FROM INERTIA FORCES TO GENERAL RELATIVITY

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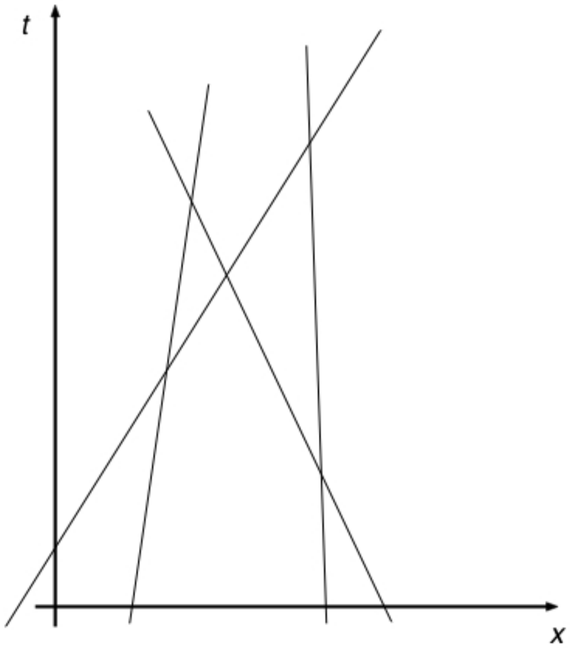
1. The *definition* of inertial frames: reference frames in which N1, N2, N3 are valid.

We can draw a spacetime diagram from the viewpoint of a reference frame S. *If S is an inertial frame* then
- the worldlines of all other *inertial* frames are *straight lines* on this diagram (see figure below)
- the worldlines of *non-inertial* frames would be *curved lines* on this diagram.



This is a beautiful, simple geometric picture which reflects the fact that there is something special about the worldlines of inertial frames (namely, they are straight).

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2. Description of motion in non-inertial frames:

$$m\vec{a} = \vec{F}_{net} - m\vec{a}_{tr} + m\vec{r} \times \dot{\vec{\omega}} + m\vec{\omega} \times (\vec{r} \times \vec{\omega}) + 2m\vec{v}_{rel} \times \vec{\omega} \qquad (1)$$

The right-hand side contains the net force and 4 other terms. In order to be able to pretend that (1) looks just like N2, we *pretend* that these 4 terms are *forces*. This is cheating; these 4 terms are *not* forces (there is nothing in nature which „causes" or „exerts" them), only algebraic terms in an equation. These fictitious „forces" are called *inertia forces*, because in all 4 terms the quantity *m* appears. (*m* is the measure of the object's *inertia*, called the inertial mass.)

At the same time $\vec{F}_{net}$ contains the *real*, physical forces, forces which are actually „caused", „exerted" by something. That they are *not* inertia forces should obviously

also be reflected in the fact that they do *not* contain mass *m* in their formula. The general rule to decide whether a force is real or fictitious seems to be:
- real forces: whose formula does not contain *m*
- inertia forces (fictitious forces): whose formula contains *m.*

A few examples to check that this last statement is indeed true (i.e. that real forces do not contain mass *m* in their formula):

$-k \cdot \vec{x}$	(spring force)
$\mu \cdot n \cdot \vec{e}_s$	(kinetic friction)
$q \cdot \vec{E}$	(electric force)
$q \cdot \vec{v} \times \vec{B}$	(magnetic force)
$\vec{T}$	(tension)
$m \cdot \vec{g}$	(gravitational force)
…	

Wait a minute! This last one, $m \cdot \vec{g}$, looks *exactly* like an inertial force! [True, the letter *m* is used in a different meaning here („gravitational mass”), but – according to experiments – it is equal to the inertial mass for any object!]

The *only logical and consistent step* (from a *mathematical* viewpoint) is to take out the $m \cdot \vec{g}$ term from $\vec{F}_{net}$ and add it to the inertia forces!

The inescapable conclusion (from a *physical* viewpoint): Gravity is not an actual force. It is fictitious, not „exerted” by anything. It is an inertia force. The only reason we „feel“ its effect is that we (who „feel“ it) are *not in an inertial frame* (quite similarly e.g. to the centrifugal force, which is only „felt“ by those who describe motion in a rotating reference frame).

