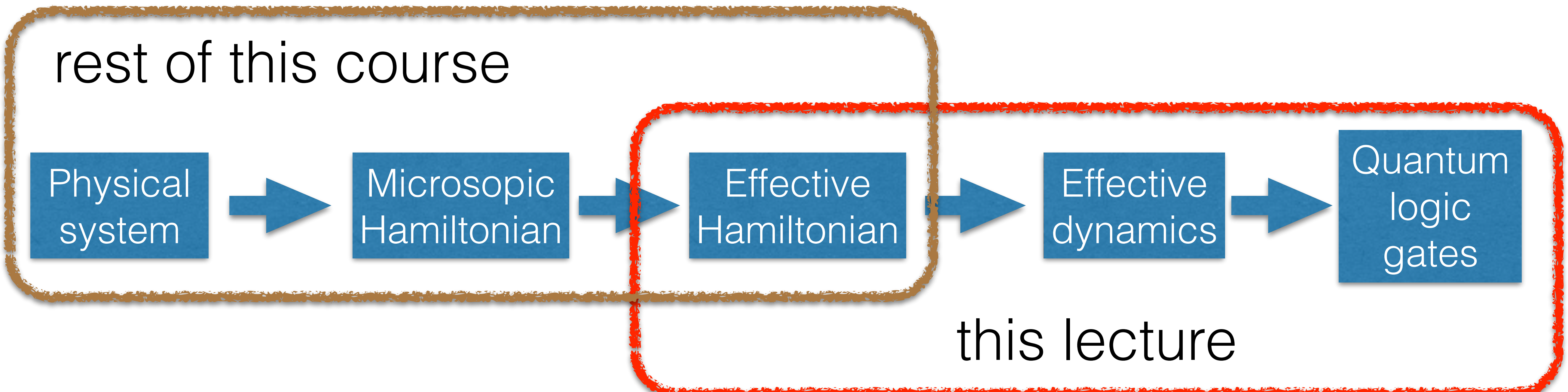
Quantum Computing Architectures

Budapest University of Technology and Economics
2018 Fall

Lecture 2
Control of quantum systems



Schedule of this course

Szerda		
augusztus 29. - Regisztrációs hét -		
szeptember 5.	lecture 01	**Introduction**
szeptember 12.		
szeptember 19.	lecture 02 (today)	
szeptember 26. TTK Dékáni szünet		
október 3.	lecture 03	**Spin qubits** (electron spin)
október 10.		
október 17.	lecture 04	
október 24.	lecture 05	
október 31.	lecture 06	**Superconducting qubits** (transmon)
november 7.	lecture 07	
november 14. TDK konferencia		
november 21.	lecture 08	
november 28.	lecture 09	
december 5.	lecture 10	

A few famous and useful model Hamiltonians

1. spin in a B-field
2. spin driven by square B-field pulses (=> single-qubit gates)
3. spin resonance (=> single-qubit gates)
4. Hubbard model and exchange interaction (=> two-qubit sqrt-of-swap)
5. Jaynes-Cummings Hamiltonian and its dispersive regime
6. driven Jaynes-Cummings Hamiltonian (=> single-qubit gates, readout)
7. two-qubit Jaynes-Cummings Hamiltonian (=> two-qubit sqrt-of-iswap)

A few famous and useful concepts

1. rotating frame
2. rotating-wave approximation
3. perturbation theory ⟶ $E_n^{(2)} = \sum_{k \neq n} \frac{\left| \langle k^{(0)} | V | n^{(0)} \rangle \right|^2}{E_n^{(0)} - E_k^{(0)}}$

Spin in a B-field

Hamiltonian:

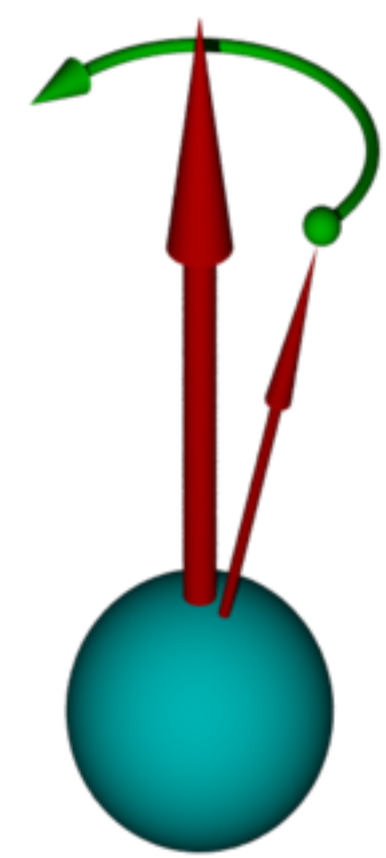
external B-field along z

$$H = \frac{1}{2} g \mu_B B_0 \sigma_z$$

g-factor,
~2 for e in vacuum

$\text{Bohr magneton} \approx 60\,\mu\text{eV/T}$

Dynamics of polarization vector: Larmor precession



Precession (Larmor) frequency @ 1 Tesla:

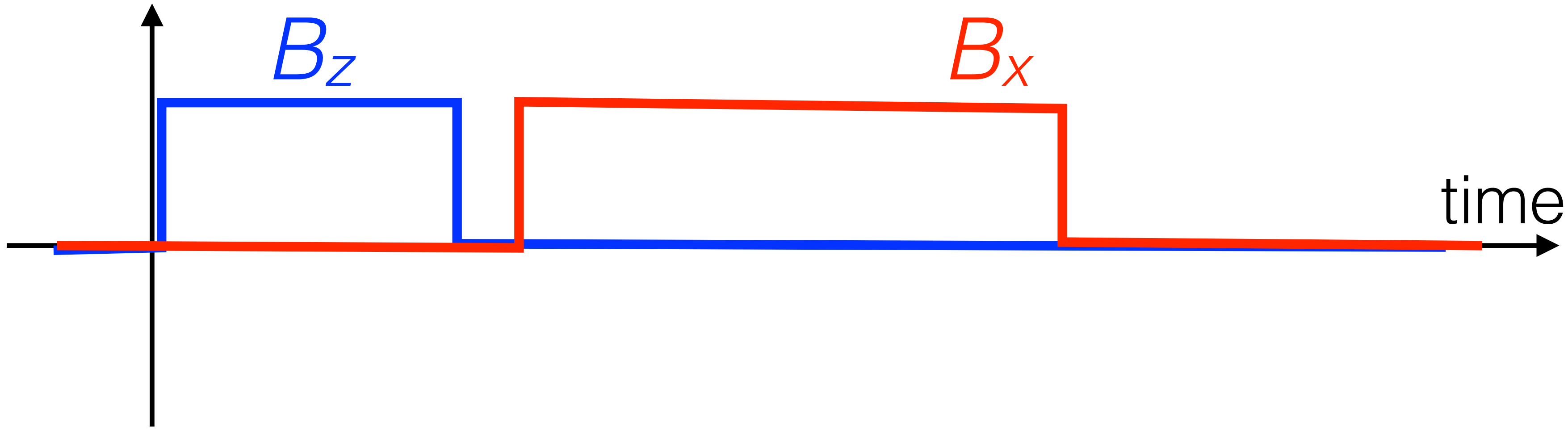
$$f_L = g\mu_B B_0/h \approx 28\,\text{GHz}$$

Homework: calculate the dynamics of the polarization vector from the TDSE.

