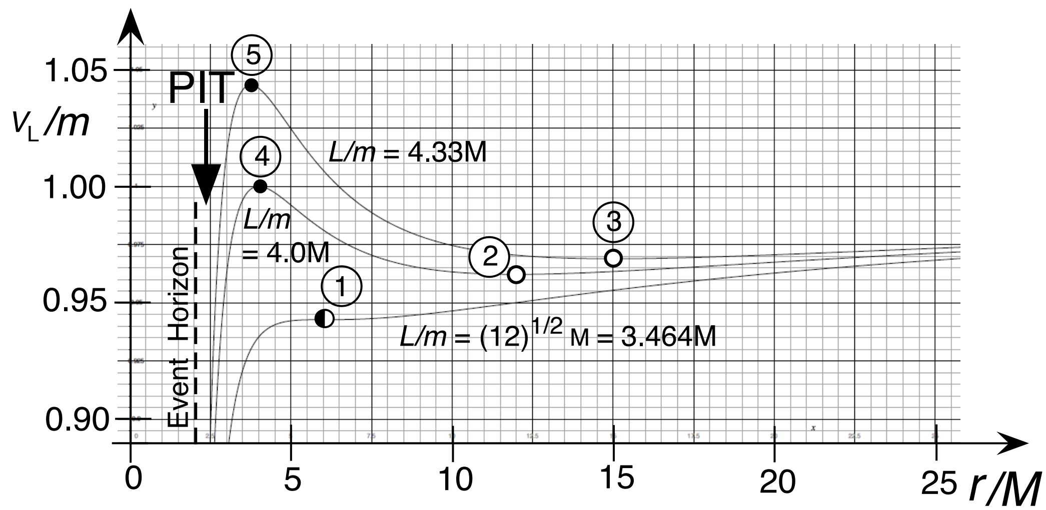

FIGURE 4 The r-coordinates of stable and unstable (knife-edge) circular orbits at points of zero slope of the effective potentials for three values of L/m. Unstable circular orbits (little filled circles numbered 4 and 5) lie between $r = 3M$ and $r = 6M$. Stable circular orbits, little open circles numbered 2 and 3, lie at r greater than $r = 6M$. Orbit numbered 1 (little half-black circle) is the limiting case, stable for increase in r; unstable for decrease in r. Section 8.5 discusses this "half-stable orbit." A forbidden map energy region (Definition 2) lies under the curve for each value of L/m.

QUERY 3. Compare Newtonian and general-relativistic orbital motion *(optional)*
The right side of (17) tells us a great deal about the difference between the stone's global motion described in global rain coordinates and its motion described by Newton.

A. Multiply out the right side of (17) and divide through by 2 to yield

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left[\left(\frac{E}{m}\right)^2 - 1\right] - \left(-\frac{M}{r} + \frac{L^2}{2m^2r^2} - \frac{ML^2}{m^2r^3}\right) \qquad \text{(global rain coordinates)} \quad (20)$$

B. Newton's expression for angular momentum, with Newton's "universal time t" is:

$$\frac{L}{m} \equiv r^2\frac{d\phi}{dt} \qquad \text{(Newton, universal time } t\text{)} \quad (21)$$

Show that Newton's expression for the square of the velocity of the stone is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2r^2} \qquad \text{(Newton)} \quad (22)$$

C. Now, Newton's expression for gravitational potential energy per unit mass (chosen to go to zero far from the center of attraction) is $U(r) = -M/r$. Write down Newton's conservation of energy equation, solve it for the radial velocity, and show the result:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \left(-\frac{M}{r} + \frac{L^2}{2m^2r^2}\right) = \frac{E}{m} - \frac{V_{\text{NewtL}}(r)}{m} \qquad \text{(Newton)} \qquad (23)$$

where **Newton's effective potential** is $V_{\text{NewtL}}(r)/m$.

D. Sketch for the Newtonian case a diagram like that of Figure 3: a plot of $V_{\text{NewtL}}(r)$ with horizontal lines for different values of E. Describe the resulting orbits and contrast them to those for motion in curved spacetime.

Of course the general relativity expression (20) is not just another version of Newton's equation (23). But look at the basic similarity of the right sides of these two equations: a constant term from which we subtract a function of the r-coordinate—the "effective potential"—that varies with the value of map angular momentum L.

> *Conclusion of this analysis:* It is the negative third term in the effective potential on the right side of (20), with r^3 in its denominator, that drives the effective potential downward as r becomes smaller as it approaches the event horizon—thereby creating the PIT in the potential labeled in Figures 3 and 4. This third term is the child of spacetime curvature.

In a stable circular orbit the stone's map energy rests at the minimum of the effective potential; the stone rides round and round the black hole without changing r-coordinate.

Stable orbit at effective potential minimum

DEFINITION 4. Stable circular orbit

A stone in a stable circular orbit has map energy E/m equal to the *minimum* of the effective potential $V_{\text{L}}(r)/m$, for example the map energy labeled 1 in Figure 3 and energies labeled 2 and 3 in Figure 4. Any incremental change in the r-coordinate at constant E/m puts the stone into the forbidden map energy region under the effective potential curve, where a stone cannot go.

We use a little open circle to locate a stable circular orbit on an effective potential energy curve. The point labeled 1 in Figure 4 is the stable circular orbit of minimum r-value analyzed in Section 8.5.

Einstein opens up a second set of r-coordinates where the effective potential also has zero slope, illustrated by point D in Figure 3 and points 4 and 5 in Figure 4. Each of these is a *maximum* of the effective potential curve; at this r-coordinate the stone experiences no tendency to move either to larger or smaller r-coordinate, so will stay at the same r-coordinate, riding round and round the black hole at constant r-coordinate. We call these **unstable** or **knife-edge** circular orbits, because slight departure from the knife-edge r-coordinate leads to decisive motion either to larger r, or else—horrors!—to smaller r that leads to the event horizon.