So far we had thought that a reference frame fixed to the surface of Earth (neglecting the rotation of the Earth) is an inertial frame: the 4 last terms in (1) are zero. But now it turns out that it is *not* an inertial frame (since $m \cdot \vec{g}$ acts in it)! In fact, the equation of motion in this frame is

$$m\vec{a} = \vec{F}^*_{net} + m\vec{g}, \qquad (2)$$

where $\vec{F}^*_{net}$ represents the sum of *„really, truly real”* forces (i.e. excluding gravity which turned out to be an inertia force).
So the reference frame fixed to the surface of Earth is a *non-inertial* frame, an *accelerating* frame. *But in which direction does it accelerate and with what acceleration (relative to an inertial frame)?*

We can easily answer this question by looking at (2) and comparing it with a simplified version of (1) that only contains translational acceleration, no rotation:

$$m\vec{a} = \vec{F}_{net} - m\vec{a}_{tr} \qquad (3)$$

Eq. (3) describes motion in a frame which accelerates with $\vec{a}_{tr}$ relative to the inertial frame.

Comparing (2) with (3) gives us the answer: when we are standing on the surface of Earth, we are accelerating with $(-\vec{g})$ relative to an inertial frame. In other words, the inertial frame accelerates with $\vec{g}$ relative to us. In other words, the inertial frame is *falling freely*! What we would normally consider to be an accelerating frame – since it „accelerates downward with g" – turns out to be an inertial frame!

Let's check this idea. Is e.g. a freely falling elevator really an inertial frame?
(1) Does it satisfy Newton's first law? Yes, it does spectacularly! (Any object which we release from rest will stay at rest.)
(2) Does it satisfy Newton's second law? Yes, it does. But we have to remember to consider only $\vec{F}^{*}_{net}$ to be the net force (i.e we must leave gravity out of it).
(3) Does it satisfy Newton's third law? Yes, it does, but again provided that we leave gravity out of the net force.

Our conclusion: *any frame of reference which floats (or falls) freely (i.e. only under the influence of gravity) is an inertial frame. Any frame in which gravity is „felt" is not an inertial frame!*

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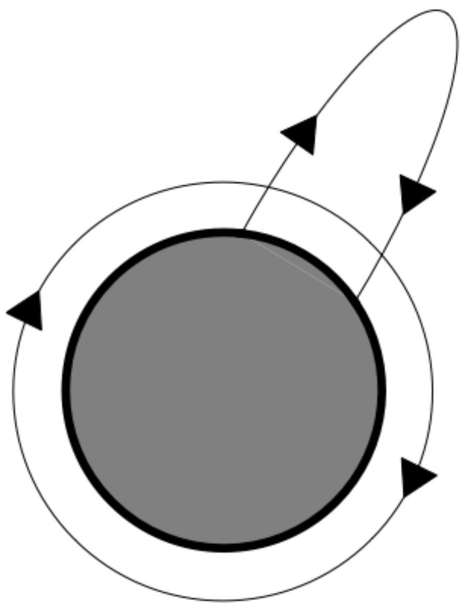
3. Let us imagine two „freely falling" objects: one is orbiting the Earth, the other is an object tossed upwards and falling back to the floor (see figure below). By careful adjustment (e.g. of the launching speed of the second object) one can achieve a situation in which the two objects meet *twice*. In other words, their worldlines intersect twice. Now, these objects are both local *inertial frames*. This implies the strange result: the worldlines of different inertial frames may intersect each other more than once!



This is in striking contrast to what we found at the very beginning of this discussion (i.e. that the worldlines of inertial frames are straight lines and hence any two can intersect each other only once).

We can have two possible conclusions:

(1) The worldlines of inertial frames are not straight, after all. They are curved lines. (This conclusion is somewhat painful, given the simplicity and beauty of the geometric picture with which we started our discussion above.)

(2) Our initial geometric picture *is* true; the worldlines of inertial frames *are* straight lines (and those of non-inertial frames *are* curved lines), but *spacetime itself is curved!* (Analogy: spacetime behaves like a curved *surface* on which „straight" (=geodesic) *lines* can meet more than once.)

This second explanation is, of course, much more „esthetically attractive" than the first.

Now the question is: what *makes* spacetime curved? Answer: this whole complication arouse from gravity and gravitational mass *m*. So it is probably the (gravitational) masses that make spacetime curved in their vicinity!

We reached our final destination, the theory of general relativity.

As summarized by John Wheeler: „Mass tells spacetime how to curve[1], spacetime tells [free] mass how to move[2]."

[1] (see the Einstein equation)
[2] on a „straight" (i.e. geodesic) line