Spin driven by square B-field pulses

$$H(t) = \frac{1}{2} g \mu_B \boldsymbol{B}(t) \cdot \boldsymbol{\sigma}$$

$$\boldsymbol{B}(t) = \begin{pmatrix} B_x(t) \\ 0 \\ B_z(t) \end{pmatrix}$$



Any rotation can be combined by an x-rotation and a z-rotation.

Any single-qubit gate can be realized by x- and z-directional B-field pulses.

Caveat: fast tuning of the magnetic field is difficult.

Homework: what is the duration of a NOT gate (`pi pulse') if a B of 1 mT is used?

Spin resonance (rotating drive)

$$H(t) = \frac{1}{2} g \mu_B B_0 \sigma_z + \frac{1}{2} g \mu_B B_{\mathrm{ac}} \left(\sigma_x \cos\omega t + \sigma_y \sin\omega t\right)$$

$$H(t) = \frac{1}{2} \hbar \omega_L \sigma_z + \frac{1}{2} \hbar \Omega \left(\sigma_x \cos\omega t + \sigma_y \sin\omega t\right)$$

Larmor frequency (ω_L) — drive strength (Ω) — drive frequency (ω)

$$\text{initial state: } \psi(t=0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{resonance condition: } \omega = \omega_L$$

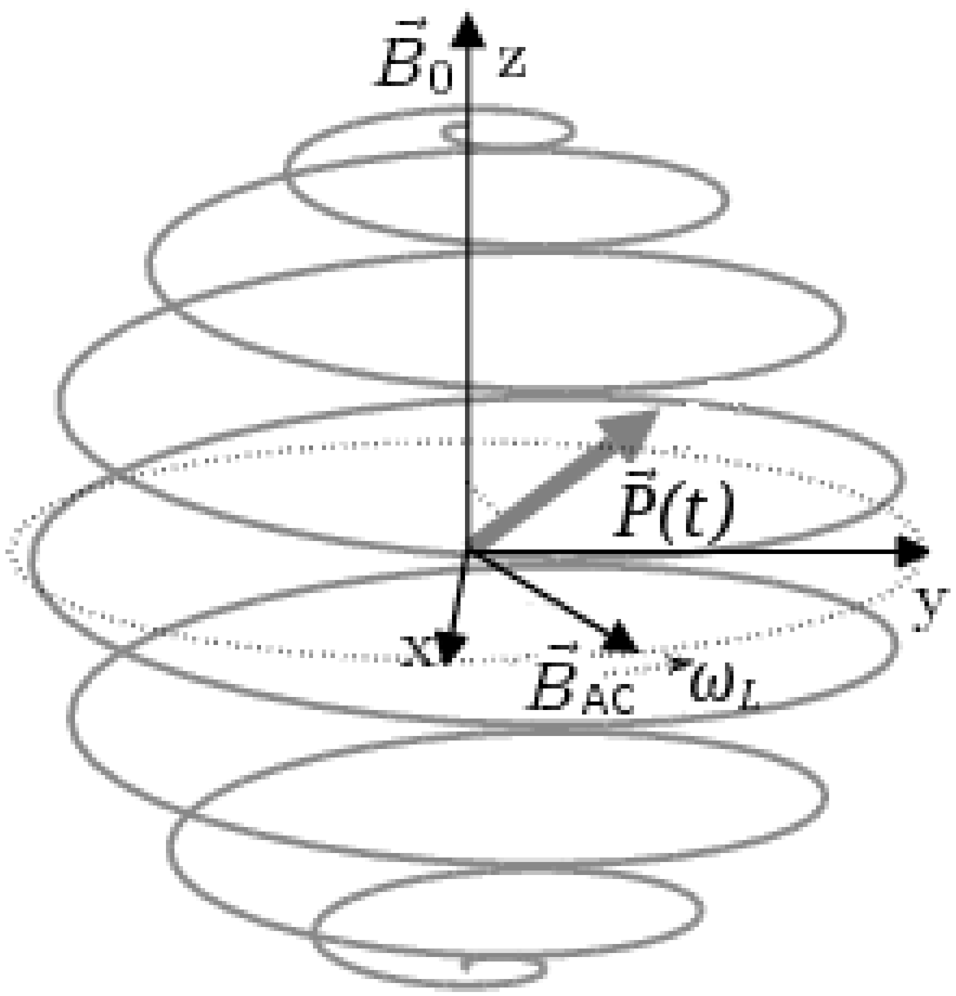
$$\psi(t) = ?$$

Spin resonance (rotating drive)

$$\psi(t) = ?$$

exactly solvable problem

time evolution of the polarization vector



precession around z
(Larmor precession)
frequency ω_L

north-south oscillation
(Rabi oscillation)
frequency Ω

Spin resonance (rotating drive)

How to solve the time-dependent Schrodinger equation?

Using the "transformation to the rotating frame".

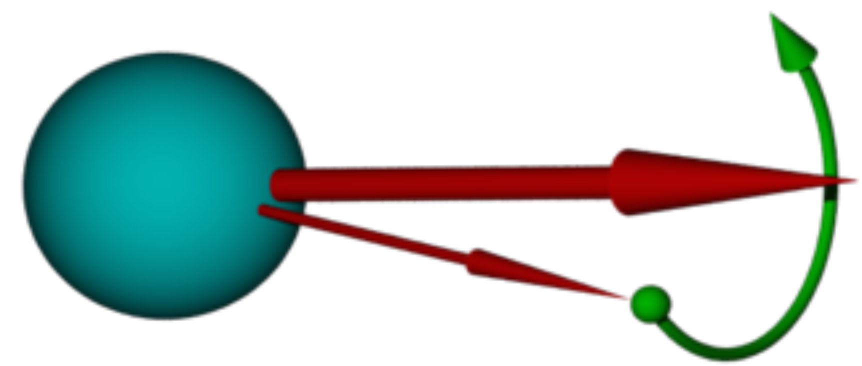
That is a time-dependent unitary transformation applied on the TDSE:

$$W(t) = e^{i\frac{H_{\text{static}}}{\hbar}t} = e^{i\frac{1}{2}\omega_L \sigma_z t}$$

$$\frac{\hbar}{i}\dot{\psi}(t) + H(t)\psi(t) = 0$$

$$\frac{\hbar}{i}\dot{\tilde{\psi}}(t) + \tilde{H}(t)\tilde{\psi}(t) = 0$$

$$\tilde{\psi}(t) = W(t)\psi(t)$$



Larmor precession around x

$$\tilde{H}(t) = W(t)H(t)W^\dagger(t) - \frac{\hbar}{i}\dot{W}(t)W^\dagger(t) = \frac{1}{2}\hbar\Omega\sigma_x$$

This is the "Hamiltonian in the rotating frame".
It is a time-independent Hamiltonian.
Hence the dynamics is exactly solvable.