DEFINITION 5. Unstable (or knife-edge) circular orbit

A stone in an unstable (or knife-edge) circular orbit has map energy

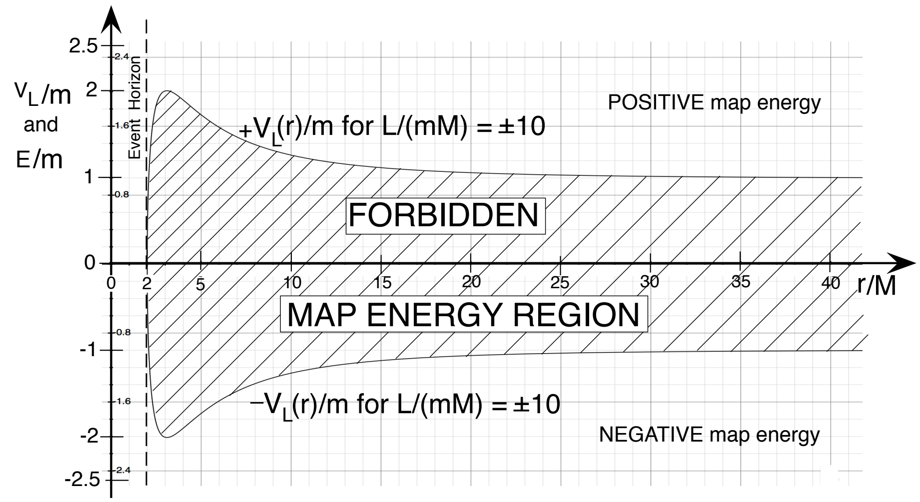

FIGURE 5 Possible negative map energy region under the $-V_{\rm L}$ curve, in addition to our everyday positive map energy region above the $+V_{\rm L}$ curve. We cannot travel between our positive map energy region and the negative map energy region, because the only worldlines that connect them must pass inward through the event horizon, then back out again. (Diagonal lines emphasize impenetrability.) So where *is* this negative map energy region?

Unstable (or knife-edge) orbit at effective potential maximum

E/m equal to the *maximum* of the effective potential $V_{\rm L}/m$, so that any incremental r-displacement in either direction puts the stone into a region with a gap between E/m and $V_{\rm L}/m$ such that this displacement increases.

We use a little filled circle to locate an unstable circular orbit on an effective potential energy curve.

Comment 4. How long on a knife edge?
Suppose that our spaceship is in a knife-edge orbit, technically an *unstable orbit*. Slight cosmic wind, firing of a projectile, or ejection of the day's trash may give our spaceship a tiny r-motion. Once displacement from the effective potential peak occurs, the slope of the effective potential urges the spaceship farther away from the point of zero slope, either outward toward larger r-coordinate or inward toward the event horizon. Sooner or later—who knows when?—a stone inevitably falls off the effective potential maximum of an unstable circular orbit.

"Why, oh why," our captain cries, "didn't I carry along a booster rocket? A tiny rocket boost to push us outward could have reversed our initially slow inward motion and allowed us to escape. But now it's too late!"

Strange results follow from equation (19), which requires that $(E/m)^2 \geq (V_{\rm L}/m)^2$ in order that $dr/d\tau$ be real. A consequence of this condition is that either $E/m \geq +V_{\rm L}/m$ or $E/m \leq -V_{\rm L}/m$. Figure 5 shows this condition. A stone cannot move, or even exist, with E/m in the region $+V_{\rm L}/m > E/m > -V_{\rm L}/m$. This is a forbidden map energy region, because

$dr/d\tau$ would be imaginary there. *Result:* The forbidden map energy region divides spacetime outside the event horizon into two isolated regions: one for positive map energy and the other for negative map energy. The stone cannot travel directly between them. This definition of a forbidden map energy region is consistent with that given in Definition 2.

Figure 3 shows only positive values of map E/m. This is the region we live in, where we carry out our measurements and observations, the upper region of positive map energy in Figure 5. What is the meaning of negative E/m in the lower region of Figure 5? Can we carry out measurements and observations there? Remember that map energy is a global map quantity, not a quantity that we can measure; its negative value tells us nothing about permitted measurements. In the exercises you show that we can construct local inertial frames in the negative map energy region, so we can carry out measurements and observations there, just as we can in the region above the forbidden map energy region.

Can we travel from the upper (positive map energy) region in Figure 5 to the lower (negative map energy) region? Our own worldline, just like the worldline of a stone, cannot pass directly through that forbidden map energy region. Figure 5 shows that the forbidden map energy region ends at the event horizon, $r = 2M$. Can we make an end run around the forbidden map energy region by moving in through the event horizon and back out again? No, sorry: Once inside the event horizon, we cannot come out again; instead we move relentlessly inward to the singularity. See exercise 11 in Section 8.7.

Objection 1. *Can light move between the upper and lower regions?*

Nope. Figure 11 in Section 11.8 shows that a corresponding forbidden region for light separates upper and lower regions. Both for stones and for light, the two regions are physically isolated.

Objection 2. *Wait! Where is this lower region? It has the same r-values as the upper region but you tell me that it lies "somewhere else," in a negative map energy region we cannot reach. Where is it?*

The answer is subtle and deep. Later we will understand that global rain coordinates do not include all of spacetime. We must find other global coordinates that include such regions. Chapter 21 treats these matters. Keep on reading!

Chapters 17 through 21 examine the spinning black hole. We will find that for the spinning black hole we may be able to travel between the corresponding upper and lower regions by dropping through the event horizon from the upper region, using rocket thrusts while inside the event horizon, then emerging outward through the event horizon into the lower region. Luc Longtin summarizes: "The non-spinning black hole is like the spinning black hole, but with its gate to other universes closed. For the spinning black hole, the gate is ajar." (initial quote, Chapter 21)

8.5 ■ PROPERTIES OF CIRCULAR ORBITS

Details! Details!

A series of Queries helps you to explore some properties of circular orbits in the everyday positive map energy region around the non-spinning black hole.

QUERY 4. Map r-values of circular orbits

A. A circular orbit is possible at every r-coordinate where the effective potential has zero slope. Take the r-derivative of both sides of (18) for a fixed L/m, set this derivative equal to zero, and show the following result:

$$r^2 - \frac{L^2}{Mm^2}r + 3\frac{L^2}{m^2} = 0 \qquad \text{(circular orbit)} \qquad (24)$$

B. Equation (24) is linear in $(L/m)^2$. Solve it to find:

$$\left(\frac{L}{m}\right)^2 = \frac{Mr^2}{r - 3M} \qquad \text{(circular orbit, } r > 3M) \qquad (25)$$

Note that this expression is valid for both stable and unstable circular orbits and is invalid for $r < 3M$, where L/m would be imaginary. Circular orbits cannot exist for $r < 3M$, and for $r = 3M$ the circular orbit is a limiting case (Item D in Query 8)

B. Equation (24) is quadratic in r. Solve it to find:

$$r = \frac{L^2}{2m^2M}\left[1 \pm \left(1 - \frac{12M^2m^2}{L^2}\right)^{1/2}\right] \qquad \text{(circular orbit, } r > 3M) \qquad (26)$$

Refer to Figure 4. Make the argument that the + sign in (26) corresponds to the minimum of the effective potential, that is to a stable circular orbit; and that the − sign corresponds to the maximum of the effective potential, that is to the unstable (knife-edge) circular orbit.

C. *Optional:* Take the second derivative of (26) and verify that the ± signs in (26) correspond, respectively, to a minimum and maximum of the effective potential.

Look more closely at equation (26) and the effective potential curve in Figure 4 with the "half-black" little circle labeled number 1. In order for the

r-coordinate to be real, the square root expression in (26) must be real. This occurs only when $|L/m| \geq (12)^{1/2}M = 3.4641M$. You can show that for the minimum map angular momentum, the global r-coordinate of the circular orbit is $r = 6M$. This is called the **innermost stable circular orbit** and is located at $r_{\mathrm{ISCO}} = 6M$.

Definition
ISCO

DEFINITION 6. Innermost stable circular orbit (ISCO)
The **innermost stable circular orbit (ISCO)**, located at $r_{\mathrm{ISCO}} = 6M$, divides r-values for unstable circular orbit in the region $3M < r < 6M$ from r-values for stable circular orbits in the region $r > 6M$. We can call the ISCO "half stable:" An increase in r at the same map energy puts the stone into a forbidden map energy region (like a stable circular orbit); a decrease in r at the same map energy puts the stone into a legal map energy region (like an unstable circular orbit).

Section 8.6 describes a so-called *toy model* of a quasar, the brightest steady source of light in the heavens. This emission comes from the loss of map energy of a stone that enters a circular orbit at large r and tumbles down through a series of "stable" circular orbits of smaller and smaller r. When the stone reaches the innermost stable circular orbit and continues to lose map energy, it spirals inward across the event horizon, after which we can no longer detect its radiation.

QUERY 5. Shell speed of a stone in a circular orbit

Compute the speed of the stone in a circular orbit measured by a shell observer, as follows.

A. Consider two ticks of the orbiting stone's clock, separated by wristwatch time $\Delta\tau$ and by zero distance measured in the stone's local frame, but separated by shell time $\Delta t_{\mathrm{shell}}$ and by shell distance $\Delta x_{\mathrm{shell}} = \bar{r}\Delta\phi$. The relation between $\Delta t_{\mathrm{shell}}$ and $\Delta\tau$ is just the special relativity expression

$$\Delta t_{\mathrm{shell}} = \gamma_{\mathrm{shell}}\Delta\tau = (1 - v_{\mathrm{shell}}^2)^{-1/2}\Delta\tau \tag{27}$$

where γ_{shell} has an obvious definition. From the value of map angular momentum, we can use (27) to calculate shell speed:

$$\begin{aligned} v_{\mathrm{shell}} &= \lim_{\Delta t_{\mathrm{shell}}\to 0}\left(\frac{\bar{r}\Delta\phi}{\Delta t_{\mathrm{shell}}}\right) = (1 - v_{\mathrm{shell}}^2)^{1/2}\ \frac{r^{\,2}d\phi}{rd\tau} \\ &= (1 - v_{\mathrm{shell}}^2)^{1/2}\ \frac{L}{mr} \qquad (\phi - \text{motion}) \end{aligned} \tag{28}$$

From this equation, show that

$$v_{\mathrm{shell}}^2 = \left[1 + \left(\frac{mr}{L}\right)^2\right]^{-1} \qquad (\phi - \text{motion}) \tag{29}$$

From equation (25) show that

$$\left(\frac{mr}{L}\right)^2 = \frac{r}{M} - 3 \qquad \text{(circular orbit)} \qquad (30)$$

Substitute this into (29) to find

$$v_{\text{shell}}^2 = \frac{M}{r-2M} = \left(\frac{M}{r}\right)\left(1-\frac{2M}{r}\right)^{-1} \qquad \text{(circular orbit, } r > 3M) \qquad (31)$$

Equation (31) is valid for both stable and unstable (knife-edge) circular orbits.

B. What is the value of the shell speed v_{shell} in the ISCO, the innermost stable circular orbit at $r = 6M$?

C. Verify that the minimum map r-coordinate for a circular orbit is $r = 3M$. (*Hint:* What is the upper limit of the shell speed of a stone?)

D. From (25) show that, as a limiting case, the map angular momentum L/m increases without limit for the knife-edge circular orbit of minimum r-coordinate.

Circular orbit of light

Comment 5. Unlimited map angular momentum?
How can the map angular momentum possibly increase indefinitely (Item D of Query 6)? It does so only as a limiting case. According to (10), the map angular momentum is equal to $L/m = r^2 d\phi/d\tau$. The relation between wristwatch time $d\tau$ and shell time dt_{shell} is given by (27), the usual time-stretch formula of special relativity. As the stone's speed approaches the speed of light, the advance of wristwatch time becomes smaller and smaller compared with the advance of shell time. In the limit, it takes zero wristwatch time for the stone to circulate once around the black hole. Because $d\tau$ is in the denominator of the expression for angular momentum, the map angular momentum L/m increases without limit. The speed of light is the limiting speed of a stone, so the speed-of-light orbit is a limiting case, reached by a stone only after an unlimited lapse of the Schwarzschild t-coordinate. This limiting case tells us, however, that light can travel in a circular orbit at $r = 3M$ (Chapter 11).