We describe qubit dynamics in the rotating frame

$$H(t) = \frac{1}{2}\hbar\omega_L\sigma_z \longrightarrow \tilde{H} = 0$$

$$H(t) = \frac{1}{2}\hbar\omega_L\sigma_z + \frac{1}{2}\hbar\Omega\left(\sigma_x \cos\omega t + \sigma_y \sin\omega t\right) \longrightarrow \tilde{H} = \frac{1}{2}\Omega\sigma_x$$

$$H(t) = \frac{1}{2}\hbar\omega_L\sigma_z + \frac{1}{2}\hbar\Omega\left(\sigma_x \cos\left(\omega t + \frac{\pi}{4}\right) + \sigma_y \sin\left(\omega t + \frac{\pi}{4}\right)\right)$$

$$\searrow \quad \tilde{H} = \frac{1}{2}\Omega\sigma_y$$

- a drive pulse rotates the polarization vector
- rotation axis depends on the phase of the drive pulse
- rotation angle depends on the product of the amplitude and duration of the pulse
- any rotation can be composed from x and y rotations
- **any single-qubit gate can be performed with spin resonance**

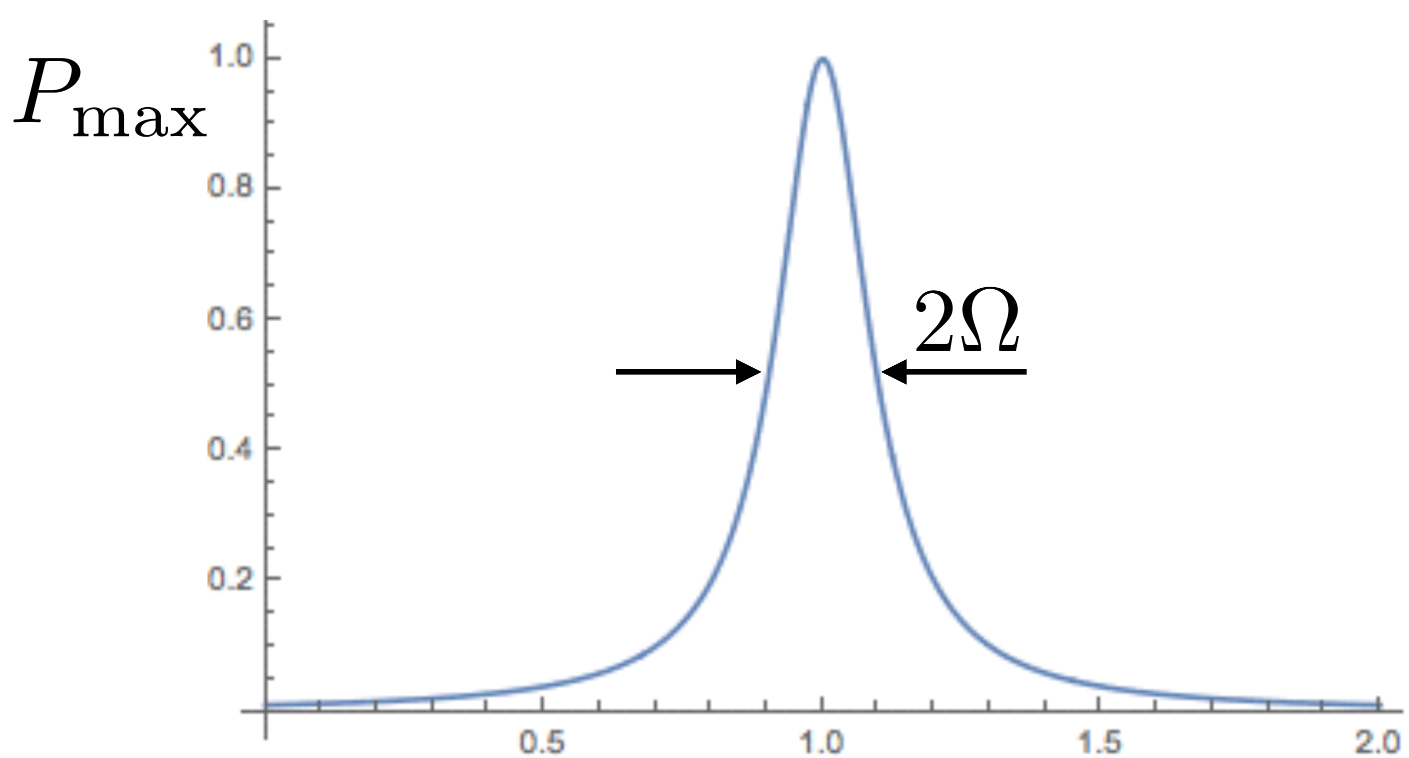
Power broadening

If driving is `off-resonant' or `detuned', then the spiral-like polarization dynamics is only partial, it doesn't reach the north pole.

$$\text{`detuning'}: \ \delta = \omega_L - \omega$$

If the initial state is the ground state, then the excited-state probability is:

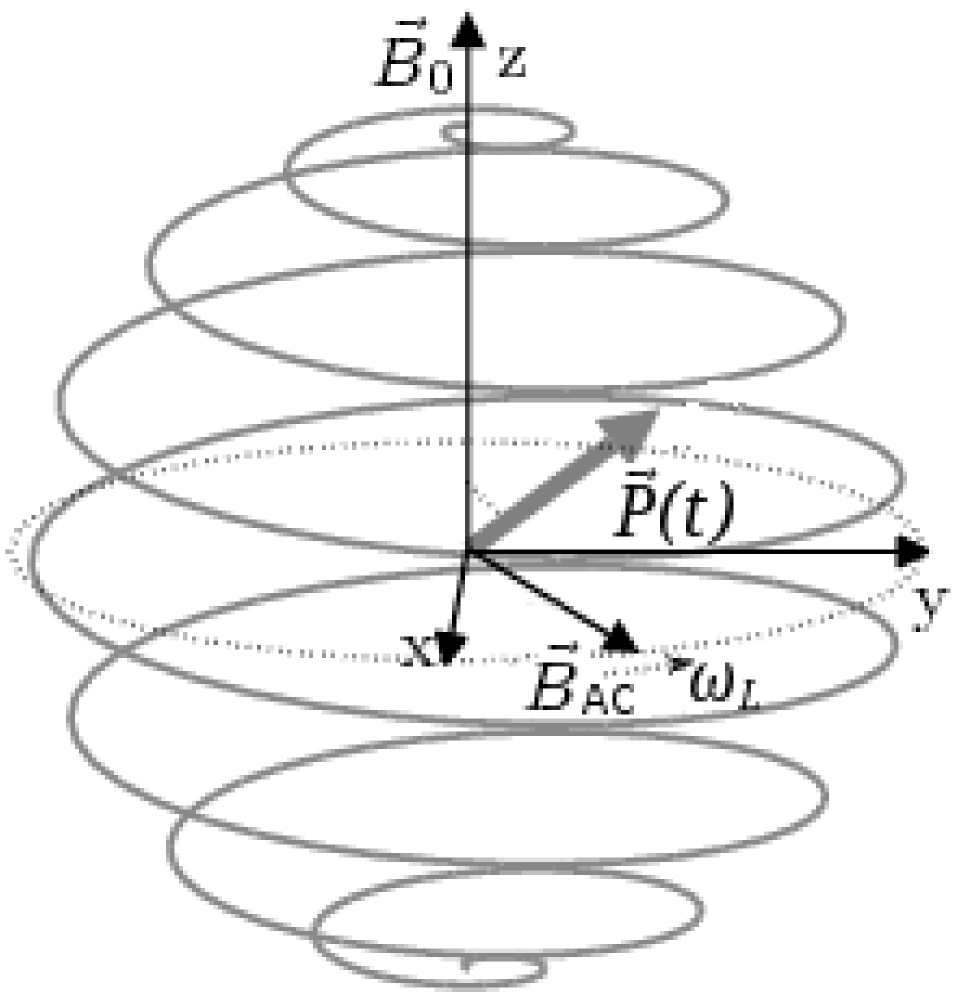
$$P_e(t) = P_{\max}(\delta) \sin^2\left(\tfrac{1}{2}\sqrt{\Omega^2 + \delta^2} t\right) = \frac{\Omega^2}{\Omega^2+\delta^2} \sin^2\left(\tfrac{1}{2}\sqrt{\Omega^2 + \delta^2} t\right)$$



relative drive frequency, ω/ω_L

Spin resonance (linear drive)

$$H(t) = \frac{1}{2} g\mu_B B_0 \sigma_z + \frac{1}{2} g\mu_B B_{\mathrm{ac}} \sigma_x \cos\omega t$$



weak driving:

$$\Omega \ll \omega_L$$

for weak driving, the qubit dynamics is approximately the same as with rotating drive

most experiments use linear drive (simpler)

From exchange interaction to sqrt-of-swap gate

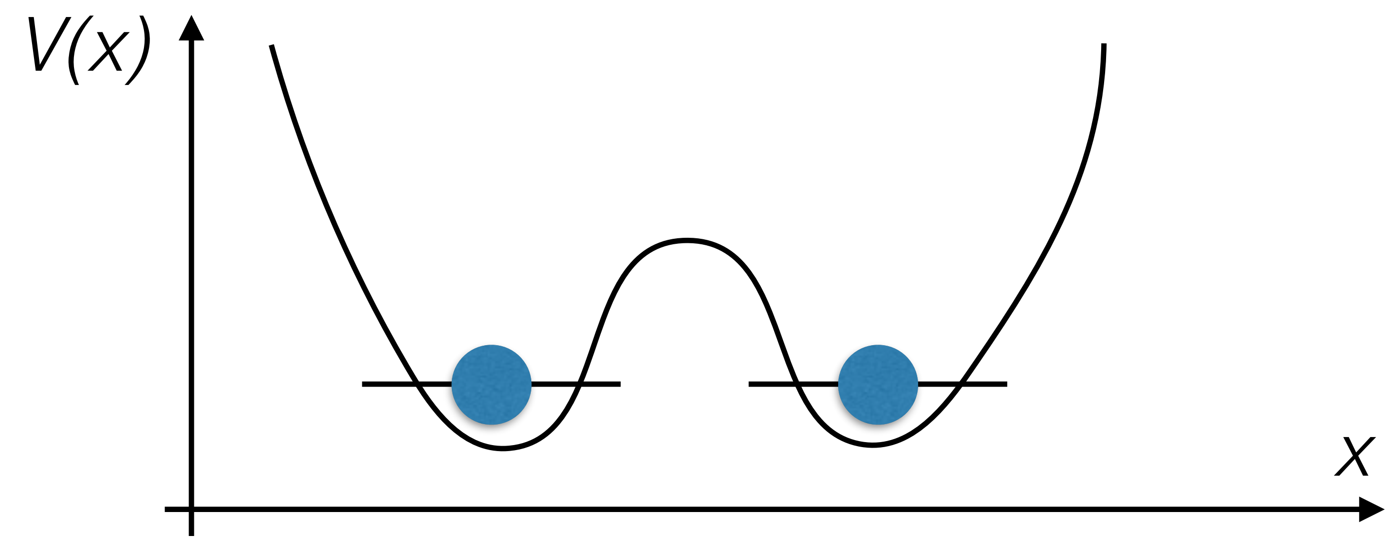
Loss & DiVincenzo, PRA 1998

- reminder: sqrt-of-swap + single-qubit gates = universal gate set

$$U_{\sqrt{\mathrm{SWAP}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} & 0 \\ 0 & \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

basis-state ordering $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

- setup: two electrons in a double well (dot)



- high/low barrier => tunneling off/on

- simple description: two-site Hubbard model

$$H_{\mathrm{Hubbard}} = H_{\text{on-site}} + H_{\mathrm{tun}} + H_{\mathrm{Coulomb}}$$

$$H_{\text{on-site}} = \varepsilon_L n_L + \varepsilon_R n_R$$

$$H_{\mathrm{tun}} = t_H \left(a^\dagger_{L\uparrow} a_{R\uparrow} + a^\dagger_{L\downarrow} a_{R\downarrow} + h.c. \right)$$