QUERY 6. Global map energy of a stone in circular orbit

Find an expression for map energy E/m in global rain coordinates for the stone in a circular orbit, as follows:

A. Use (25) and (15) with $dr = 0$ for a circular orbit. Show that the result is:

$$\frac{E}{m} = \frac{r-2M}{r^{1/2}(r-3M)^{1/2}} \qquad \text{(circular orbit, } r > 3M) \qquad (32)$$

B. Does (32) go to the values you expect in three cases: Case 1: $r \gg M$? Case 2: $r \to 3M^+$ (r decreases from above)? Case 3: $r < 3M$?

QUERY 7. Map energy and map angular momentum of a stone in the ISCO

A. Show that the map angular momentum of the ISCO is $L_{\text{ISCO}}/(mM) = 3.464\ 101\ 615$.

B. Show that the map energy of the ISCO is $E_{\text{ISCO}}/m = 0.942\ 809\ 042$.

QUERY 8. Shell energy of a stone in a circular orbit

A. Use the special relativity relation $E_{\text{shell}}/m = (1 - v^2_{\text{shell}})^{-1/2}$ for the local shell frame plus (31) for v^2_{shell} to show that

$$\frac{E_{\text{shell}}}{m} = \left(\frac{r-2M}{r-3M}\right)^{1/2} \qquad \text{(circular orbit, } r > 3M) \tag{33}$$

B. From (32) and (33), verify that

$$\frac{E_{\text{shell}}}{m} = \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{E}{m} \qquad \text{(circular orbit } r > 2M) \tag{34}$$

This agrees with equation (12) in Section 6.3 for a diving stone.

C. Far from the black hole, that is for $r \gg M$, set $\epsilon = M/r$. Use our standard approximation (inside the front cover) to show that at large r-coordinate equation (33) becomes:

$$\frac{E_{\text{shell}}}{m} \approx 1 + \frac{M}{2r} \qquad \text{(circular orbit, } r \gg M) \tag{35}$$

D. Take (31) to the same limit and show that (35) becomes:

$$E_{\text{shell}} \approx m + \frac{1}{2} m v^2_{\text{shell}} \qquad \text{(circular orbit, } r \gg M) \tag{36}$$

Would Newton be happy with your result? Would Einstein?

QUERY 9. Orbiter wristwatch time for one circular orbit

A. From (25) and (11) verify the following wristwatch time for one circular orbit ($\Delta\phi = 2\pi$),

$$\frac{\Delta\tau}{M} = \frac{2\pi(r/M)^2}{L/(mM)} = 2\pi\frac{r}{M}\left(\frac{r-3M}{M}\right)^{1/2} \qquad \text{(one circular orbit)} \qquad (37)$$

B. Explain why $\Delta\tau \to 0$ as $r \to 3M$.

C. For a black hole with $M = 10M_{\text{Sun}}$, find the wristwatch time in seconds for one circular orbit for the three values $r/M = 10, 6, 4$.

D. For a non-spinning black hole of mass $M \approx 4 \times 10^6 M_{\text{Sun}}$ equal to the black hole at the center of our galaxy, find the wristwatch time in seconds for one circular orbit for the three values $r/M = 10, 6, 4$.

E. *Optional:* Solve (37) for $(r/M - 3)$ and put $r \approx 3M$ in the expression on the right side of your result. Find the value of $(r/M - 3)$ when $\tau = 1$ microsecond for a black hole of mass $M = 10M_{\text{Sun}}$. What is the numerical value of the observed distance $2\pi r$ around this circumference in meters—a directly-measurable distance (Section 3.3). So now we have an astronaut who traverses this large, measurable circumference in a microsecond. To do this, she must move at many times the speed of light. Can this be right? Explain your answer.

QUERY 10. Shell time for one circular orbit

Verify the following expressions for the periods of one circular orbit.

A. From equations (27), (31), and (37), show that the local shell time for one circular orbit is:

$$\Delta t_{\text{shell}} = 2\pi r\left(\frac{r-2M}{M}\right)^{1/2} \qquad \text{(one circular orbit)} \qquad (38)$$

For the minimum (knife-edge) orbit, with $r = 3M$, explain why the shell period is equal to the circumference of the orbit.

B. For a circular orbit of very large r-coordinate, explain why global rain ΔT, shell Δt_{shell} , and orbiter wristwatch time $\Delta\tau$ all have the same value for one orbit, namely $2\pi r^{3/2}/M^{1/2}$.

8.6 ■ TOY MODEL OF A QUASAR

Beacon of the heavens

Quasar

A **quasar** ("quasi-stellar object") is an astronomical object that pours out electromagnetic radiation of many frequencies at a prodigious rate. The quasar is the brightest steady source of light in the heavens, so we can see it farther away than any other steady source. At the center of a quasar is, almost certainly, a spinning black hole (Chapters 17 through 21), but here we make a first quick model of a quasar using a non-spinning black hole. This sort of rough, preliminary analysis is called a **toy model**.

Toy model

Accretion disk

A simple model of quasar emission postulates an **accretion disk**, a gas disk that swirls around the black hole in its equatorial plane. Interactions among the molecules and atoms in this gas cloud are complicated. We assume simply that interactions among neighboring atoms and ions heats the accretion disk to high temperature and that the resulting electromagnetic emission is what we observe far from the quasar. The radiated energy we observe comes from the change in orbital map energy of each atom as it moves sequentially from a large-r circular orbit to smaller-r circular orbits with smaller and smaller map energy. We also assume that significant map energy change takes place over many orbits, so sequential orbits are nearly circular, each with its nearly constant values of E/m and L/m.

Radiating away change in map energy

Comment 6. Losing map angular momentum

What is the mechanism of this orbit change? The sequence of circled numbers 3-2-1 in Figure 4 shows that "our circulating atom" decreases both its map energy and its map angular momentum as it occupies a set of circular orbits of decreasing r-values. Map angular momentum of an isolated system is conserved, so the lost map angular momentum of our circulating atom must be transported outward, away from the black hole. What mechanism can transport map angular momentum outward? Equation (31) tells us that atoms in adjacent circular orbits have slightly different shell speeds, with atoms in the higher orbit moving more slowly. One might think (incorrectly) that *friction* between our atom and atoms in a higher orbit increases the velocity—and therefore the map angular momentum—of atoms in the higher orbit, and so on outward. However, direct friction turns out to be far too small to account for the outward transport of map angular momentum. The mechanism may depend on our model that the accretion disk consists of highly ionized atoms, a plasma, threaded with magnetic field lines. Magnetic fields greatly increase interactions between ions, so might account for the outward transport of map angular momentum in a quasar. We simply do not know.

Transport map angular momentum outward

Innermost stable circular orbit

Eventually our atom's circular orbit drops to $r = 6M$, the innermost stable circular orbit (Definition 6 in Section 8.5). At this point our atom continues to lose map angular momentum, so that it drops out of the last stable circular orbit and spirals inward across the event horizon. Once our atom crosses the event horizon, any further radiation moves only inward and cannot reach us, the external observers.

QUERY 11. Map energy given up by "our atom."

The prodigious radiation we observe from quasars is all emitted before orbiting atoms and ions cross the event horizon.

A. Start with an atom in a circular orbit at large r-coordinate, moving slowly so its initial map energy is approximately equal to its mass, $E/m \approx 1$, from (32). Now think of its map energy later, as the atom moves in the stable circular orbit of minimum r-coordinate, $r = 6M$. Using (32), find the map energy E/m of the atom in this minimum-r circular orbit to three significant digits. How much map energy has the atom given up during the process of dropping gradually from large r-coordinate to the smallest stable circular orbit? [My answers: $E_{\text{final}} = 0.943m$ so $\Delta E = 0.057m$.]

B. Suppose that the atom emits as electromagnetic radiation all the map energy it gives up (from Item A) as it spirals down to the circular orbit at $r = 6M$. Show that the map energy of that total amount of radiation emitted is $\Delta E = 0.057m$. Since initially we had $E/m = 1$, therefore 0.057, or 5.7%, is also the fraction of initial map energy that is radiated as the atom spirals inward to the lowest stable circular orbit.

Measure map energy at far from the black hole

Map energy E/m is a constant of motion, independent of position. Suppose that the map energy radiated by the atom during its descent finds its way outward. Then the same map energy ΔE arrives at the distant r-coordinate from which the atom departed earlier with $E/m \approx 1$. Moreover, very far from the black hole spacetime is flat; so map energy is equal to shell energy there, equation (34). Therefore the group of shell frame observers far from the black hole see—can in principle measure—a total radiated energy of $\Delta E = 0.057m$, which is 5.7 percent of the stone's initial map energy.

Comment 7. How much emitted energy?
No nuclear reaction on Earth—except particle-antiparticle annihilation—releases as much as one percent of the rest energy of its constituents. Chapter 18 shows that for a black hole of maximum spin, the fraction of initial mass radiated away by a stone that spirals down from a large r-coordinate to an innermost stable circular orbit is 42 percent of its rest energy. No wonder quasars are such bright beacons in the heavens!

Rate of emitted radiation

Now let our atom drop into the black hole from the innermost stable circular orbit at $r = 6M$. How much does the mass of the black hole increase? Equation (28) in Section 6.5 says that the total mass of the black hole increases by the map energy E/m of the object falling into it. This allows us to connect the rate of increase of the mass of a quasar and its brightness to the rate at which it is swallowing matter from outside. Let dm/dT be the rate at which mass falls into the black hole from far away and dM/dT be the rate at which the mass of the black hole increases. Then Item B in Query 11 tells us that the rate of radiated energy is

$$\text{Rate of radiated energy} \approx 0.057\,\frac{dm}{dT} \qquad (\,dm = \text{mass falling in}) \tag{39}$$

so that the mass M of the black hole increases at the rate:

$$\frac{dM}{dT} = (1 - 0.057)\frac{dm}{dT} = 0.943\frac{dm}{dT} \qquad (M = \text{ mass of black hole}) \tag{40}$$

QUERY 12. Power output of a quasar
During every Earth-year, a distant quasar swallows $m = 10M_{\text{Sun}}$ = ten times the mass of our Sun. Recall that watts equals joules/second and, from special relativity, $\Delta E[\text{joules}] = \Delta m[\text{kilograms}]c^2[\text{meters}^2/\text{second}^2]$.

A. How many watts of radiation does this quasar emit, according to our toy model?
B. Our Sun emits radiation at the rate of approximately 4×10^{26} watts. The quasar is how many times as bright as our Sun?
C. Compare your answer in Item B to the total radiation output of a galaxy, approximately 10^{11} Sun-like stars.

QUERY 13. How long does a quasar shine?
We see most quasars with large redshifts of their light, which means they began emission not long after the Big Bang, about 14×10^9 years ago. A typical quasar is powered by a black hole of mass less than 10^9 solar masses. Explain, from the results of Query 12, what this says about the lifetime during which the typical quasar shines.

Objection 3. *We have talked about t and T global coordinates and different kinds of local times near a black hole: shell time, diver time, orbiter time. Is it possible for me to travel to a black hole and use it, in some way, to live longer than I can live here on Earth?"*

As with many profound questions, the answer is both "Yes," and "No." With or without a black hole, you may live longer—on your wristwatch—between any two events than your twin—on her wristwatch—who takes a different worldline through spacetime between these two events (Twin "Paradox," Section 1.6). However, you cannot escape the iron rule that *your aging is identical to the total time lapse on your wristwatch,* no matter where you travel or at what rates you move along the way. When your wristwatch says 100 years after birth, you have aged 100 years. This is the total time lapse that you *experience.* In this sense, relativity does not provide a way for you to burst the bonds of human aging. Sorry!