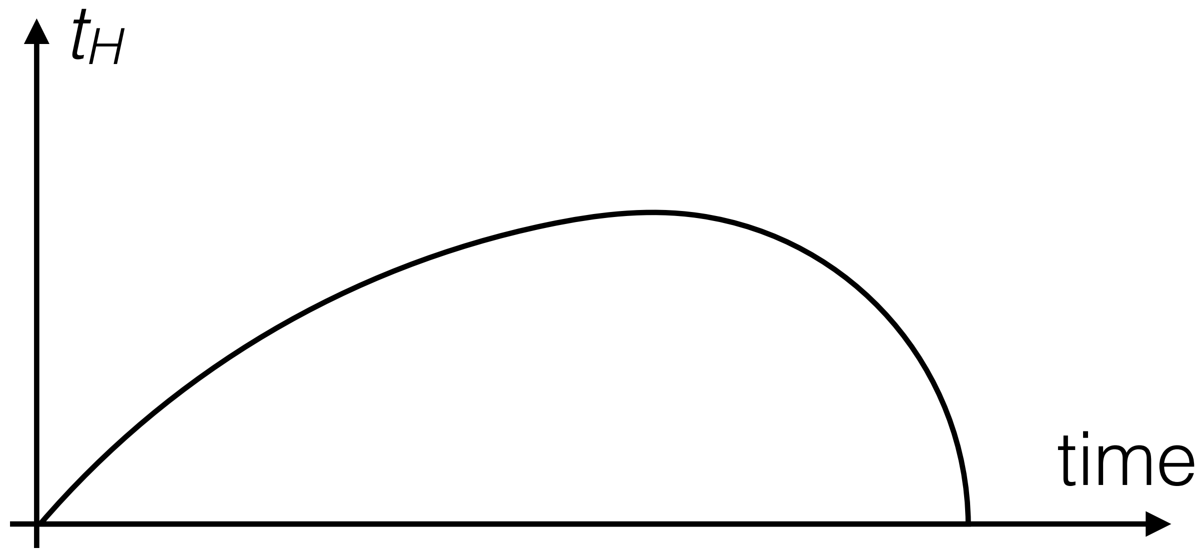
$$H_{\mathrm{Coulomb}} = U(n_{L\uparrow} n_{L\downarrow} + n_{R\uparrow} n_{R\downarrow})$$

$$n_{L\uparrow} = a^\dagger_{L\uparrow} a_{L\uparrow}, \text{ etc.}$$

- on-site energies = zero
- tunable tunnel amplitude
- strong Coulomb repulsion

$$t_H \ll U$$

The statement



$$\mathcal{A} = \int_{t_0}^{t_1} dt\, t_H^2(t)$$

If $\frac{4\mathcal{A}}{\hbar U} = \frac{3\pi}{2}$ then $\psi(t_1) = U_{\sqrt{\mathrm{SWAP}}}\psi(t_2)$.

The proof

- 2 electrons in the Hubbard model => 6 states: (2,0), (1,1)x4, (0,2)

basis: $|2,0\rangle,\ |{\downarrow},{\downarrow}\rangle,\ |{\downarrow},{\uparrow}\rangle,\ |{\uparrow},{\downarrow}\rangle,\ |{\uparrow},{\uparrow}\rangle,\ |0,2\rangle$

$$H_{\mathrm{Coulomb}} = \begin{pmatrix} U & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U \end{pmatrix} \qquad H_{\mathrm{tun}} = \begin{pmatrix} 0 & 0 & t_H & -t_H & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ t_H & 0 & 0 & 0 & 0 & t_H \\ -t_H & 0 & 0 & 0 & 0 & -t_H \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_H & -t_H & 0 & 0 \end{pmatrix}$$

Exercise: calculate these matrices.

The proof (contd.)

- unitary transformation to Singlet-Triplet (S-T) basis + reordering the basis

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

basis:

$|2,0\rangle$

$|0,2\rangle$

$|S(1,1)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\right)$

$|T_0(1,1)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle\right)$

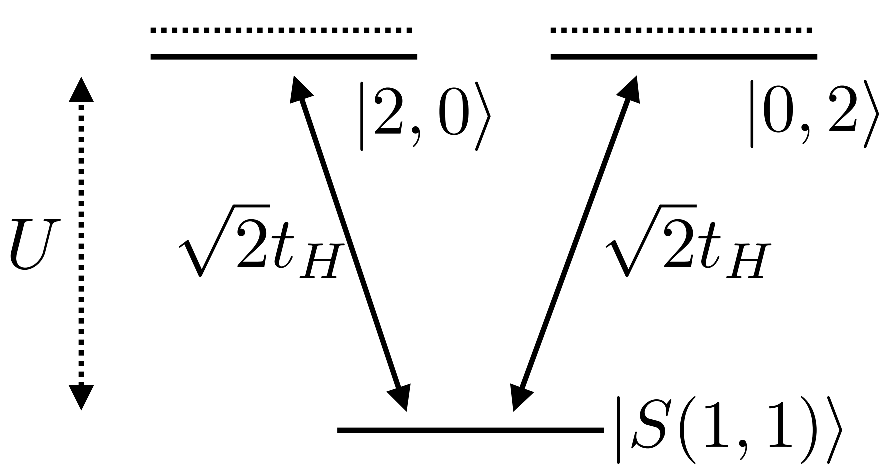
$|T_-\rangle = |\downarrow,\downarrow\rangle$

$|T_+\rangle = |\uparrow,\uparrow\rangle$

$$H'_{\text{Hubbard}} = W H_{\text{Hubbard}} W^\dagger = \begin{pmatrix} U & 0 & \sqrt{2}t_H & 0 & 0 & 0 \\ 0 & U & \sqrt{2}t_H & 0 & 0 & 0 \\ \sqrt{2}t_H & \sqrt{2}t_H & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

irrelevant subspace

relevant subspace



do perturbation theory:

$$E_n^{(2)} = \sum_{k \neq n} \frac{\left|\langle k^{(0)}|V|n^{(0)}\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}} \Rightarrow E_{S(1,1)}^{(2)} = -\frac{4t_H^2}{U}$$

The proof (contd.)

- solve the dynamics for this (approximate Hamiltonian):

$$H'_{\text{Hubbard}} \approx \begin{pmatrix} -\frac{4t_H^2}{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} S \\ T_0 \\ T_- \\ T_+ \end{matrix}$$

$$\varphi = \frac{1}{\hbar} \int_{t_0}^{t_1} dt \, \frac{4t_H^2}{U}$$

- transform back to product basis:

$$U'(t) = e^{-i\frac{H'_{\text{Hubbard}}}{\hbar}t} \Rightarrow U(t) = W^\dagger U'(t) W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \frac{1}{2}e^{i\varphi} & \frac{1}{2} - \frac{1}{2}e^{i\varphi} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2}e^{i\varphi} & \frac{1}{2} + \frac{1}{2}e^{i\varphi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- it gives sqrt-of-swap if:

$$\varphi = \frac{3\pi}{2}$$

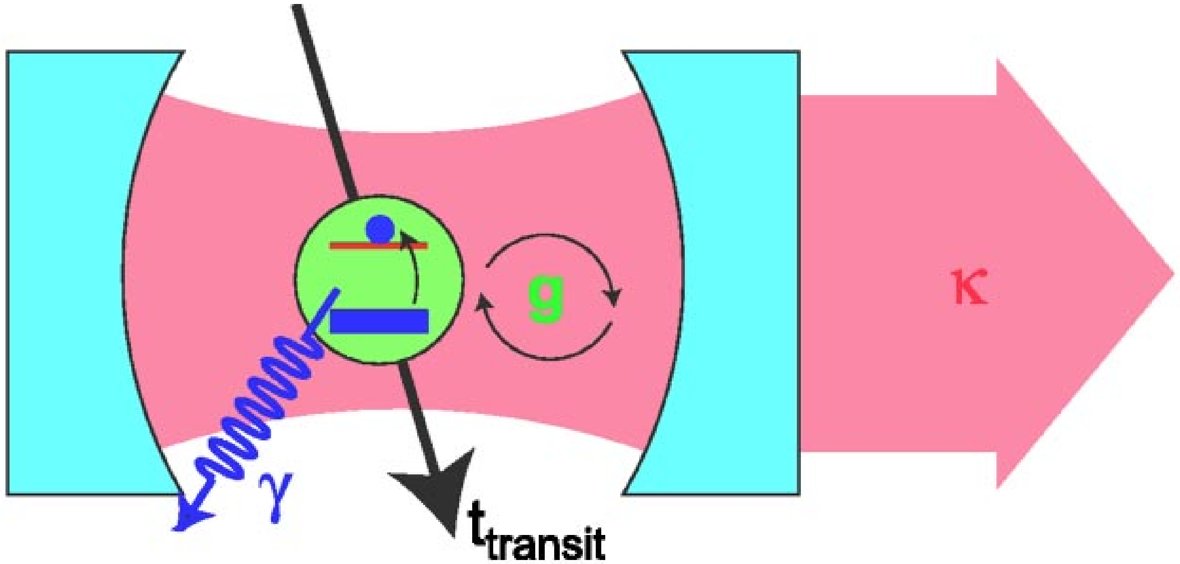
$$U_{\sqrt{\text{SWAP}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} & 0 \\ 0 & \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Q.E.D.

Exercise: do the calculations that were omitted here.

Jaynes-Cummings Hamiltonian

Setup: qubit interacting with a harmonic oscillator



oscillator frequency (`resonator frequency')

$$H = \hbar\omega_{\mathrm{r}}\left(a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\Omega}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + \sigma^+ a) + H_\kappa + H_\gamma.$$

qubit-oscillator coupling strength

oscillator = resonator = cavity = one mode of a microwave resonator

qubit = e-charge, e-spin, superconducting qubit

'strong coupling' regime: $\gamma, \kappa \ll g$

many back-and-forth oscillations of an energy quantum between qubit and oscillator are possible

Blais et al., Phys. Rev. A 69, 062320 (2004)

Typical parameter values in `cavity/circuit quantum electrodynamics'

Parameter	Symbol	3D optical	3D microwave	1D circuit
Resonance or transition frequency	$\omega_r/2\pi$, $\Omega/2\pi$	350 THz	51 GHz	10 GHz
Vacuum Rabi frequency	g/π, g/ω_r	220 MHz, 3×10^{-7}	47 kHz, 1×10^{-7}	100 MHz, 5×10^{-3}
Transition dipole	d/ea_0	~1	1×10^3	2×10^4
Cavity lifetime	$1/\kappa, Q$	10 ns, 3×10^7	1 ms, 3×10^8	160 ns, 10^4
Atom lifetime	$1/\gamma$	61 ns	30 ms	2 μs
Atom transit time	t_{transit}	$\geqslant50$ μs	100 μs	∞
Critical atom number	$N_0=2\gamma\kappa/g^2$	6×10^{-3}	3×10^{-6}	$\leqslant6\times10^{-5}$
Critical photon number	$m_0=\gamma^2/2g^2$	3×10^{-4}	3×10^{-8}	$\leqslant1\times10^{-6}$
Number of vacuum Rabi flops	$n_{\mathrm{Rabi}}=2g/(\kappa+\gamma)$	~10	~5	$\sim10^2$

strong coupling achieved in circuit QED

we assume strong coupling from now on

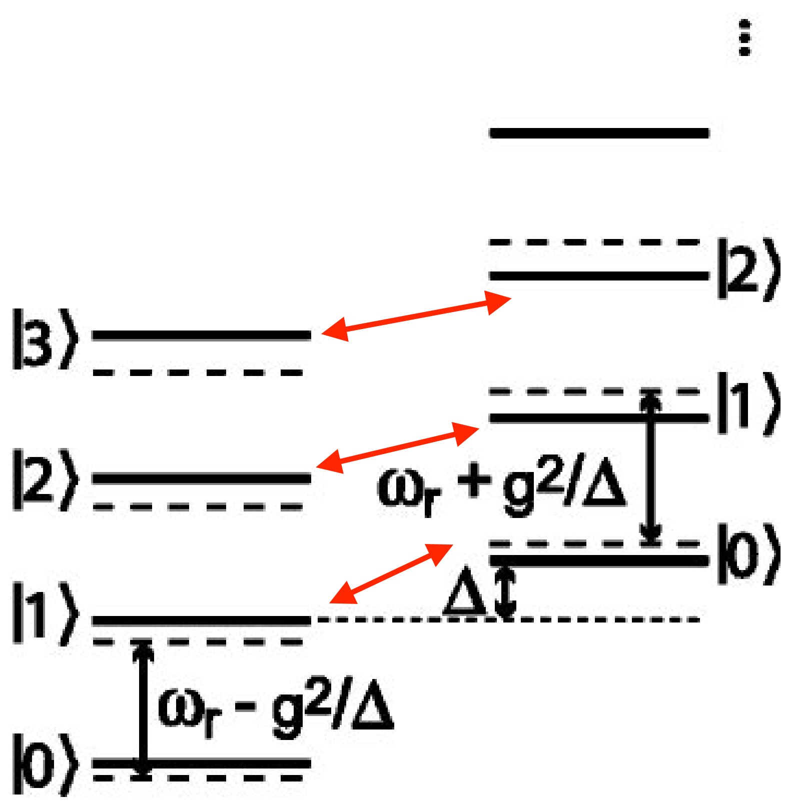
Table I. from Blais et al., Phys. Rev. A 69, 062320 (2004)