8.7 ■ EXERCISES

1. Shell time for one orbit

An observer in a circular orbit at a given map r-coordinate moves at speed v_{shell} past the shell observer. Equation (31) gives the value of this shell speed. Query 9 gives the wristwatch time for one orbit. What is the shell time for one orbit?

A. Show that this shell time for one orbit is

$$\frac{\Delta t_{\text{shell}}}{M} = \frac{2\pi r/M}{v_{\text{shell}}} = 2\pi \frac{r}{M}\left(\frac{r-2M}{M}\right)^{1/2} \qquad \text{(one circular orbit)} \quad (41)$$

(*Hint:* Recall the definition in Section 3.3 of r—the "reduced circumference"—as the measured circumference of a concentric shell divided by 2π.)

B. Compare Δt_{shell} for one orbit in (41) with $\Delta\tau_{\text{shell}}$ for one orbit from (37). Which is longer at a given r-value? Give a simple explanation.

C. What is the map angular momentum L of the orbiter, written as r times an expression involving v_{shell}? (The answer is *not* mrv_{shell}.)

D. The text leading up to Definition 4 in Section 8.5 shows that the smallest r-coordinate for a stable circular orbit is $r = 6M$; equation (31) determines that in this orbit the orbiter's shell speed $v_{\text{shell}} = 0.5$, half the speed of light. Assume the central attractor to be Black Hole Alpha, with $M = 5000$ meters. The following equation gives, to one significant digit, the values of some measurable quantities for the innermost stable circular orbit. Find these values to three significant digits.

$$\begin{aligned} \Delta t_{\text{shell}} &\approx 4\times 10^5 \text{ meters} && \text{(shell time for one orbit)} \\ \Delta\tau_{\text{orbiter}} &\approx 3\times 10^5 \text{ meters} && \text{(wristwatch time for one orbit)} \\ L/m &\approx 2\times 10^4 \text{ meters} && \end{aligned} \tag{42}$$

E. The orbiter of Item D completes one circuit of the black hole in approximately one millisecond on her wristwatch. If you ignore tidal effects, does this extremely fast rotation produce *physical discomfort* for the orbiter? If she closes her eyes, does she get dizzy as she orbits?

2. When are Newton's Circular Orbits Almost Correct?

Your analysis of the Global Positioning System (GPS) in Chapter 4 calculated values of r-coordinate and orbital speed of a GPS satellite in circular orbit using Newton's mechanics, with the prediction that the general relativistic analysis gives essentially the same values of r-coordinate and speed for this application. Under what circumstances are circular orbits predicted by Newton indistinguishable from circular orbits predicted by Einstein? Answer this question using the following outline or some other method.

A. Find Newton's expression similar to equation (26) for the r-coordinate of a stable circular orbit, starting with equation (23).

B. Recast equation (26) for the general-relativistic prediction of r for stable orbits in the form

$$r = r_{\text{Newt}}(1-\epsilon) \tag{43}$$

where r_{Newt} is the r-coordinate of the orbit predicted by Newton and ϵ is the small fractional deviation of the orbit from Newton's prediction. This expression neglects differences between the Newtonian and relativistic values of L when expressed in the same units. Use the approximation inside the front cover to derive a simple algebraic expression for ϵ as a function of r_{Newt}.

C. Set your expression for ϵ equal to 0.001 as a criterion for good-enough equality of the r-coordinate according to both Newton and Einstein. Find an expression for r_{min}, the smallest value of the r-coordinate for which this approximation is valid.

D. Find a numerical value for r_{min} in meters for our Sun. Compare the value of r_{min} with the r-coordinate of the Sun's surface.

E. What is the value of ϵ for the r-coordinate of the orbit of the planet Mercury, whose orbit has an average r-coordinate 0.387 times that of Earth?

F. What is the value of ϵ for the r-coordinate of a 12-hour orbit of GPS satellites around Earth?

3. Map ΔT for one orbit

Convert lapse of wristwatch time $\Delta\tau$ for one circular orbit from (37) to lapse ΔT for one circular orbit using the following outline or some other method:

A. Show that for a circular orbit, equation (13) becomes:

$$\frac{\Delta T}{\Delta\tau}(\text{one orbit}) = \frac{E}{m}\left(1-\frac{2M}{r}\right)^{-1} = \frac{E}{m}\frac{r}{(r-2M)} \tag{44}$$

B. Into this equation, substitute for E/m from (32) to obtain

$$\frac{\Delta T}{\Delta\tau}(\text{one orbit}) = \left(\frac{r}{r-3M}\right)^{1/2} \tag{45}$$

C. Use this result plus (37) to show that

$$\Delta T(\text{one orbit}) = \Delta\tau\frac{\Delta T}{\Delta\tau} = 2\pi\frac{r^{3/2}}{M^{1/2}} \tag{46}$$

Does any observer measure this lapse Δt for one orbit?

4. Kepler's Laws of Planetary Motion

Johannes Kepler (1571-1630) provided a milestone in the history of astronomy: his **Three Laws of Planetary Motion**, deduced from a huge stack of planetary observations made by his mentor Tycho Brahe (1546-1601) and expressed in our notation.

1. A planet orbits around the Sun in an elliptical orbit with the Sun at one focus of the ellipse.

2. The r-coordinate vector from the Sun to the planet sweeps out equal areas in equal lapses of T-coordinate.
3. The square of the period of the planet is proportional to the cube of the planet's mean r-coordinate from the Sun.

A. Show by a simple symmetry argument that Kepler's Second Law describes circular orbits around a black hole.

B. From equation (46) show that Kepler's Third Law is also valid for *circular* orbits around a black hole (when expressed in global rain coordinates).

C. Kepler's Third Law is sometimes called the **1-2-3 Law** from the exponents in the following equation. Use equation (46) to show that for circular orbits, in our regular notation using meters,

$$M \equiv M^1 = \omega^2 r^3 \tag{47}$$

where $\omega \equiv 2\pi/\Delta T$, with ΔT for one orbit.

Comment 8. Is Kepler's First Law Valid?
Figure 4 in Section 9.3 shows that Kepler's First Law is definitely *not* valid for non-circular orbits near a non-spinning black hole. Chapter 11 shows that the orbit of the planet Mercury differs *slightly* from the planetary orbit analyzed by Newton. The predicted value of this deviation of Mercury's orbit was an early validation of Einstein's general relativity.

5. Longest Life Inside the event Horizon

Objection 12 in Section 7.8 asked, "Can I increase my lifetime inside the event horizon by blasting rockets in either ϕ direction to add a ϕ-component to my global velocity?" You are now able to answer this question using your new knowledge of map angular momentum. Suppose that you ride on a stone that moves between the event horizon and the singularity.

A. What equation in the present chapter leads to the following expression for your wristwatch lifetime inside the horizon?

$$\tau\,[2M \rightarrow 0] = \int_0^{2M} \left[\left(\frac{E}{m}\right)^2 + \left(\frac{2M}{r} - 1\right)\left(1 + \frac{L^2}{m^2 r^2}\right)\right]^{-1/2} dr \tag{48}$$

Note, first, that the square-bracket expression on the right side of (48) is in the denominator of the integrand. Second, note that this equation describes any motion of the observer whatsoever, free-fall or not. Free-fall motion has constant E and L. For motion that is not free-fall, the value of E or L (or both) can change along the worldline of the stone.

B. Can any non-zero value of L along your worldline increase your wristwatch lifetime inside the event horizon?

C. What value of E gives you the maximum wristwatch lifetime inside the event horizon?

D. By what practical maneuvers can you achieve the value of E determined in Item C?

E. Show that the maximum value of wristwatch time from the event horizon to the singularity is πM meters. *Hint:* Make the substitution $(r/2M)^{1/2} = \sin\theta$.

F. Chapter 7 found the mass of a "20-year black hole" for a raindrop. Find the numerical value of (*fraction*) in the following equation:

$$\begin{aligned} &(\text{mass of "20-year black hole" in Item E}) \\ &= (fraction) \times (\text{mass of "20-year black hole" for a raindrop}) \end{aligned} \tag{49}$$

6. Forward Time Travel Using a *Stable* Circular Orbit

You are on a panel of experts asked to evaluate a proposal from the Space Administration to "travel forward in time" using the difference in rates between a clock in a stable circular orbit around a black hole and our clocks remote from the black hole. Give your advice about the feasibility of the scheme, based on the following analysis or one of your own.

A. Consider two sequential ticks of the clock of a satellite in a stable circular orbit around a black hole. Use a result of Exercise 1 to show that

$$\frac{\Delta\tau_{\text{orbiter}}}{\Delta T} = \left(\frac{r-3M}{r}\right)^{1/2} \tag{50}$$

B. What is the value of the ratio $\Delta\tau_{\text{orbiter}}/\Delta T$ in the stable circular orbit of smallest r-coordinate, $r = 6M$?

C. What rocket speed in flat spacetime gives the same ratio of rocket clock time to "laboratory" time as the stable circular orbit of smallest r-coordinate?

D. Does the proposed time travel method require rocket fuel to put the rocket in orbit and to escape the black hole?

E. Based on this analysis, do you recommend in favor of—or against—the Space Administration's proposal for forward time travel using stable circular orbits around a black hole?

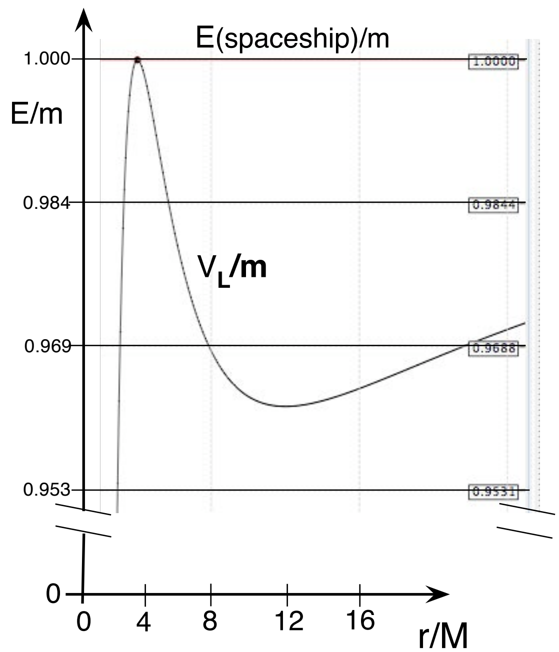

FIGURE 6 Insertion into a knife-edge orbit at $r = 4M$ with map energy $E/m \approx 1$, equal to that of a spaceship moving slowly at large r-coordinate in a direction chosen to give it the value of L/m required to establish the peak value for V_L/m.

7. Forward Time Travel Using a *Knife-Edge* Circular Orbit

Whatever your own vote on the forward time travel proposal of Exercise 6, the majority on your panel rejects the proposal because it requires extra rocket thrust for insertion into and extraction from the circular orbit at $r = 6M$. The Space Administration returns with a new proposal that uses a knife-edge circular orbit, assuming that an automatic device can fire small rockets to balance the satellite safely on the knife-edge of the effective potential. The Space Administration notes that such an orbit can be set up to require *very small* rocket burns, both for insertion into and extraction from a knife-edge circular orbit. As an example, they present Figure 6 for the case of nonrelativistic distant velocity, so that the map energy of the satellite is $E/m \approx 1$. While still far from the black hole, the spaceship captain uses rockets to achieve the value of L required so that $V_L(r)/m = E/m = 1$ on the peak shown in Figure 6. They boast that the time stretch factor is increased enormously by high satellite shell speed in the knife-edge orbit without the need for rocket burns to achieve that speed.