`Dispersive qubit readout' in circuit QED

$$H = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\Omega}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + \sigma^+ a) + H_\kappa + H_\gamma.$$

qubit-oscillator detuning: $\Delta \equiv \Omega - \omega_r$

`large detuning regime' or `dispersive regime': $g/\Delta \ll 1$



$$|g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

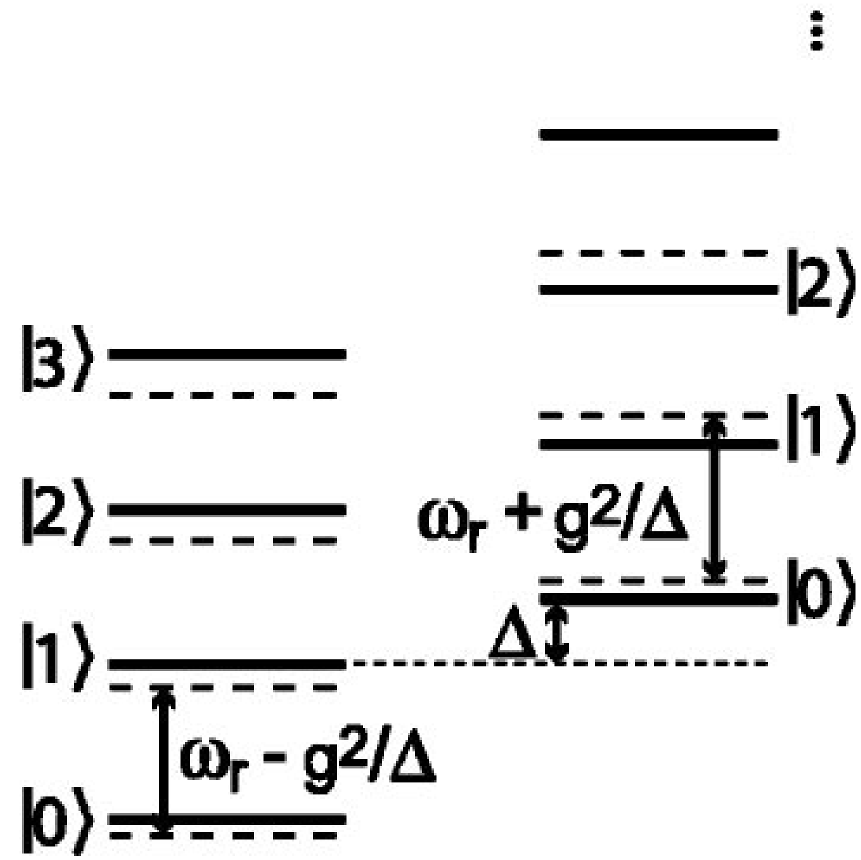
do perturbation theory:

$$E_n^{(2)} = \sum_{k\neq n} \frac{\left|\langle k^{(0)}|V|n^{(0)}\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}} \Rightarrow E_{g1}^{(2)} = -\frac{g^2}{\Delta}$$

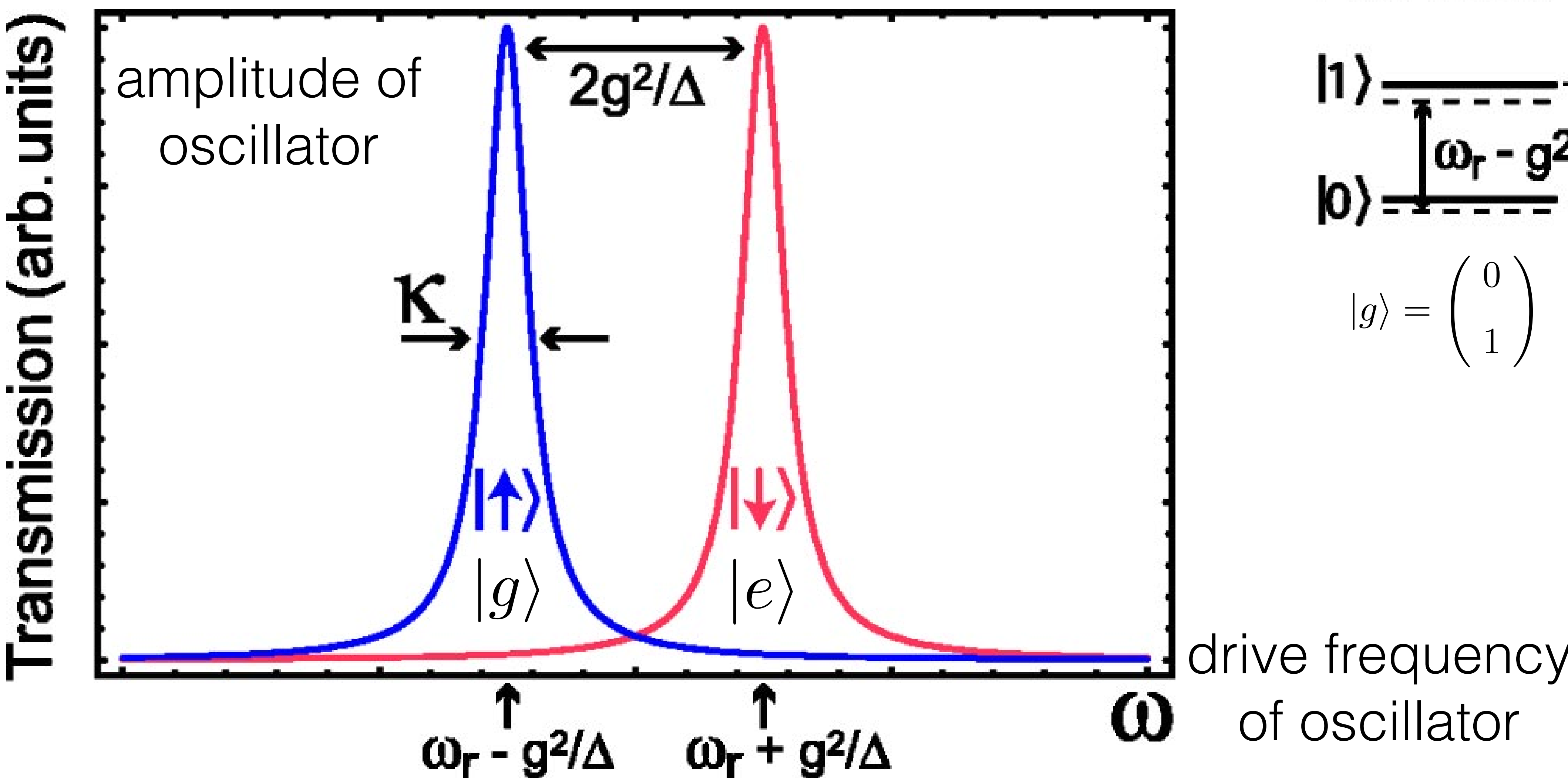
In the dispersive regime, the resonator acquires a qubit-state dependent shift of its eigenfrequency.

`Dispersive qubit readout' in circuit QED

$$H = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\Omega}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + \sigma^+ a) + H_\kappa + H_\gamma.$$



$$|g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



In the dispersive regime, the qubit can be read out by probing the oscillator.

An alternative way to derive the `dispersive cavity shift'

- start from Jaynes-Cummings Hamiltonian:

$$H = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\Omega}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + \sigma^+ a) + H_\kappa + H_\gamma.$$

- do a `small' unitary transformation: $U = \exp\left[\frac{g}{\Delta}(a\sigma^+ - a^\dagger\sigma^-)\right]$

- expand the result up to second order in *g:*

$$UHU^\dagger \approx \hbar\left[\omega_r + \frac{g^2}{\Delta}\sigma^z\right]a^\dagger a + \frac{\hbar}{2}\left[\Omega + \frac{g^2}{\Delta}\right]\sigma^z.$$

↑ qubit-state-dependent cavity eigenfrequency

A sqrt-of-iSWAP gate in circuit QED

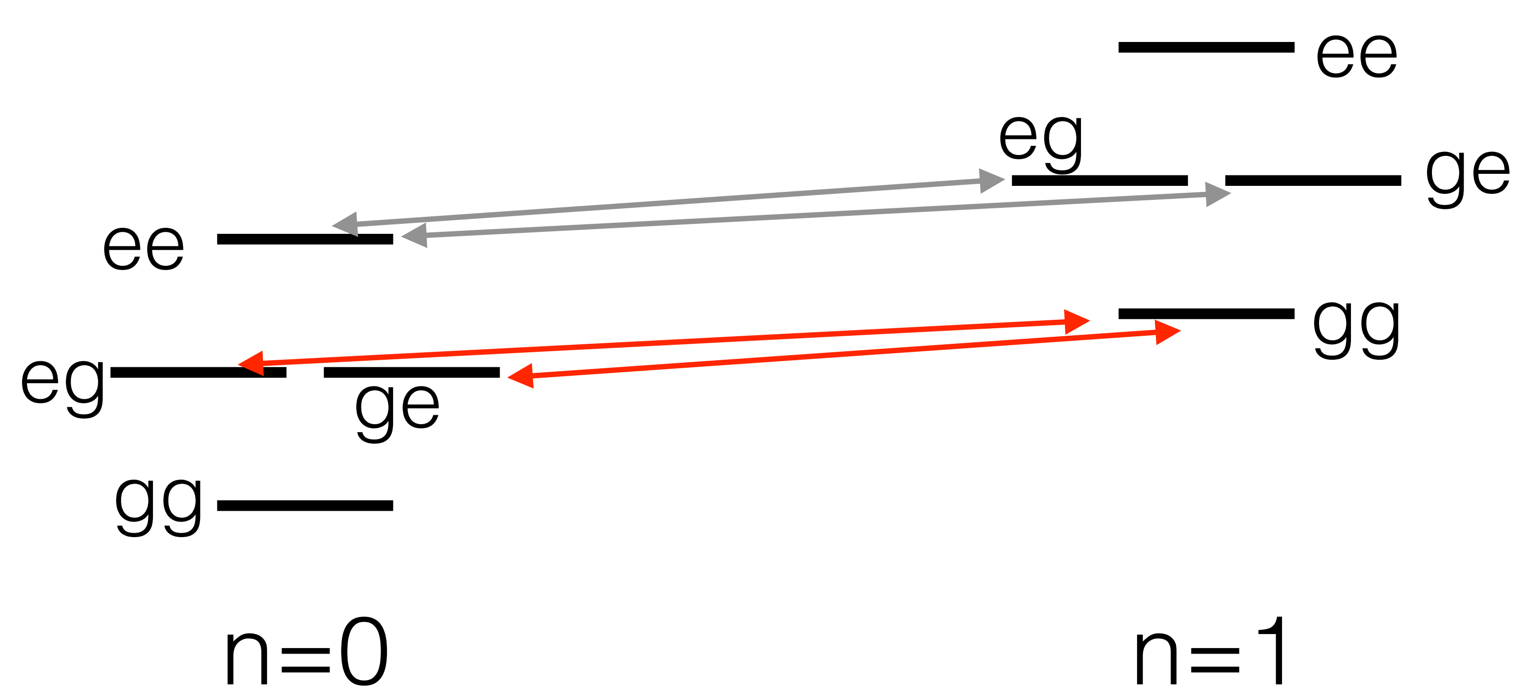
- Setup: two qubits (i and j) interacting with the same oscillator
- Do the unitary transformation + expansion from the last slide

$$H_{2q} \approx \hbar\left[\omega_r + \frac{g^2}{\Delta}(\sigma_i^z + \sigma_j^z)\right]a^\dagger a + \frac{1}{2}\hbar\left[\Omega + \frac{g^2}{\Delta}\right](\sigma_i^z + \sigma_j^z)$$

$$+\hbar\frac{g^2}{\Delta}(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+). \qquad (32)$$

qubit-qubit interaction

qubit-qubit interaction:
from
`virtual photon exchange'



A sqrt-of-iSWAP gate in circuit QED

$$H_{2q} \approx \hbar\left[\omega_{\mathrm{r}} + \frac{g^2}{\Delta}(\sigma_i^z + \sigma_j^z)\right]a^\dagger a + \frac{1}{2}\hbar\left[\Omega + \frac{g^2}{\Delta}\right](\sigma_i^z + \sigma_j^z)$$
$$+ \hbar\frac{g^2}{\Delta}(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+)\,. \qquad (32)$$

In a frame rotating at the qubit's frequency Ω, H_{2q} generates the evolution

$$U_{2q}(t) = \exp\left[-i\frac{g^2}{\Delta}t\left(a^\dagger a + \frac{1}{2}\right)(\sigma_i^z + \sigma_j^z)\right]$$
$$\times\begin{pmatrix} 1 & & & \\ & \cos\frac{g^2}{\Delta}t & i\sin\frac{g^2}{\Delta}t & \\ & i\sin\frac{g^2}{\Delta}t & \cos\frac{g^2}{\Delta}t & \\ & & & 1 \end{pmatrix} \otimes \mathbb{1}_r, \qquad (33)$$

Up to phase factors, this corresponds at $t = \pi\Delta/4g^2$ to a $\sqrt{i\mathrm{SWAP}}$ operation.

Together with single-qubit gates, it forms a universal gate set.

Turning the sqrt-of-iSWAP gate On and Off

$$H_{2q} \approx \hbar\left[\omega_r + \frac{g^2}{\Delta}(\sigma_i^z + \sigma_j^z)\right]a^\dagger a + \frac{1}{2}\hbar\left[\Omega + \frac{g^2}{\Delta}\right](\sigma_i^z + \sigma_j^z)$$

$$+\hbar\frac{g^2}{\Delta}(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+). \qquad (32)$$

qubit-qubit interaction: always On

- the effect of the qubit-qubit interaction on dynamics is suppressed at \`large qubit-qubit detuning', that is, if:

$$g^2/\Delta \ll |\Omega_i - \Omega_j|$$

- the sqrt-of-iSWAP gate can be turned Off by detuning the two qubits from each other

Summary of key results

1. spin resonance => single-qubit gates
2. Hubbard model and exchange interaction => two-qubit sqrt-of-swap
3. qubit readout with a dispersively coupled oscillator
4. two-qubit sqrt-of-iswap via virtual photon exchange