A. The condition shown in Figure 6 means that $V_{\rm L}(r)/m = 1$ at the peak of the effective potential (18). This equation plus equation (26) are two equations in the two unknowns r and L. Solve them to find $r = 4M$ and $L/m = 4M$. *Optional:* Describe in words how the commander of the spaceship sets the desired value of L while still far away, without changing the remote non-relativistic speed $v_{\rm far}$.

B. What is the factor $d\tau/dt_{\rm shell}$ for the spaceship in this orbit? What speed in flat spacetime gives the same time-stretch ratio?

C. Does the spaceship require a significant rocket burn to leave its knife-edge circular orbit and return to a remote position? What will be its shell speed at that distant location?

8. "Free" data-collection orbit

After its long interstellar trip, the spaceship approaches the black hole at relativistic speed, that is $E/m > 1$. The commander does not want to use a rocket burn to change spaceship map energy, but rather only its direction of motion (hence its map angular momentum) to enter a knife-edge circular orbit with the same map energy it already has.

A. Draw a figure similar to Figure 6 for this case.

B. Show that the astronauts can find a knife-edge circular orbit on which to perch, no matter how large the incoming far-away speed with respect to the black hole.

Once in an unstable circular orbit, small rocket thrusts keep the spaceship balanced at the peak of the effective potential. After they finish collecting data, the astronauts push-off outward and return toward home base at the same speed at which they approached, even if this speed is relativistic. In summary, once launched toward a black hole the explorers need little rocket power to go into an unstable circular orbit, to balance in that orbit while they study the black hole, then to return home. Further details in Chapter 9.

9. Nandor Bokor disproves relativity.

Nandor Bokor looks at Exercise 1 and shouts, "Aha, now I can disprove relativity!" Parts A through D below are steps in Nandor's reasoning, not separate questions to be answered. Resolve Nandor's disproof without criticizing him.

A. Nandor Bokor says, "Before I begin my disproof of relativity, recall that we have always had a choice about the shell frame. *First choice:* In order to be inertial, the local shell frame must be in free fall. In this case we drop the local shell frame from rest as we begin the experiment and must complete the experiment so quickly that the shell frame's

r-coordinate changes a negligible amount. *Second choice:* The local shell frame is at rest and therefore has a local gravitational acceleration. In that case we must complete our experiment or observation so quickly that local gravity does not affect the outcome. Usually our choice does not change the experimental result, but I am being super-careful here and will take the first choice, so that shell and orbiter frames are both inertial.

B. "Assume, then, that the shell frame is inertial," Nandor continues. "Equation (42) says that during one revolution of the orbiter its measured time lapse is $\Delta t_{\text{orbiter}} \approx 3 \times 10^5$ meters, while the measured shell clock time lapse is $\Delta t_{\text{shell}} \approx 4 \times 10^5$ meters. Note that these are both observed readings—measurements—and they are *different*. When the orbiter returns after one orbit the two inertial frames—orbiter and shell—overlap again.

C. "Now we have two truly equivalent inertial reference frames that overlap twice so we can compare their clock readings directly. (This is different from special relativity, in which one of the two frames—in the Twin Paradox, Section 1.6—is not inertial during their entire separation.) In the present orbiting case, neither observer can tell which of the two inertial frames s/he is in from inside his or her inertial frame."

D. Nandor concludes, "You tell me, Dude, which of the two equivalent inertial clocks—the orbiter's frame clock or the shell observer's frame clock—runs slow compared with the clock in the other frame. You can't! Equation (42) claims a difference where no difference is possible. Good-bye relativity!"

10. Equations of motion in Schwarzschild global coordinates

Start with the Schwazschild metric, equation (5) in Section 3.1, and show that equations (11), are (15) are the same in both global coordinate systems, but (16) takes the simpler form:

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \frac{E}{m} \qquad \text{(stone, Schwarzschild)} \qquad (51)$$

Comment 9. Why not Schwarzschild?

Why don't we take advantage of the simpler equation (51) by using Schwarzschild coordinates to describe the motion of the free stone? Because we already know—equation (22) in Section 6.4—that neither light nor a stone moves inward through the event horizon in a finite lapse of the Schwarzschild t-coordinate. In theory, Schwarzschild coordinates would not cause a problem with circular orbits in the present chapter because these orbits exist only outside the event horizon—indeed, only in the region $r > 3M$. But Chapter 9 treats more general trajectories of a stone, some of which move inward across the event horizon.

11. Life under the forbidden map energy region

If we could find some way to travel from our normal upper, positive map energy region in Figure 5 to the lower, negative map energy region (which extends outward far from the black hole), could we live a normal life there? What does "normal life" mean? We reduce "normal life" to essentials: that the equations of motion for a stone are real! Limit attention to motion outside the event horizon:

A. Show that the first two equations of motion (11) and (15) are the same for E/m under the forbidden region as for E/m above the forbidden region.

B. Show that the third equation of motion (16) tells us that $dT/d\tau$ is negative under the forbidden region, so that global T runs backward along the worldline of the stone. But T is a unicorn, not a measured quantity, so the third equation of motion is also valid under the forbidden region.

Where are we when we are under the forbidden map energy region in Figure 5? This is our first hint that our everyday lives may not have access to all regions of spacetime. Alice had it right: Wonderland—and black holes—become "curiouser and curiouser."

8.8 ■ REFERENCES

Initial Emily Dickinson poem from R. W. Franklin, *The Poems of Emily Dickinson, Variorum Edition* 1998, The Belknap Press of Harvard University. This poem is variation E of the poem with Franklin number 1570, written about 1882. Reprinted and modified with permission of Harvard University.

GRorbits interactive software program that displays orbits of a stone and light flash is available at `http://stuleja.org/grorbits/`

Last sentence of the final exercise: *Alice in Wonderland* by Lewis Carroll, first sentence of Chapter 2.